Geometric Analysis of the Averaged Euler Equations

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Abstract

This report concerns the geometric analysis of the averaged Euler equations of ideal incompressible hydrodynamics. The equations are a system of conservative PDEs that model the motion of fluids at length scales greater than some given scale in the problem (for instance, the smallest scale possible within the numerical discretization). Remarkably, this conservative model is able to capture the vortex merger phenomenon without the addition of viscous dissipation. We prove sharp well-posedness results for the system by studying the geometry of the group of volume preserving diffeomorphisms; it is rigorously shown that geodesics of an H^1 right invariant metric on this group are solutions to the averaged Euler equations.

1 Introduction

The averaged Euler (or Euler-α) equations are a system of hyperbolic PDEs which model the motion of ideal incompressible fluids at length scales larger than some small length scale $\alpha > 0$; on a compact oriented n-dimensional Riemannian manifold M with metric compatible connection ∇, they may be expressed as

$$\begin{gathered}\partial_t U^\epsilon + (1-\epsilon)^{-1}\left[\nabla^\epsilon_U (1-\epsilon\triangle)U^\epsilon - \epsilon(\nabla U^\epsilon)^T \cdot \triangle U^\epsilon\right] = (1-\epsilon\triangle)^{-1}\,\mathrm{grad}\, p,\\ \mathrm{div}\, U^\epsilon = 0, \quad U^\epsilon(0) = U^\epsilon_0,\end{gathered} \tag{1.1}$$

with appropriate boundary conditions and with $\epsilon = \alpha^2$ (here, $\triangle$ is the rough Laplacian which we shall discuss in subsection 2.3). This system of PDEs appeared with $\epsilon = \alpha^2$ in [9] as a model of large scale fluid motion, but with $\epsilon = \alpha_1$, this system is also the model for ideal non-Newtonian fluids of second grade (see [5] and the references therein for the history).

This report will survey only the geometric analysis of the inviscid equations (1.1). See [14] for the use of (1.1) as a conservative model of vortex merger in two spatial dimensions. Although we shall not discuss the viscous version of these equations herein, there is a growing list of literature that the reader is referred to [7, 3, 4] and references therein.

2 Averaged Euler Equation on compact boundaryless Riemannian manifolds

2.1 Preliminaries

Let $(M, \langle \cdot, \cdot \rangle)$ be a compact oriented Riemannian n dimensional manifold without boundary and define $\mathcal{D}^s(M)$ to be the set of all bijective maps $\eta : M \to M$ such that η and η^{-1} are of Sobolev class H^s. For $s > \frac{n}{2}+1$, $\mathcal{D}^s(M)$ is a C^∞ infinite dimensional Hilbert manifold which, about each η, is locally diffeomorphic to the Hilbert space $H^s_\eta(TM) := \{X \in H^s(M, TM) : \pi \circ X = \eta\}$ where $\pi : TM \to M$. The condition $s > \frac{n}{2}+1$ ensures that $\mathcal{D}^s(M) \subset H^s(M, M)$ is open (see [10], Proposition 2.3.1).

A local chart is given by $\omega_{\exp} : H^s_\eta(TM) \to \mathcal{D}^s(M)$, $\omega_{\exp}(X) = \exp \circ X$, where exp is the Riemannian exponential map of $\langle \cdot, \cdot \rangle$. The manifold $\mathcal{D}^s(M)$ is a topological group with composition being the group operation. The ω-lemma asserts that for each $\eta \in \mathcal{D}^s(M)$, right composition $\alpha_\eta : \mathcal{D}^s(M) \to \mathcal{D}^s(M)$ is C^∞, while for all $\eta \in \mathcal{D}^{s+r}(M)$, left composition $\omega_\eta : \mathcal{D}^s(M) \to \mathcal{D}^s(M)$ is C^r.

2.2 Weak L^2 structure

The weak L^2 right invariant Riemannian metric on $\mathcal{D}^s(M)$ is given by

$$\langle X_\eta, Y_\eta \rangle_0 = \int_M \langle X_\eta(x), Y_\eta(x) \rangle_{\eta(x)} \mu(x), \tag{2.1}$$

where $\eta \in \mathcal{D}^s(M)$, $X_\eta, Y_\eta \in T_\eta \mathcal{D}^s(M)$, and $\langle \cdot, \cdot \rangle$ and μ are the Riemannian metric and volume element on M.

2.3 The Laplacian

Letting $\triangle = d\delta + \delta d$ denote the Laplace-de Rham operator[1], we define the H^s metric as follows. Let $X, Y \in T_e \mathcal{D}^s(M)$ and set

$$\langle X, Y \rangle_s = \int_M \langle X(x), (1 + \triangle^s) Y(x) \rangle \mu(x). \tag{2.2}$$

Extending $\langle \cdot, \cdot \rangle_s$ to $\mathcal{D}^s(M)$ by right invariance gives a smooth invariant metric on $\mathcal{D}^s(M)$. We shall be particularly interested in the metric $\langle \cdot, \cdot \rangle_1$.

[1] We identify vector fields and 1-forms on M.

In order to obtain formulas for the unique Levi-Civita covariant derivative of $\langle\cdot,\cdot\rangle_1$, it is convenient to express the metric (2.2) in terms of the rough Laplacian $\hat{\triangle} = \mathrm{Tr}\nabla\nabla$. We will need the relationship between the rough Laplacian and the Laplace-de Rham operator so that we may express (2.2) in terms of $\hat{\triangle}$. Let ∇^* denote the L^2 formal adjoint of ∇ so that for any $X \in C^\infty(TM)$ and $S, T \in C^\infty(E)$, E a vector bundle over M, $\langle \nabla^*_X S(x), T(x)\rangle_0 = \langle S(x), \nabla_X T(x)\rangle_0$. Then $\nabla^*_X = -\nabla_X + \mathrm{div}X$. To see this, note that

$$\begin{aligned}\langle \nabla^*_X S, T\rangle_0 = \int \langle S, \nabla_X T\rangle\mu &= \int X\langle S, T\rangle\mu - \langle \nabla_X S, T\rangle_0 \\ &= \int \langle S, T\rangle \mathrm{div}X\mu - \langle \nabla_X S, T\rangle_0.\end{aligned}$$

If $\mathrm{div}X = 0$, then $\nabla^*_X = -\nabla_X$ which we shall often make use of.

Next, let $\tau \in C^\infty(T^*M \otimes TM)$, let $\{e_i\}$ be a local orthonormal frame on M, and let $\sigma \in C^\infty(TM)$ with support in the domain of definition of the local frame $\{e_i\}$. Then

$$\langle \nabla^*\tau, \sigma\rangle_0 = \langle \tau, \nabla\sigma\rangle_0 = \langle \tau\langle e_i\rangle, \nabla_{e_i}\sigma\rangle_0 = \langle \nabla^*_{e_i}(\tau\langle e_i\rangle), \sigma\rangle_0.$$

We may choose the frame $\{e_i\}$, so that locally $\nabla e_i = 0$ and hence $\mathrm{div}e_i = 0$. Then

$$\nabla^*\tau = \nabla^*_{e_i}\tau\langle e_i\rangle = -\nabla_{e_i}(\tau\langle e_i\rangle) = -(\nabla_{e_i}\tau)\langle e_i\rangle = -\nabla\tau(e_i, e_i),$$

where the last equality follows from our choice of frame, since $\nabla_{e_i}(\tau\langle e_i\rangle) = (\nabla_{e_i}\tau)\langle e_i\rangle = \nabla\tau\langle e_i, e_i\rangle$. Hence $\nabla^*\tau = -\nabla\tau(e_i, e_i)$, and since $\nabla X \in C^\infty(T^*M \otimes TM)$, we have that

$$\hat{\triangle} = -\nabla^*\nabla.$$

With the notation established, we write Bochner's formula relating $\hat{\triangle}$ with $\triangle$ on 1-forms as

$$\triangle\alpha = \hat{\triangle}\alpha + \alpha\langle Ric\langle\cdot\rangle\rangle, \tag{2.3}$$

where $Ric\langle X\rangle := R(e_i, X)e_i$, R being the curvature of ∇ on M. Because the Ricci tensor is a self-adjoint operator with respect to the metric on TM, for $X \in C^\infty(TM)$, we have that

$$\triangle X = \nabla^*\nabla X + Ric\langle X\rangle.$$

2.4 Weak H^1 metric

Using (2.2), the H^1 metric at the identity may be re-expressed as

$$\begin{aligned}\langle X, Y\rangle_1 &= \langle X, (1 + Ric)Y\rangle_{L^2} + \langle X, \nabla^*\nabla Y\rangle_{L^2} \\ &= \langle X, (1 + Ric)Y\rangle_{L^2} + \langle \nabla X, \nabla Y\rangle_{L^2}\end{aligned} \tag{2.4}$$

for all $X, Y \in T_e\mathcal{D}^s_\mu(M)$. The metric (2.4) extends smoothly by right translation in the following way. Let $X_\eta, Y_\eta \in T_\eta\mathcal{D}^s_\mu(M)$. Then

$$\begin{aligned}\langle X_\eta, Y_\eta\rangle_1 = &\int_M \langle X_\eta(x), Y_\eta(x) + Ric\langle Y_\eta \circ \eta^{-1}\rangle \circ \eta(x)\rangle_{\eta(x)} \\ &+\langle \nabla(X_\eta \circ \eta^{-1}) \circ \eta(x), \nabla(Y_\eta \circ \eta^{-1}) \circ \eta(x)\rangle_{\eta(x)}\mu.\end{aligned} \tag{2.5}$$

From the implicit function theorem, the set of all volume preserving H^s diffeomorphisms of M, $\mathcal{D}^s_\mu(M) := \{\eta \in \mathcal{D}^s(M) : \eta^*(\mu) = \mu\}$, is a submanifold of $\mathcal{D}^s(M)$ with the induced right invariant H^1 Riemannian metric, as well as a subgroup. For each $\eta \in \mathcal{D}^s_\mu(M)$, the metric (2.5) defines a smooth orthogonal projection $P_\eta : T_\eta\mathcal{D}^s(M) \to T_\eta\mathcal{D}^s_\mu(M)$ defined by

$$P_\eta(X) = (P_e(X \circ \eta^{-1})) \circ \eta, \quad X \in T_\eta\mathcal{D}^s(M), \tag{2.6}$$

where P_e is the H^1 orthogonal projection onto the 1-forms $\{\alpha \in H^s : \alpha \in \ker\delta\}$ in the Hodge decomposition

$$H^s(T^*M) = \ker\delta \oplus_{H^1} dH^{s+1}(M). \tag{2.7}$$

Remark 2.1 *We remark here that it is essential to use the Laplace-de Rham operator in defining the metric (2.5) in order for the Hodge decomposition to hold. Using the rough Laplacian instead to define the H^1 metric would not provide an orthogonal decomposition in the H^1 topology of divergence-free vector fields and gradients of functions, unless the manifold M is either flat or Einstein, as can be seen from (2.3).*

3 H^1 covariant derivative and its geodesic flow

3.1 Weak H^1 Riemannian connection

Next, we compute the Riemannian covariant derivative on $\mathcal{D}^s(M)$ of the H^1 right invariant metric restricted to vectors tangent to $\mathcal{D}^s_\mu(M)$. Using the Hodge decomposition, we define the induced covariant derivative $\tilde{\nabla}^1$ on $\mathcal{D}^s_\mu(M)$. We then prove the local well-posedness of the geodesic equations of $\tilde{\nabla}^1$.

Theorem 3.1 ([16]) *The unique Levi-Civita covariant derivative ∇^1 of $\langle\cdot,\cdot\rangle_1$ restricted to vector fields in $T\mathcal{D}^s_\mu(M)$ is given by*

$$\nabla^1_X Y = \nabla^0_X Y + A(X,Y) + B(X,Y) + C(X,Y), \tag{3.1}$$

where for any $\eta \in \mathcal{D}^s_\mu(M)$,

$$\begin{aligned}
A_\eta(X_\eta, Y_\eta) = \frac{1}{2}(1 + Ric_\eta - \hat{\triangle}_\eta)^{-1} \Big[&\nabla^*\{\nabla X_\eta[T\eta]^{-1}\nabla Y_\eta[T\eta]^{-1}[T\eta]^{-1^t} \\
&+\nabla Y_\eta[T\eta]^{-1}\nabla X_\eta[T\eta]^{-1}[T\eta]^{-1^t} + (\nabla X_\eta[T\eta]^{-1})(\nabla Y_\eta[T\eta]^{-1})^t[T\eta]^{-1^t} \\
&+(\nabla Y_\eta[T\eta]^{-1})(\nabla X_\eta[T\eta]^{-1})^t[T\eta]^{-1^t} - (\nabla X_\eta[T\eta]^{-1})^t(\nabla Y_\eta[T\eta]^{-1})[T\eta]^{-1^t} \\
&- (\nabla Y_\eta[T\eta]^{-1})^t(\nabla X_\eta[T\eta]^{-1})[T\eta]^{-1^t}\}\Big],
\end{aligned}$$

$$B_\eta(X_\eta, Y_\eta) = \frac{1}{2}\,(1 + Ric_\eta - \hat{\triangle}_\eta)^{-1}\Big\{ - \mathrm{Tr}[R(\nabla X_\eta T\eta^{-1}\langle\cdot\rangle, Y_\eta) \cdot$$

$$
\begin{aligned}
&+R(\nabla Y_\eta T\eta^{-1}\langle\cdot\rangle, X_\eta)\cdot + R(X_\eta,\cdot)\nabla Y_\eta T\eta^{-1}\langle\cdot\rangle + R(Y_\eta,\cdot)\nabla X_\eta T\eta^{-1}\langle\cdot\rangle]\\
&+\nabla^*[R(X_\eta, T\eta^{-1^t})Y_\eta + R(Y_\eta, T\eta^{-1^t})X_\eta]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
C_\eta(X_\eta, Y_\eta) &= (1+Ric_\eta - \hat{\triangle}_\eta)^{-1}\left[(\nabla_{X_\eta}Ric)\langle Y_\eta\rangle + (\nabla_{Y_\eta}Ric)\langle X_\eta\rangle\right.\\
&\left.-\frac{1}{2}\left[\langle(\nabla Ric\langle\cdot\rangle\langle X_\eta\rangle, Y_\eta\rangle^\sharp + \langle(\nabla Ric\langle\cdot\rangle\langle Y_\eta\rangle, X_\eta\rangle^\sharp\right] - Ric_\eta\langle[X_\eta, Y_\eta]\rangle\right],
\end{aligned}
\tag{3.2}
$$

where $X_\eta, Y_\eta \in T_\eta\mathcal{D}^s_\mu(M)$,

$$Ric_\eta\langle X_\eta\rangle = Ric\langle X_\eta\circ\eta^{-1}\rangle\circ\eta$$

is the right-translated Ricci tensor,

$$\hat{\triangle}_\eta = -\nabla^*[\nabla(\cdot)(T\eta)^{-1}(T\eta)^{-1^t}],$$

and $(\cdot)^\sharp$ *is the operator mapping* 1*-forms to vector fields through the given metric on* M*.*

Remark 3.1 *Note that for* $X_\eta \in H^s_\eta(TM)$*, the operators* $[T\eta]^{-1}$*,* $[T\eta]^{-1^t}$*, and* ∇X_η *induce the following pointwise operators*

$$
\begin{aligned}
[T\eta(x)]^{-1} &: T_{\eta(x)}M \to T_xM,\\
[T\eta(x)]^{-1^t} &: T_xM \to T_{\eta(x)}M,\\
(\nabla X_\eta)(x) &: T_xM \to T_{\eta(x)}M.
\end{aligned}
$$

Now, on $H^{s+1}(M)$, $\triangle = d\delta = -\text{div grad}$, so an explicit formula for $P_e : T_e\mathcal{D}^s(M) \to T_e\mathcal{D}^s_\mu(M)$ is obtained as follows. Suppose that $V \in H^s(TM)$, and let $p \in H^{s+1}(M)$ solve $\triangle p = \text{div}V$. Then

$$P_e(V) = V - \text{grad}\triangle^{-1}\text{div}V.$$

We shall denote the orthogonal projection onto $dH^{s+1}(M)$ by

$$Q_e(V) = \text{grad}\triangle^{-1}\text{div}V. \tag{3.3}$$

$\mathcal{D}^s_\mu(M)$ thus becomes a weak Riemannian submanifold of $\mathcal{D}^s(M)$ with the metric (2.5), and the induced covariant derivative

$$\tilde{\nabla}^1 = P\circ\nabla^1$$

is inherited from $\mathcal{D}^s(M)$.

3.2 Geodesic flow of $\tilde{\nabla}^1$

Theorem 3.2 ([16]) *If $\eta(t)$ is a geodesic of $\tilde{\nabla}^1$, then $U(t) = \dot{\eta} \circ \eta^{-1}(t)$ is a vector field on M which satisfies the mean motion equations of an ideal fluid,*

$$\begin{array}{c} \partial_t U(t) + (1+\triangle)^{-1}\left[\nabla_{U(t)}(1+\triangle)U(t) + \langle \nabla U(t)\langle\cdot\rangle, \triangle U(t)\rangle^{\sharp}\right] = -\text{grad } p(t) \\ \text{div} U(t) = 0, \;\; U(0) = U_0, \end{array} \tag{3.4}$$

where $p(t)$ is the pressure function which is determined from $V(t)$. Laplacian

Lemma 3.1 ([16]) *Let $\hat{\triangle}_{(\cdot)} : \cup_{\eta \in \mathcal{D}^s_\mu(M)} H^s_\eta(TM) \downarrow \mathcal{D}^s_\mu(M) \longrightarrow \cup_{\eta \in \mathcal{D}^s_\mu(M)} H^{s-2}_\eta(TM) \downarrow \mathcal{D}^s_\mu(M)$ be given by*

$$\hat{\triangle}_\eta = -\nabla^*[\nabla(\cdot)(T\eta)^{-1}(T\eta)^{-1^t}]$$

and the identity on $\mathcal{D}^s_\mu(M)$. Then $\hat{\triangle}_{(\cdot)}$ is a C^1 bundle map.

Lemma 3.2 ([16]) *The operator $(1 + Ric_{(\cdot)} - \hat{\triangle}_{(\cdot)})^{-1} : \cup_{\eta \in \mathcal{D}^s_\mu(M)} H^{s-2}_\eta(TM) \downarrow \mathcal{D}^s_\mu(M) \longrightarrow \cup_{\eta \in \mathcal{D}^s_\mu(M)} H^s_\eta(TM) \downarrow \mathcal{D}^s_\mu(M)$ is a C^1 bundle map.*

For the following theorem, recall that $TT\mathcal{D}^s_\mu(M)$ is identified with H^s maps $\mathcal{Y} : M \to TTM$ covering some $X_\eta \in T_\eta \mathcal{D}^s_\mu(M)$.

Theorem 3.3 ([16]) *For $s > \frac{n}{2} + 1$, there exists a neighborhood of $e \in \mathcal{D}^s_\mu(M)$ and an $\epsilon > 0$ such that for any $V \in T_e\mathcal{D}^s_\mu(M)$ with $\|V\|_s < \epsilon$, there exists a unique geodesic $\dot{\eta} \in C^1((-2,2), T\mathcal{D}^s_\mu(M))$ satisfying*

$$\tilde{\nabla}^1_{\dot{\eta}}\dot{\eta} = 0, \quad \eta(0) = e, \; \dot{\eta}(0) = V,$$

with smooth dependence on V.

Remark 3.2 *Together with Theorem 3.2, we have proven the local well-posedness of the Cauchy problem for the Euler-α equations (3.4) on M.*

This implies the following facts.

Corollary 3.1 ([16]) *Let $\eta \in \mathcal{D}^s_\mu(M)$ be in a sufficiently small neighborhood of e. Then, there exists a vector field V on M such that $exp_e(V) = \eta$. In other words, the Euler-α flow with initial condition V reaches η in time 1.*

As another corollary, we immediately have the H^1 analog of Theorem 12.1 of [6].

Corollary 3.2 ([16]) *For $s > \frac{n}{2} + 1$, let $\eta(t)$ be a geodesic of the right invariant H^1 metric on $\mathcal{D}^s_\mu(M)$. If $\eta(0) \in \mathcal{D}^{s+k}_\mu(M)$ and $\dot{\eta}(0) \in T_{\eta(0)}\mathcal{D}^{s+k}_\mu(M)$ for $0 \le k \le \infty$, then $\eta(t)$ is H^{s+k} on M for all t for which $\eta(t)$ was defined in $\mathcal{D}^s_\mu(M)$.*

This has the important consequence that the time of existence of a geodesic does not depend on s, so that a geodesic with C^∞ initial conditions is a curve in

$$\mathcal{D}_\mu(M) = \cap_{s>n/2} \mathcal{D}^s_\mu(M),$$

where $\mathcal{D}_\mu(M)$ is the ILH (inverse limit Hilbert) Lie group of C^∞ diffeomorphisms.

Our method also gives sharp well-posedness for the Camassa-Holm equation.

Theorem 3.4 *[16, 8] The Cauchy problem for the 1D CH equation, given by*

$$\ddot{\eta} = -\left[(1-\partial_y^2)^{-1}\partial_y\left((\dot{\eta}\circ\eta^{-1})^2 + \frac{1}{2}(\dot{\eta}\circ\eta^{-1})_y^2\right)\right]\circ\eta \tag{3.5}$$

with initial conditions

$$\eta(0) = e, \quad \dot{\eta}(0) = u_0,$$

has a unique solution $(\eta, \dot{\eta})$ in $\mathcal{D}^s(\mathbb{S}^1) \times H^s(\mathbb{S}^1)$ for $s > \frac{3}{2}$ on a finite time interval where the solution has C^1 dependence on time and smooth dependence on initial data.

4 Curvature of the H^1 metric

4.1 Curvature of $\tilde{\nabla}^1$

We define the (weak) curvature $\tilde{R}^1$ of the induced metric $\langle\cdot,\cdot\rangle_1$ on $\mathcal{D}^s_\mu(M)$ as

$$\begin{gathered}\tilde{R}^1_\eta : T_\eta\mathcal{D}^s_\mu(M) \times T_\eta\mathcal{D}^s_\mu(M) \times T_\eta\mathcal{D}^s_\mu(M) \to T_\eta\mathcal{D}^s_\mu(M),\\ \tilde{R}^1_\eta(X_\eta, Y_\eta)Z_\eta = (\tilde{\nabla}^1_X\tilde{\nabla}^1_Y Z)_\eta - (\tilde{\nabla}^1_Y\tilde{\nabla}^1_X Z)_\eta - (\tilde{\nabla}^1_{[X,Y]}Z)_\eta,\end{gathered}$$

where $\eta \in \mathcal{D}^s_\mu(M)$, and X, Y, Z are smooth extensions of X_η, Y_η, Z_η in a neighborhood of η.

Theorem 4.1 ([16]) *The curvature $\tilde{R}^1$ of the induced H^1 metric on $\mathcal{D}^s_\mu(M)$ is a trilinear operator which is continuous in the H^s topology for $s > \frac{n}{2} + 2$.*

Remark 4.1 *One might try to argue that the boundedness in H^s of $\tilde{R}^1$ follows immediately from the regularity of the geodesic spray, but this argument fails for the following reason. Let $\mathcal{U} \subset \mathcal{D}^s_\mu(M)$ be sufficiently small so as to allow a trivialization of $T\mathcal{D}^s_\mu(M)$, and let $\mathcal{A}^1$ be the local connection 1-form defining the H^1 covariant derivative $\tilde{\nabla}^1$. The fact that the geodesic spray of $\tilde{\nabla}^1$ is C^1 implies that $\mathcal{A}^1$ is a C^1 map as well. Now the curvature can be defined as $d\mathcal{A}^1 + \mathcal{A}^1 \wedge \mathcal{A}^1$, and it may seem that for all $\eta \in \mathcal{U}$, $d\mathcal{A}^1(\eta)$ is then necessarily a continuous operator from H^s into H^s. This is not the case, however, as the exterior derivative is defined in terms of the H^1-Frechet derivative, while the fact that $\mathcal{A}^1$ is C^1 is verified using the H^s-Frechet derivative. It is for this reason, that curvatures of strong metrics are trivially bounded operators in the strong topology of the manifold, while for weak metrics, one must verify any boundedness claims.*

4.2 Jacobi equations

We can now prove the existence of solutions to the Jacobi equation

$$\tilde{\nabla}^1_{\dot{\eta}}\tilde{\nabla}^1_{\dot{\eta}}Y + \tilde{R}^1_{\eta}(Y,\dot{\eta})\dot{\eta} = 0 \tag{4.1}$$

along the geodesic $\eta(t)$ of the H^1-metric which solves the Euler-α equation in Lagrangian coordinates. Note that the Euler-α equations may equivalently be written as

$$\tilde{\nabla}^1_{\dot{\eta}}\dot{\eta} = 0, \tag{4.2}$$

for $\eta(t)$ a curve in $\mathcal{D}^s_\mu(M)$. The Jacobi equation (4.1) is the linearization of (4.2) along the geodesic.

Theorem 4.2 *[16] Let $s > \frac{n}{2} + 2$ and let $Y_e, \dot{Y}_e \in T_e\mathcal{D}^s_\mu(M)$. Then there exists a unique H^s vector field $Y(t)$ along η that is a solution to (4.1) with initial conditions $Y(0) = Y_e$ and $\tilde{\nabla}^1_{\dot{\eta}}Y(0) = \dot{Y}_e$.*

5 Stability and Curvature

In this section, we define the notion of Lagrangian linear stability.

5.1 Lagrangian stability

For $k \geq 1$, a fluid motion η is Lagrangian H^k (linearly) stable if every solution of the Jacobi equation (4.1) along η is bounded in the H^k norm.

Theorem 5.1 ([16]) *If $\eta(t)$ is a geodesic of $\tilde{\nabla}^1$ on $\mathcal{D}^s_\mu(M)$ whose pressure function $p(t)$ is constant for all t and if the sectional curvature of R^1 is nonpositive, then η is H^k Lagrangian unstable for $k \geq 1$.*

If η is a geodesic in $\mathcal{D}^s_\mu(M)$, two points $\eta(t_1)$ and $\eta(t_2)$ are conjugate with respect to η if there exists a nonzero Jacobi field $Y(t)$ along η such that $Y(t_1) = Y(t_2) = 0$. Such Jacobi fields are thus stable perturbations of the initial flow.

Corollary 5.1 ([16]) *Let η be a pressure constant geodesic in $\mathcal{D}^s_\mu(M)$. If the sectional curvature of R^1 is nonpositive, then there are no conjugate points along η.*

5.2 Lagrangian Stabilization

We now present new results on the sectional curvature of the group of area-preserving diffeomorphisms of a two-torus with a right invariant H^1 metric in view of the application to the Lagrangian stability analysis following Arnold [1].

Theorem 5.2 ([15]) *The explicit formulas for the H^1_α right invariant inner product and, the coadjoint action and Levi-Civita covariant derivative uniquely associated to it on $\mathcal{D}_\mu(T^2)$ have the following form:*

$$\langle e_k, e_l \rangle = A^\alpha(k)\delta_{k,-l} \tag{5.1}$$

$$[e_k, e_l] = (k \times l)e_{k+l} \tag{5.2}$$

$$ad^*_{e_l} e_k = b_{k,l} e_{k+l}, \quad \textit{where} \quad b_{k,l} = (k \times l)\frac{A^\alpha(k)}{A^\alpha(k+l)} \tag{5.3}$$

$$\nabla_{e_k} e_l = d_{k,k+l} e_{k+l}, \quad \textit{where} \quad d_{k,k+l} = \frac{k \times l}{s}\left(1 - \frac{A^\alpha(k) - A^\alpha(l)}{A^\alpha(k+l)}\right). \tag{5.4}$$

Further the Riemannian curvature operator of the H^1_α metric is given by

$$\begin{aligned} R_{k,l,m,n} \quad \equiv \quad & \langle R(e_k, e_l)e_m, e_n \rangle = (-d_{l+m,k+l+m} d_{m,l+m} \\ & + d_{k+m,k+l+m} d_{m,k+m} + (k \times l) d_{m,k+l+m}) A^\alpha(k+l+m)S. \end{aligned} \tag{5.5}$$

We shall analyze the sectional curvature K_1 of the H^1_α metric in the plane defined by the stream functions

$$\xi = \cos(k, x) \text{ and } \eta = \cos(l, x).$$

The function $\cos(k, x)$ is a stationary solution to both the Euler and Euler-α equations, and thus presents us with an opportunity to examine the stabilizing effect produced by the regularized H^1_α geodesic motion.

Theorem 5.3 ([15]) *Let $K^1(\xi, \eta)$ denote the sectional curvature on $(\mathcal{D}_\mu(\mathbb{T}^2), H^1_\alpha(\mathbb{T}^2)$ where $\xi = \cos(k, x)$ and $\eta = \cos(l, x)$. For $|\epsilon|$ sufficiently small, let $l = k + \epsilon$. Then for any k, there exists $0 < \alpha_0(k) < 1$, such that for all $\alpha > \alpha_0(k)$, $K^1(\xi, \eta) > 0$.*

6 Averaged Euler Equation on compact Riemannian manifolds with boundary

Letting N denote the normal bundle on ∂M, it is proven in [13] that the set

$$\mathcal{N}^s_\mu(M) = \{\eta \in \mathcal{D}^s_\mu(M) \mid T\eta|_{\partial M} \cdot n \in H^{s-\frac{3}{2}}_\eta(N), \text{ for all } n \in H^{s-\frac{1}{2}}(N)\}$$

is a new subgroup of $\mathcal{D}^s_\mu(M)$. Here H^s_η denotes the space of sections of N covering the diffeomorphism η.

It is then proven that geodesic motion of the right invariant H^1_α pseudo-metric on $\mathcal{N}^s_\mu(M)$ given at $e \in \mathcal{N}^s_\mu(M)$ by (H_n is the second fundamental form of ∂M)

$$\frac{1}{2}\int_M \left[g\langle u, u\rangle + \alpha^2 g\langle \nabla u, \nabla u\rangle\right]\mu + \alpha^2 \int_{\partial M} H_n\langle u, u\rangle\gamma$$

is smooth in the strong H^s topology.

The tangent space of $\mathcal{N}_\mu^s(M)$ at e consists of divergence-free vector fields of class H^s satisfying the free-slip or normal boundary conditions

$$g\langle u, n\rangle = 0, \quad (\nabla_n u)^{\tan} + S_n\langle u\rangle = 0 \text{ on } \partial M; \tag{6.1}$$

hence, geodesics of the right invariant H^1_α pseudo-metric are solutions of the Euler-α equations with free-slip or normal boundary conditions

$$\begin{gathered} \dot{v} + \nabla_u v - \alpha^2[\nabla u]^T \cdot \triangle u = -\text{grad } p \text{ in } M, \\ v = (1-\alpha^2\triangle)u, \quad \text{div } u = 0, \\ g\langle u, n\rangle = 0, \quad (\nabla_n u)^{\tan} + S_n\langle u\rangle = 0, \end{gathered} \tag{6.2}$$

where, S_n is the symmetric operator associated to the second fundamental form H_n of the boundary ∂M, and where we suppress the dependence on α of u.

For each $\eta \in \mathcal{D}_\mu^s(M)$, we may use the L^2 Hodge decomposition to define the projection $P_\eta : T_\eta\mathcal{D}^s(M) \to T_\eta\mathcal{D}_\mu^s(M)$ given by

$$P_\eta(X) = (P_e(X \circ \eta^{-1})) \circ \eta,$$

where $X \in T_\eta\mathcal{D}_\mu^s(M)$, and P_e is the L^2 orthogonal projection onto the divergence-free vector fields on M. Recall that this projection is given by

$$P_e(v) = v - \text{grad}p(v) - \text{grad}b(v),$$

where p is the solution of the boundary value problem

$$\begin{aligned} \triangle p(v) &= \text{div } v && \text{in } M \\ p(v) &= 0 && \text{on } \partial M, \end{aligned}$$

and b solves

$$\begin{aligned} \triangle b(v) &= 0 && \text{in } M \\ g\langle \text{grad}b(v), n\rangle &= g\langle v - \text{grad}p, n\rangle && \text{on } \partial M, \end{aligned}$$

where n is the orientation preserving normal vector field on ∂M. The function p is the pressure associated with v, while the function b is a smooth extension of the normal component of v along ∂M to the interior of M. Subtraction of grad $b(v)$ is necessary as volume preserving diffeomorphisms of a manifold with boundary leave the boundary invariant.

6.1 The subgroup $\mathcal{N}_\mu^s(M)$

Theorem 6.1 ([13]) *The set $\mathcal{N}_\mu^s(M)$ is a subgroup of $\mathcal{D}_\mu^s(M)$ for $s > \frac{n}{2}+1$, such that*

$$\begin{aligned} T_e\mathcal{N}_\mu^s(M) = \{u \in T_e\mathcal{D}_\mu^s(M) \mid (\nabla_n u|_{\partial M})^{\tan} + S_n\langle u\rangle = 0 \in H^{s-\frac{3}{2}}(T\partial M) \\ \text{for all } n \in H^{s-\frac{1}{2}}(N)\}, \end{aligned}$$

where $S_n : T\partial M \to T\partial M$ is the symmetric linear operator satisfying

$$g\langle S_n\langle u\rangle, v\rangle = H_n\langle u, v\rangle, \quad u, v \in H^{s-\frac{3}{2}}(T\partial M),$$

where H_n is the second fundamental form of ∂M given by $H_n\langle u, v\rangle = -g\langle \nabla_u n, v\rangle$.

We define $P^\alpha : T_e\mathcal{D}^s_\mu(M) \to T_e\mathcal{N}^s_\mu(M)$ to be the H^2-orthogonal projector. If there is no boundary, then P^α is the same as P, the usual Hodge L^2 orthogonal projection. For the case of normal (free-slip) boundary conditions, we have

$$P^\alpha = (1 - \alpha^2\triangle)^{-1} P_e (1 - \alpha^2\triangle)$$

where $D(1 - \alpha^2\triangle) = D(L)$ and

$$D(L) = \{v \in H^2(TM) : \operatorname{div} v = 0, g\langle u, n\rangle = 0 \text{ on } \partial M, (\nabla_n u)^{\tan} = 0\}. \tag{6.3}$$

We then define $P^\alpha_\eta : T_\eta\mathcal{D}^s(M) \to T_\eta\mathcal{N}^s_\mu(M)$ by

$$P^\alpha_\eta(X) = \left[(1 - \alpha^2\triangle)^{-1} P_e (1 - \alpha^2\triangle)(X \circ \eta^{-1})\right] \circ \eta. \tag{6.4}$$

7 Mean Hydrodynamics on the Subgroup $\mathcal{N}^s_\mu(M)$

7.1 H^1 Metric on $\mathcal{N}^s_\mu(M)$

In this section, we shall consider geodesic motion of the weak H^1_α right invariant (pseudo) metric on the group $\mathcal{N}^s_\mu(M)$ which is defined as follows. For $X, Y \in T_e\mathcal{N}^s_\mu(M)$, we set

$$\langle X, Y\rangle_1 = \int_M \left(g\langle X(x), Y(x)\rangle + \alpha^2 g\langle \nabla X(x), \nabla Y(x)\rangle\right) \mu(x) + \alpha^2 \int_{\partial M} H_n\langle X(x), Y(x)\rangle \gamma(x) \tag{7.1}$$

and extend $\langle\cdot,\cdot\rangle_1$ to $\mathcal{N}^s_\mu(M)$ by right invariance. Here n is the outward unit normal on ∂M and γ is the induced volume measure on ∂M.

7.2 Euler-Poincaré equations on $T_e\mathcal{N}^s_\mu(M)$

Theorem 7.1 (Euler-Poincaré for $\mathcal{N}^s_\mu(M)$ [13]) *Equip $\mathcal{N}^s_\mu(M)$ with the right invariant metric $\langle\cdot,\cdot\rangle_1$. Then, a curve $\eta(t)$ in $\mathcal{N}^s_\mu(M)$ is a geodesic of this metric if and only if $u(t) = T_{\eta(t)}R_{\eta(t)^{-1}}\dot\eta(t) = \dot\eta(t) \circ \eta(t)^{-1}$ satisfies*

$$\frac{d}{dt}u(t) = -P^\alpha \circ ad^*_{u(t)} u(t) \tag{7.2}$$

where ad^*_u is the formal adjoint of ad_u with respect to the metric $\langle\cdot,\cdot\rangle_1$ at the identity, i.e.,

$$\langle ad^*_u v, w\rangle_1 = \langle v, [u, w]\rangle_1$$

for all $u, v, w \in T_e\mathcal{N}^s_\mu(M)$.

7.3 The geodesic spray on $\mathcal{N}^s_\mu(M)$

We can now prove the analogue of Theorem 3.3 of [16] on the subgroup $\mathcal{N}^s_\mu(M)$.

Theorem 7.2 ([13]) *For $s > \frac{n}{2}+1$, there exists a neighborhood of $e \in \mathcal{N}^s_\mu(M)$ and an $\epsilon > 0$ such that for any $V \in T_e\mathcal{N}^s_\mu(M)$ with $\|V\|_s < \epsilon$, there exists a unique geodesic $\dot\eta \in C^1((-2,2), T\mathcal{N}^s_\mu(M))$ satisfying*

$$\tilde\nabla^1_{\dot\eta}\dot\eta = 0, \quad \eta(0) = e, \ \dot\eta(0) = V,$$

with smooth dependence on V.

8 The limit of zero viscosity on $\mathcal{N}^s_\mu(M)$

The **Navier-Stokes α-model** is obtained by adding viscous diffusion to the Euler-α model. With the normal (free slip) boundary conditions, the equations are given by

$$\partial_t u - \nu\triangle u + (1-\alpha^2\triangle)^{-1}\left[\nabla_u(1-\alpha^2\triangle)u - \alpha^2\nabla u^t\cdot\triangle u\right] = -(1-\alpha^2\triangle)^{-1}\text{grad } p. \quad (8.1)$$

In [7], global well-posedness of (8.1) was established, as well as estimates on the dimension of the global attractor. Having proven the smoothness of the geodesic spray of the Euler-α equations, we follow [6] and use the product formula approach to prove the existence of the limit of zero viscosity of (8.1). In the case that $\alpha = 0$, this limiting procedure is believed to be valid only for compact manifolds without boundary (e.g., for flows with periodic boundary conditions), as the Navier-Stokes equations and the Euler equations do not share the same boundary conditions on manifolds with boundary. When, $\alpha \neq 0$, however, as we shall discuss in the last section, a certain type of elasticity is added into the Euler-α model, and the mean motion of the fluid exhibits normal stress effects. Because of this, we may prescribe zero velocity boundary conditions even in the inviscid limit, and thus extend the limit of zero viscosity theorems for the averaged Euler equations to manifolds with boundary.

The following is the Euler-α version of Theorem 13.1 of [6].

Theorem 8.1 ([13]) *Let $S : T\mathcal{N}^s_\mu(M) \to TT\mathcal{N}^s_\mu(M)$ be the Euler-α vector field. For each s, let $T : T_e\mathcal{N}^s_\mu(M) \to T_e\mathcal{N}^s_\mu(M)$ be a given map, where the integer $\sigma \geq 2$, and assume that T is a bounded linear map that generates a strongly-continuous semi-group $F_t : T_e\mathcal{N}^s_\mu(M) \to T_e\mathcal{N}^s_\mu(M)$, $t \geq 0$, and satisfies $\|F_t\|_s \leq e^{\beta t}$ for some $\beta > 0$ and some s. Extend F_t to $T\mathcal{N}^s_\mu(M)$ by*

$$\tilde F_t(X_\eta) = TR_\eta \cdot F_t \cdot TR_{\eta^{-1}}(X_\eta)$$

for $X_\eta \in T_\eta\mathcal{N}^s_\mu(M)$, and let $\tilde T$ be the vector field $\tilde T : T\mathcal{N}^s_\mu(M) \to TT\mathcal{N}^{s-\sigma}_\mu(M)$ associated to the flow $\tilde F_t$.

Then $S + \nu\tilde T$ generates a unique local uniformly Lipschitz flow on $T\mathcal{N}^s_\mu(M)$ for $\nu \geq 0$, and the integral curves $c^\nu(t)$ with $c^\nu(0) = x$ extend for a fixed time $\tau > 0$ independent of ν and are unique. Further,

$$\lim_{\nu\to 0} c^\nu(t) = c^0(t)$$

for each t, $0 \leq t < \tau$, the limit being in the H^s topology, $s > (n/2) + 1 + 2\sigma$.

All of the results just stated for the subgroup $\mathcal{N}_\mu^s(M)$ also hold for the much simpler case of H_α^1 geodesic motion on the subgroup of $\mathcal{D}_\mu^s(M)$ that keeps the boundary pointwise fixed (see [8]).

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