Approximating quantum group link invariants on quantum computers

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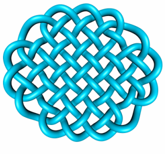
What is a link?

- A knot is a closed nonintersecting curve in $\mathbb{R}^3$

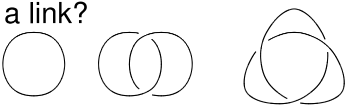


- A link is a knot with many components

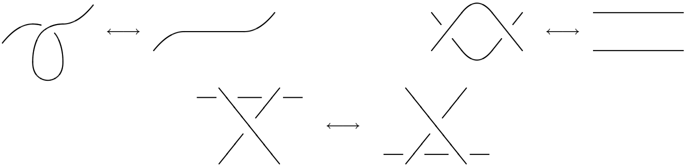
- Links are equivalent (isotopic) if they can be deformed into one another

$\simeq$

Fundamental problem

- Do two descriptions describe equivalent links?
- How to describe a link?
- Link diagrams: 

Theorem: Two link diagrams represent equivalent links iff they are connected by a sequence of Reidemeister moves:



- Problem: may have to introduce many more crossings
  - no polynomial upper bound known

Polynomial invariants

- Link invariant = function on links $L \mapsto f(L)$ such that

$$L \text{ equivalent to } M \Rightarrow f(L) = f(M)$$

- Should be computable (in principle) from some description
- This talk: polynomial invariants of links coming from “quantum" deformations of Lie groups (quantum groups)

$$\begin{tikzcd} & \begin{array}{c}\text{Jones } \; J(q)\\ SU(2)\end{array} & \\ \begin{array}{c}\text{HOMFLYPT } \; H_d(q)\\ SU(d)\end{array} \arrow[ur] & & \begin{array}{c}\text{Kauffman } \; K_d(q)\\ SO(d), Sp(2d)\end{array} \arrow[ul] \end{tikzcd}$$

Polynomial invariants

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Jones $J(q)$

$SU(2)$

HOMFLYPT $H_d(q)$

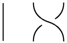
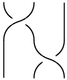
$SU(d)$

Kauffman $K_d(q)$

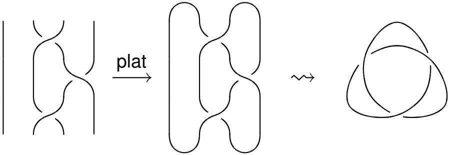
$SO(d), Sp(2d)$

Describing links with braids

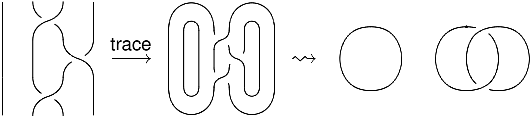
- Braid group B_n
  - generated by counterclockwise twists $\{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}$

$\sigma_1 =$  $\sigma_2^{-1} =$  $\sigma_2^{-1}\sigma_1 =$ 

  - Trajectories in 2+1 dimensional spacetime
- Get links by closing braids
  - plat closure:

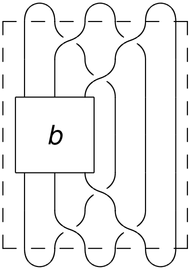


  - trace closure:



Trace from plat

- Trace closure expressed as plat closure of related braid: [Jones '87]



- Suffices to find algorithms for approximating plat closure

Generalites

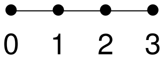
- Given $b \in B_n$ (n even), let $f(q)$ be invariant of plat closure
- General formula (for invariants of interest here):

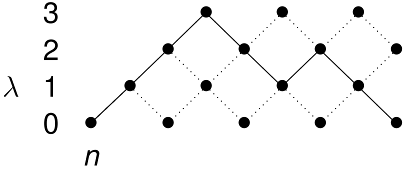
$$\left|f\left(e^{2\pi i/\ell}\right)\right|^2 = d_\ell^n \left| \langle \cup \quad \cup \quad \cup \left| U(b,\ell) \right| \cap \quad \cap \quad \cap \rangle \right|^2$$

- $U(b,\ell)$ unitary representation of B_n ($\ell \in \mathbb{N}$)
- $| \cap \quad \cap \quad \cap \rangle$ has $n/2$ caps
- Upper bound (by unitarity): $|f(q)|^2 \leq d_\ell^n$ (exp. large)
- Efficiently sample r.v. X with $\mathbb{E}X = d_\ell^{-n} |f\left(e^{2\pi i/\ell}\right)|^2$ if
  - Can prepare and measure $| \cap \quad \cap \quad \cap \rangle$
  - Can efficiently implement $U(\sigma_i, \ell)$ for each generator
- w.h.p., obtain approximation of $d_\ell^{-n} \left| g(e^{2\pi i/\ell}) \right|^2 \pm \delta$ on QC in poly(length of braid, $1/\delta$) time

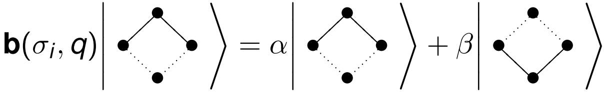
Jones representations of B_n

- $\mathcal{B} = \text{span}\left\{ \left| 0 = \lambda^{(0)} \to \lambda^{(1)} \to \cdots \to \lambda^{(\text{n})} = 0 \right\rangle \right\}$
- $|\lambda^{(i)} = \lambda^{(i-1)}| = 1$, $0 \le \lambda^{(i)} \le k$, $(q = e^{2\pi i/(k+2)})$
- Example $k = 3$, $q = e^{2\pi i/5}$, $n = 8$:





- Braid generator σ_i acts locally on $(\lambda^{(i-1)}, \lambda^{(i)}, \lambda^{(i+1)})$, e.g.



- α, β functions of $(\lambda^{(i-1)}, \lambda^{(i)}, \lambda^{(i+1)})$ only
- Just a phase for straight segments

Approximating Jones polynomial on quantum computer [AJL, FKW]

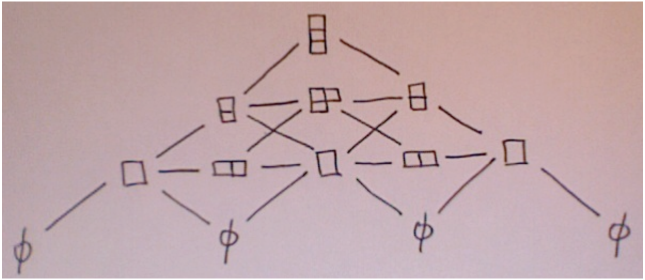
- Quantum circuits for Jones representation of B_n
  - Embed representation into $\left(\mathbb{C}^2\right)^{\otimes n}$
  - Encode paths into qubits $|\text{up}\rangle|\text{down}\rangle|\text{up}\rangle|\text{up}\rangle$
  - Coherently compute $\lambda^{(i)}$'s
  - Braiding by local controlled unitaries (efficient)
  - Coherently uncompute $\lambda^{(i)}$'s
  - Therefore can apply unitary $U(b, \ell)$ efficiently for $b \in B_n$
- Can efficiently prepare (and measure) state

$$\Big|\ \frown \quad \frown \quad \frown\ \Big\rangle = |\text{up}\rangle|\text{down}\rangle|\text{up}\rangle|\text{down}\rangle|\text{up}\rangle|\text{down}\rangle$$

- Can thus estimate $d_\ell^{-n}|J(e^{2\pi i/\ell})|^2$
- Also possible to estimate phase with swap test

Kauffman polynomial representation (BMW-algebra)

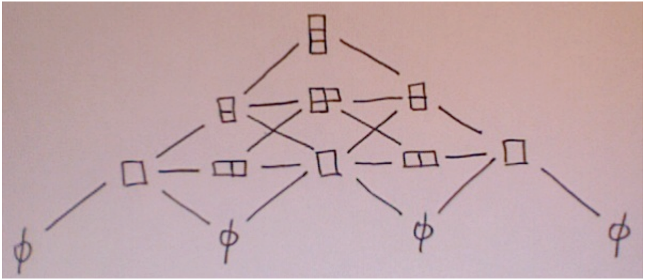
- Basic structure of representations remains the same for $K_d(e^{2\pi i/\ell})$ [Wenzl, Leduc & Ram]
- Just paths on a more complicated path of Young diagrams with at most $|d|$ rows and $n - 2k$ boxes at n'th step
- Add or remove a box at each step
- Example: n=6



- Root of unity $e^{2\pi i/\ell}$ truncates some diagrams from graph

Implementing the representation on QC

- Each step in path has $\leq 2|d|$ choices
- Encode paths into $(\mathbb{C}^{2|d|})^{\otimes n}$
- Braid generators act locally as in Jones representation implementation
- Braid group generators efficiently applied to $O(\log 2|d|)$ local qubits



Further generalizations

- Common features to Jones and Kauffman:
  - Braid group representations expressed on path bases for centralizer algebras of tensor powers $V^{\otimes n}$ of defining representation V of some quantum group
  - Jones - $U_q(\mathfrak{sl}_2)$ - Temperley-Lieb algebra
  - Kauffman - $U_q(\mathfrak{so}_d)$, $U_q(\mathfrak{sp}_{-2d})$ - BMW-algebra
  - Underlying representation is self-dual ($V = V^*$) - cupcaps correspond to contraction operator (projection onto trivial irrep in $V \otimes V^* = V \otimes V$)
  - Bratelli diagram for iterated tensor products is multiplicity-free
- Many other invariants possible:
  - Arbitrary quantum group $U_q(\mathfrak{g})$, $\mathfrak{g}$= simple Lie algebra
  - Arbitrary self-dual representation generating multiplicity-free diagram (these are classified by Howe)

Future

- Other invariants?
  - Reshetikin-Tureav graph/3-manifold invariants...

- HOMFLYPT invariant of *oriented links* - but requires more work since $V \neq V^*$
  - Trace closure done by [WY]

- Complexity?
  - BQP
  - DQC1 (one pure qubit)

- Real applications? New useful algorithms?