Non-commutative probability theory description of strongly correlated electronic systems

Razvan Teodorescu

Theoretical Division (T-13) &
Center for Nonlinear Studies

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Classical and Quantum Problems

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Is there a unifying statistical formalism ?

Lessons from the past I: QFT

- Quantum Field Theory $n-$point amplitudes

$$\langle\Omega_i|\mathrm{T}\left\{:\exp\left[\frac{i}{\hbar}\int_{-\infty}^{\infty}\mathcal{L}(\hat{\phi})d\tau\right]\hat{\phi}(x_1)\hat{\phi}(x_2)\ldots\hat{\phi}(x_n):\right\}|\Omega_o\rangle$$

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- Stochastic Field Theory $n-$point functions

$$Z^{-1}\int\mathcal{D}[\phi]\exp\left[\frac{i}{\hbar}\int_{-\infty}^{\infty}\mathcal{L}(\phi)d\tau\right]\phi(x_1)\phi(x_2)\ldots\phi(x_n)$$

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Stochastic interpretation of boundary operators

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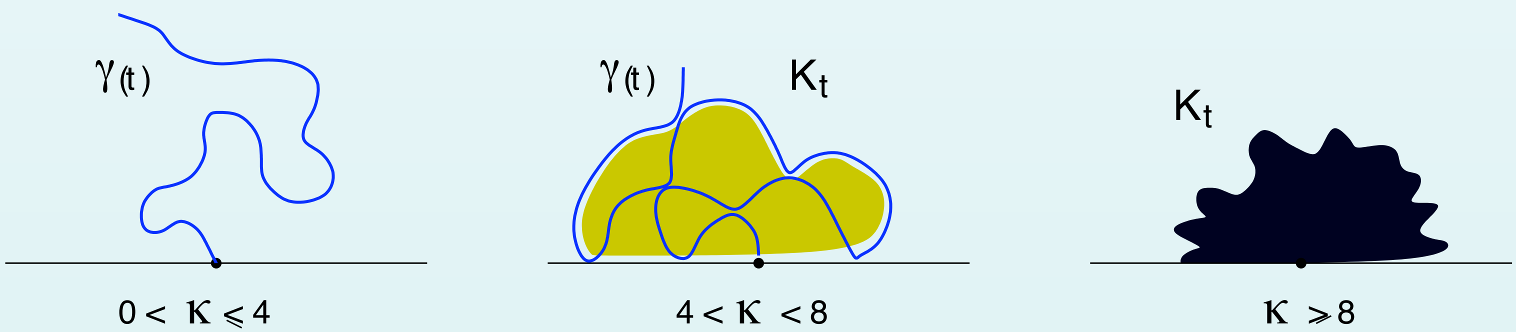

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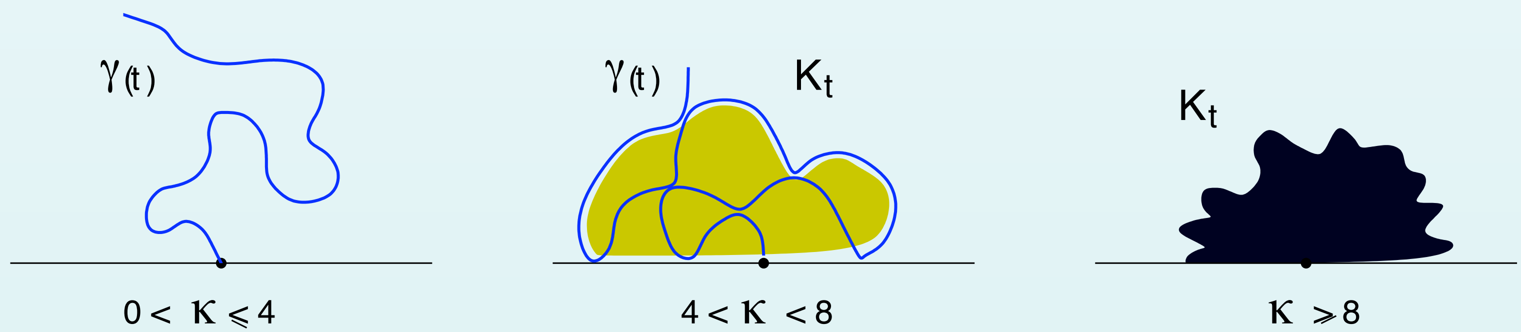

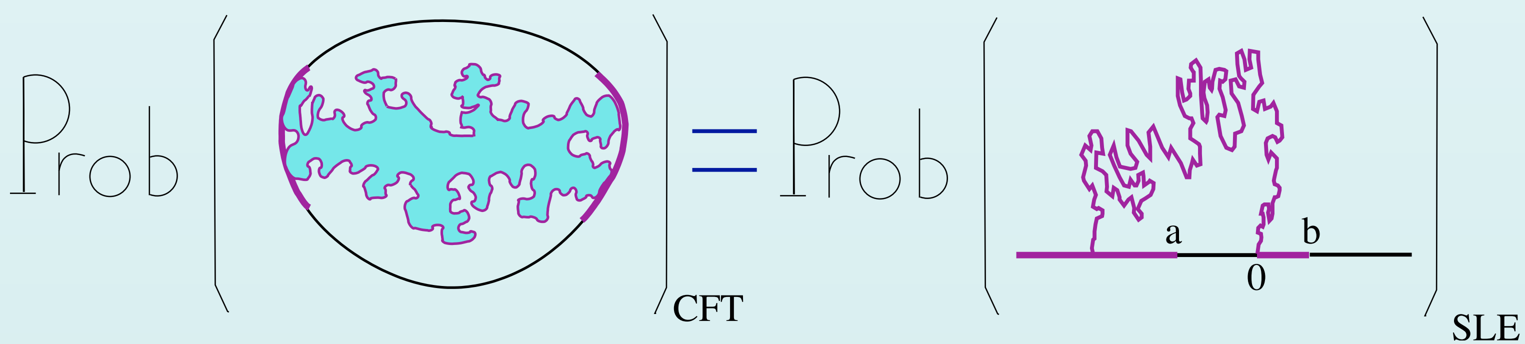

Lessons from the past III: Witten TFT - Kontsevich RMT

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$$A(X) \equiv \int dM e^{i\mathrm{Tr}[\frac{M^3}{3} - XM]}, \quad M \text{ hermitian}$$

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- The topological amplitude is directly related to the Jones polynomial

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- $Free$ non-commutative r.v.: if $\phi(A_i) = 0$,

$$\phi(A_{i_1} A_{i_2} \ldots A_{i_k}) = 0, \quad A_{i_j} \neq A_{i_{j+1}}.$$

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$$\boxed{\text{Random } N \times N \text{ matrices become } free \text{ in the large } N \text{ limit !}}$$

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- $d-$dimensional lattice (cubic ...), n states (orbitals) at each state $|x, i\rangle, i = 1, ..., n$

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$$H = H_0 + H_d, \quad H_0 = \sum_{n,\langle x,y\rangle} t_{x,y}|x,n\rangle\langle y,n|, \quad H_d = \sum_{x,i,j} f^{ij}|x,i\rangle\langle x,j|$$

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- Random matrices with symmetry group $SU(1,1)$

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- XY spin chain:

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$$P[0,\ell]\sim\det[I-K_\ell],\quad K_\ell(x,y),\ \text{sine kernel on } L^2[0,\ell]$$

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- Efficient method for simulating quantum dynamics on classical variables