A Classical vs. Quantum Paradigm for the Topological Model

Eric C. Rowell, Texas A&M U.
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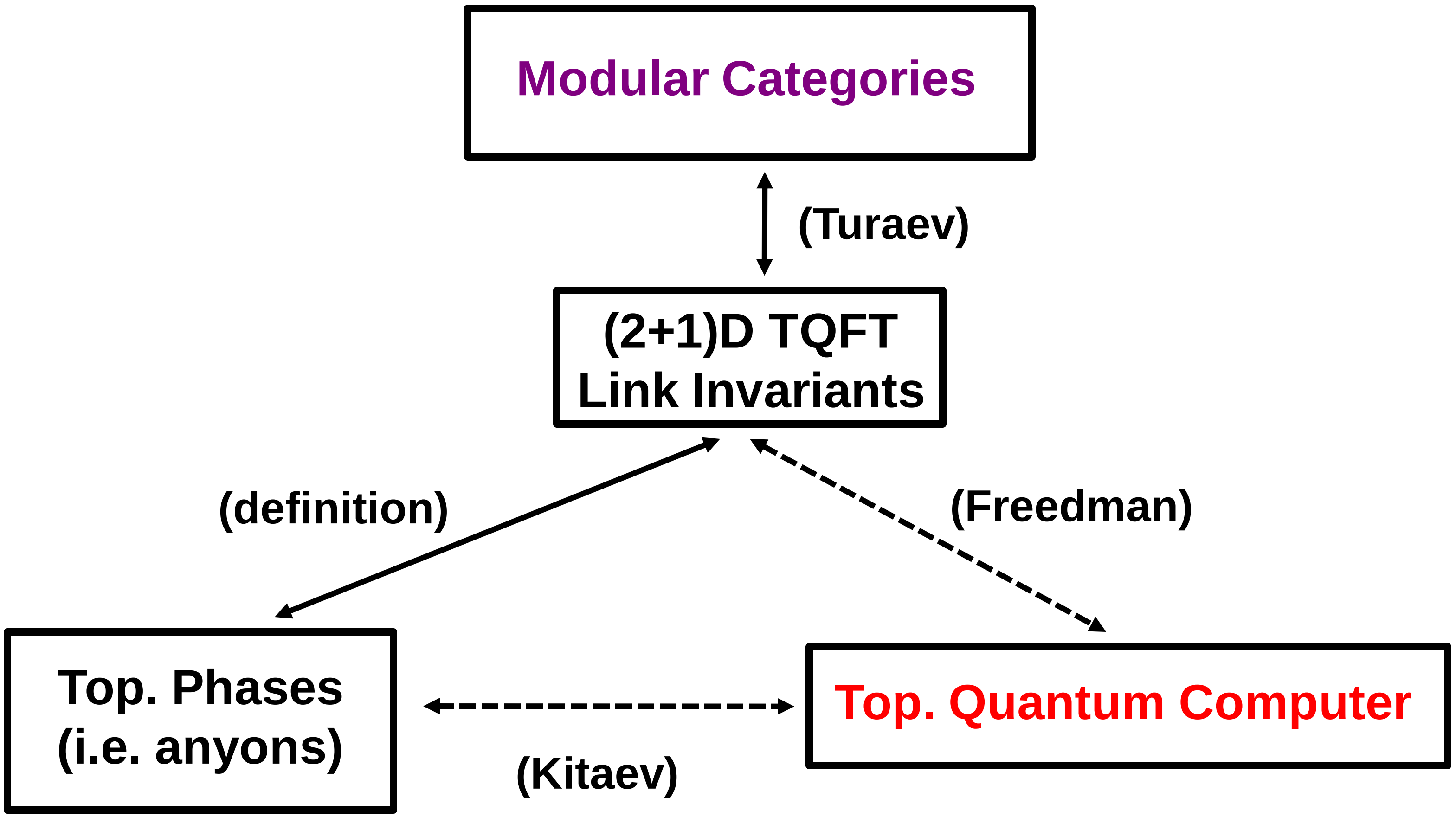
What is a Topological Phase?

[Das Sarma, Freedman, Nayak, Simon, Stern]

"…a system is in a **topological phase** if its low-energy effective field theory is a topological quantum field theory…"

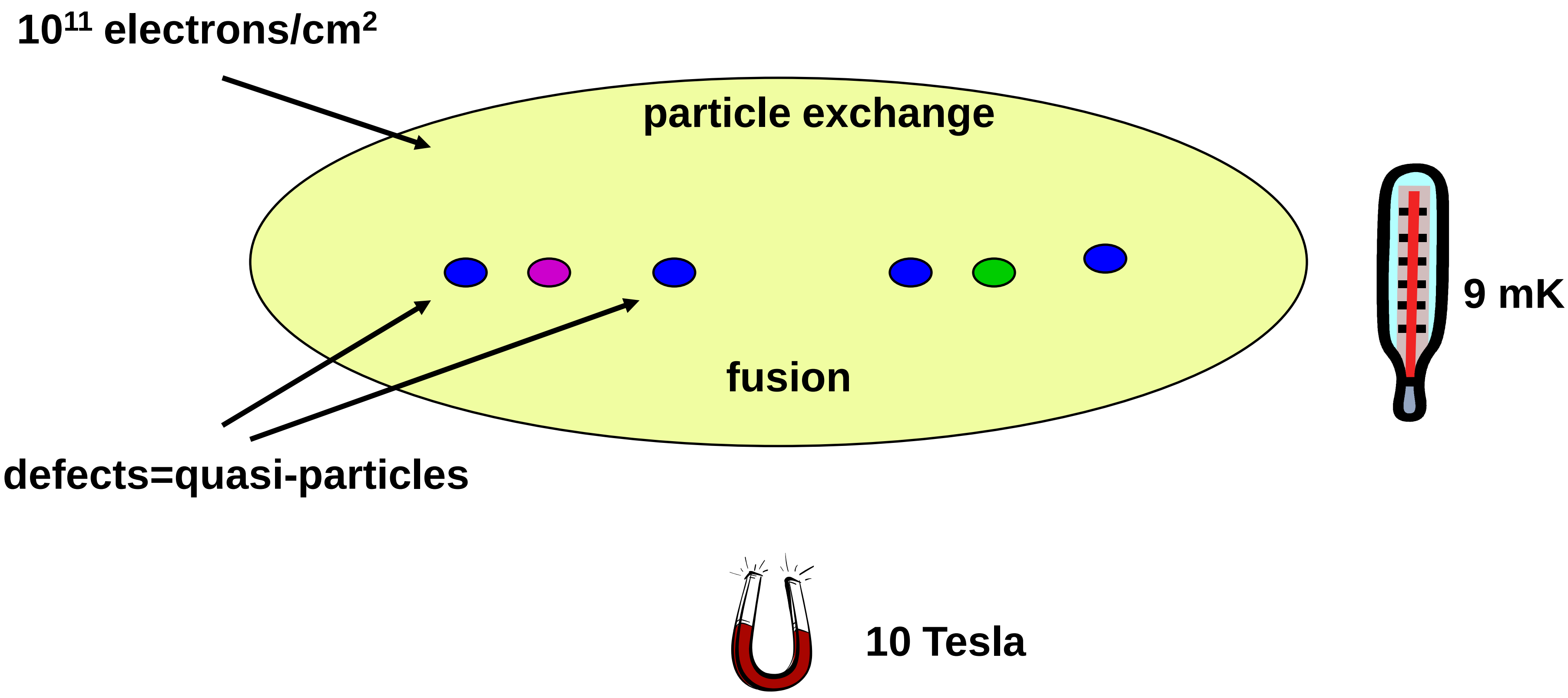
Working definition…

Motivation

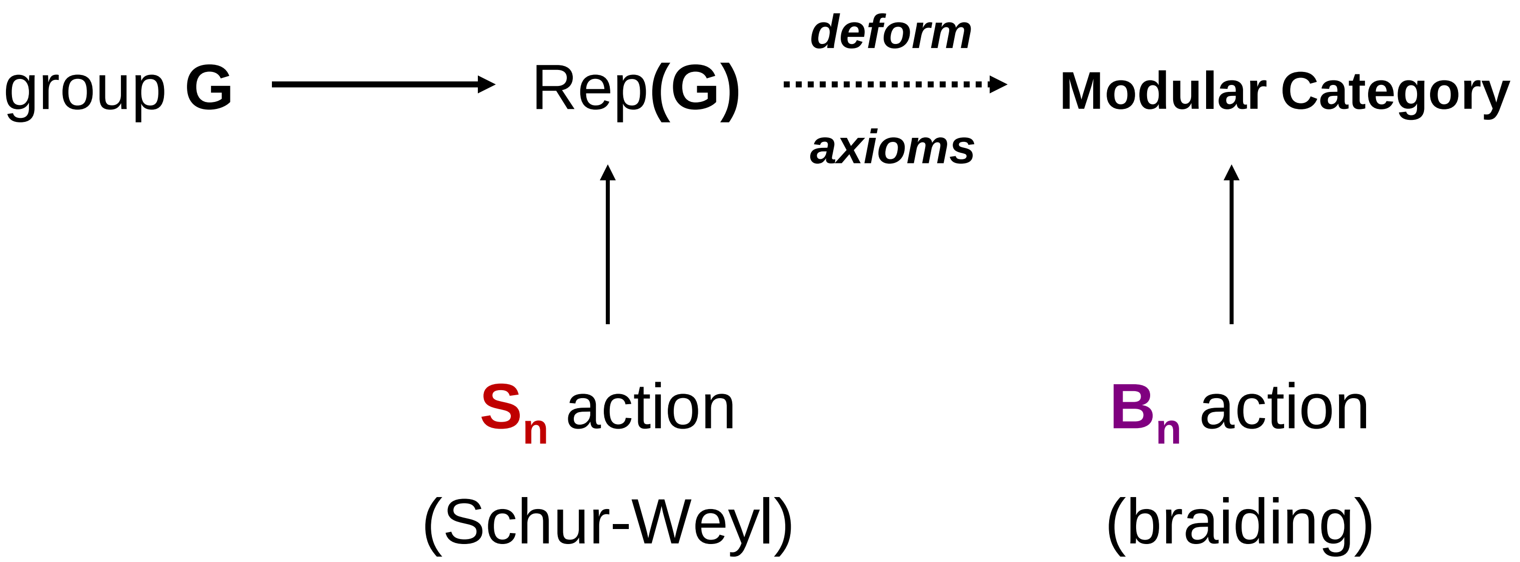


Modular Categories
(Turaev)
(2+1)D TQFT
Link Invariants
(definition)
(Freedman)
Top. Phases
(i.e. anyons)
Top. Quantum Computer
(Kitaev)

Topological Phases: FQHE



Modular Categories



Modular Category **C**

- Objects: $X \in \mathrm{Obj}(\mathbf{C})$

- Morphisms: Hom(X,Y) f.d. vector spaces

Thesis: *Modular Categories encode topological phases of matter.*

Required Objects

Modular Category	Physics
Rank=n: simple obj. $\{X_0=\mathbf{1},X_1,\ldots,X_{n-1}\}$	distinguishable particle species
$X_0=\mathbf{1}$	Vacuum
X^*	Antiparticle
$X\otimes Y$	Fusion

Morphisms

Modular Category	Physics
$\lvert\psi\rangle \in \mathbf{End(X)}$	state vector
b_X: $\mathbf{1} \to X \otimes X^*$	particle/antiparticle creation
d_X: $X^* \otimes X \to \mathbf{1}$	particle/antiparticle annihilation
$\mathbf{c}_{X,Y}$: $X \otimes Y \cong Y \otimes X$	exchange

Further Attributes

Modular Category	**Physics**
$X_i \otimes X_j \cong \oplus_k \mathbf{N_{ij}^k} X_k$	fusion channels
$\mathbf{\Phi_X}: \mathbf{B_n} \to \mathbf{U}(\mathbf{End}(X^{\otimes n}))$ $\boldsymbol{\sigma_i} \to \mathbf{Id}^{\otimes(i-1)} \otimes \mathbf{C}_{X,X} \otimes \mathbf{Id}^{\otimes(n-i-1)}$	particle exchange
$\mathbf{S_{ij}}=\mathrm{tr}(\mathbf{C}_{ji}\mathbf{C}_{ij})$, $\det(\mathbf{S})\neq 0$	loops distinguish species

Example: "Fibonacci MC"

- Simple classes: **1**, Ψ

- Fusion rules: $\mathbf{1}\otimes\mathbf{X}=\mathbf{X}$, $\Psi\otimes\Psi = \mathbf{1}\oplus\Psi$

- $\mathbf{C}_{\Psi,\Psi}=e^{-4\pi \mathbf{i/5}}\mathbf{p_1}+e^{3\pi\mathbf{i/5}}\mathbf{p}_{\Psi} \in \mathbf{End}(\Psi\otimes\Psi)$

- $\mathbf{S}= \begin{bmatrix} \mathbf{1} & \tau \\ \tau & \mathbf{-1} \end{bmatrix}$ τ = golden ratio

Basic Constructions

quantum group · ***semisimplify***

$$\mathbf{g} \longrightarrow U_q\mathbf{g} \longrightarrow \mathrm{Rep}(U_q\mathbf{g}) \longrightarrow \mathcal{C}(\mathbf{g},q)$$

Lie algebra

$q^L = -1$

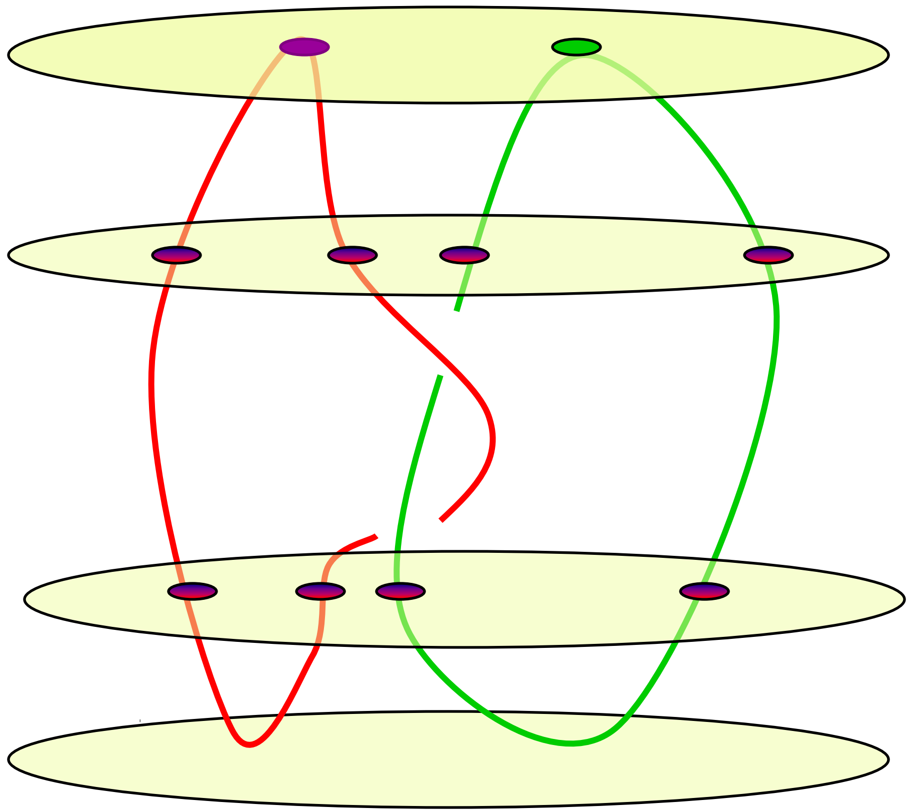
(or $SU(N)_k$, etc.)

quantum double

$$G \longrightarrow DG \longrightarrow \mathrm{Rep}(DG)$$

finite group

Physics/Quant. Comp.



Computation

output

apply gates

initialize

vacuum

Physics

measure (left particle)

particle **exchange**

create particle/anti-particle pairs

What does a TQC compute?

Answer: Link invariants/Braid traces

- Topology:

$$\mathbf{Inv}(L)=\Pr(\ \ =\text{vacuum}\ \ )$$

- Algebra:

$$\mathrm{Tr}_{\mathbf{c}}(\Phi_{\mathbf{X}}(\beta)) = \mathbf{Inv}(\hat{\beta})$$

Question: What is the computational complexity of **Inv**?

Examples

Lie Type	Algebra	Invariant
A_1	Temperley-Lieb	Jones poly'l
A_{N-1}	Iwahori-Hecke	HOMFLY-PT poly'l
B_k,C_k, or D_k	BMW	Kauffman poly'l
G_2	"Spiders"	Kuperberg's Invar.

Braid Group Reps.

- Let ***D*** be a ***unitary*** MC

$$\text{unitary rep. } \Phi_X\!: B_n \to \mathbf{U}(\mathbf{End}(X^{\otimes n}))$$

Question: what is the closure of $\Phi_X(B_n)$?

- Observe: $\overline{\Phi_X(B_n)}$ is a compact Lie group.

Density Property

Suppose $\mathbf{End}(X^{\otimes n}) \cong \bigoplus_i \mathbf{H}_{i,n}$, so that

$$\mathbf{U}(\mathbf{End}(X^{\otimes n})) \cong \prod_i \mathbf{U}(\mathbf{H}_{i,n}).$$

If $\overline{\Phi_X(\mathbf{B_n})} \supseteq \prod_i \mathbf{SU}(\mathbf{H}_{i,n})$ for all $n \geq N$, we say

$\Phi_X(\mathbf{B_n})$ is <u>*dense*</u>.

Property *F*

A unitary MC has *property F* if:

$\Phi_X(B_n) \subset \mathbf{U}(\mathbf{End}(X^{\otimes n}))$

is ***finite*** for ***all*** n and X.

Or, more generally, for **any** braided fusion category…

Computational Power

$\{U_1, U_2, \ldots, U_s\}$ ***universal*** **if**

$\overline{\{\mathrm{Id}^{\otimes a} \otimes U_i \otimes \mathrm{Id}^{\otimes b}\}}$ **contains all unitaries**

Question: which TQC models are universal?

Example 1

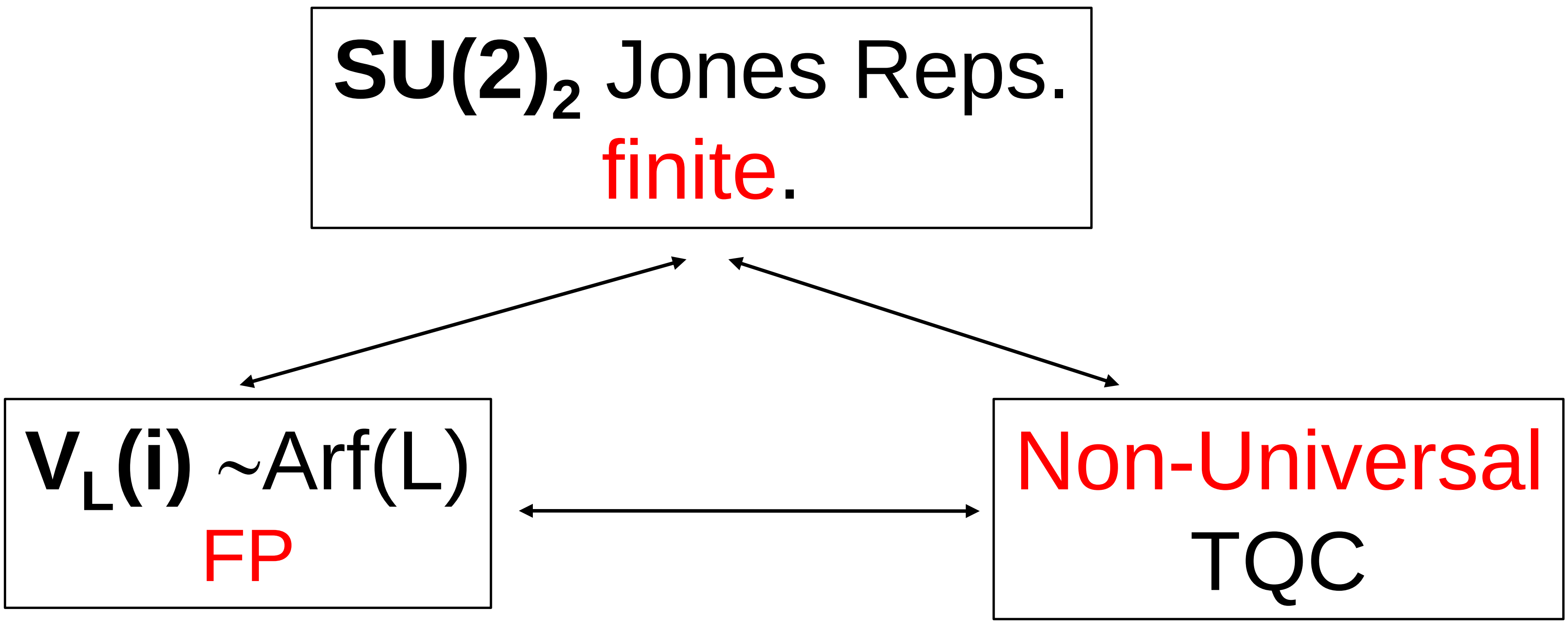
$\mathbf{SU(2)_3}$ Jones Reps.
dense.

$\mathbf{V_L(e^{2\pi i/5})}$
#P-hard

Universal
TQC

See [Freedman, Larsen, Wang ‘02]

Example 2



See [Jones '86, '87]

Naïve Paradigm

$\Phi_X(B_n)$

dense/finite

Link. Inv.

quantum/classical

#P-hard/FP

TQC

Univ./Non-Univ.

Upsetting The Apple Cart

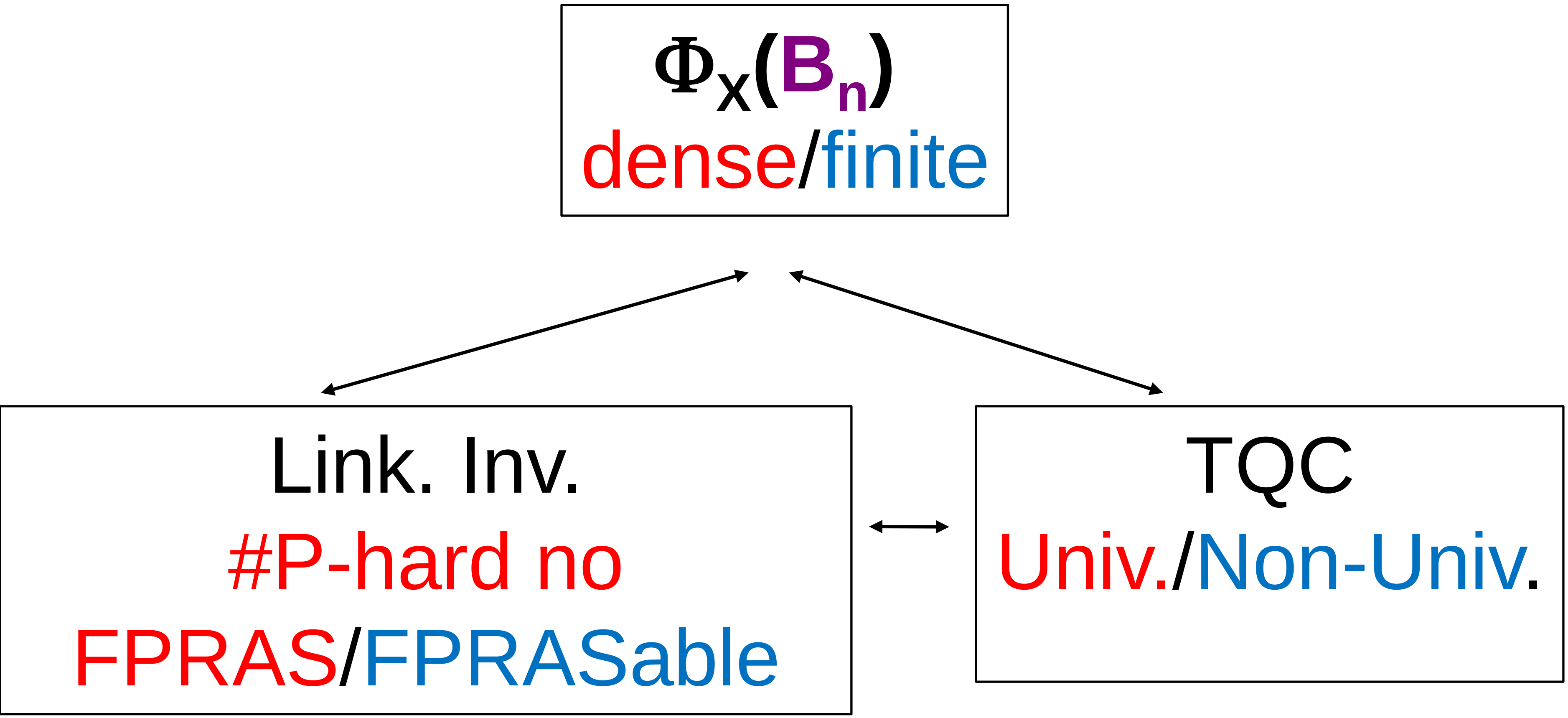
Rep(DG): $\Phi_X(B_n)$
finite.

[Etingof,R,Witherspoon]

Inv. #P
$|\mathrm{Hom}(\pi_1(S^3 \backslash L), G)|$

Non-Universal
TQC

Refined Paradigm



FPRAS

- Algorithm, poly'l in $1/\varepsilon$, n: input an n-bit instance x, output **Y** with

$$\Pr(1/(1+\varepsilon) \le \mathbf{Y}/\mathbf{f(x)} \le 1+\varepsilon) \ge 3/4$$

Conjecture 1

Jones Polynomial $\mathbf{V_L(q)}$ at $\mathbf{q=e^{2\pi i/R}}$ **not** FPRASable for **R**$\notin\{1,2,3,4,6\}$ (assuming $RP\neq NP$).

- Evidence: [Goldberg,Jerrum] showed Tutte poly'l evaluations at most rational points not FPRASable. Jones and Tutte related… [Jaeger,Vertigan,Welsh]

$\pi_1(S^3\backslash L)$

- Finitely generated:

$\pi_1(S^3\backslash L)\cong\langle a_1,\ldots,a_n:R_1,\ldots,R_m\rangle$ n+m bounded by crossings+components

$$\pi_1(S^3\backslash \text{trefoil})\cong\langle a,b:\ a^2=b^3\rangle$$

$$\pi_1(S^3\backslash \text{Hopf link})\cong\langle a,b:\ ab=ba\rangle$$

$$\pi_1(S^3\backslash \text{figure-8 knot})\cong\langle a,b:\ bab^{-1}ab=aba^{-1}ba\rangle$$

Conjecture 2

a) $\mathbf{H_L(G)}$:=$|\mathrm{Hom}(\pi_1(S^3 \backslash L), G)|$ **is** FPRASable

b) **G** *solvable*, $\mathbf{H_L(G)}$ in FP.

Random walks on groups?

If **G** is semidirect product of $\mathbf{Z_n}$ and $\mathbf{Z_m}$, can show b) is true.

Evidence

Construct.	Prop. F?	Invariant	Complexity
C(sl_2,q)	L=2,3,4,6	Arf, $H_1(M,\mathbf{Z}/3\mathbf{Z})$ Jones	P if L=2,3,4,6 #P-hard else
C(sl_n,q)	L=n+1,4,6	classical HOMFLYPT	P if L=n+1,4,6 #P-hard else
C(so_{2k+1},q)	L=4k+2	$H_1(M,\mathbf{Z}/N\mathbf{Z})$ Kauffman	P if L=4k+2 #P-hard else
Rep(D**G**)	Yes	$H_L($**G**$)$	P if **G** solvable ? else?

Algorithms

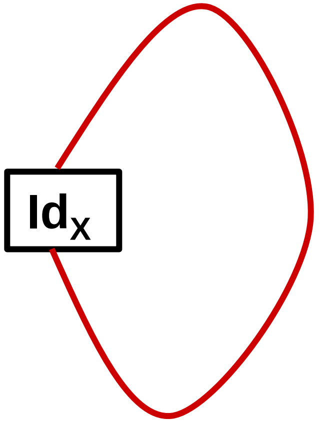
- Input $|G|^n$ n-tuples, check $R_1, \ldots R_m$ exponential
- If G *abelian*, $H_L(G) = |G|^{comp(L)}$
- If G *nilpotent*, K *a knot*, $H_K(G) = |G|$ [Eisermann]
- If G *solvable*, better algorithms? [Matei,Suciu]

Computations for Trefoil, Figure 8, and Hopf link

G	$H_{tref}(G)$	$H_{fig8}(G)$	$H_{Hopf}(G)$
Sym(3)	12	6	18
Alt(4)	36	36	48
Sym(4)	96	48	120
Alt(5)	360	300	300
Sym(5)	600	600	840

Dimension Functions

- Categorical dimension:

$$\dim(X)=\mathrm{Tr}(\mathrm{Id}_X) =$$ 

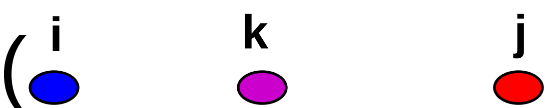
- FP-dimension:

$$\mathrm{FPdim}(X_i) = \text{largest eigenvalue of}$$

N_i =fusion matrix of X_i.

If unitary, FPdim=dim.

Related Facts

- $d_i d_j = \Sigma_k \mathbf{N_{ij}^k} d_k$ $\quad$ ($X_i \rightarrow d_i$ gives a character)
- $\dim \mathbf{End}(X_i^{\otimes \mathbf{n}}) \approx (d_i)^{n-1}$
- Pr(

) = $\mathbf{N_{ij}^k} d_k (d_i d_j)^{-1}$

Conjecture 3

A unitary MC has

property ***F*** $\Leftrightarrow$ $\dim(\mathbf{X}_i)^2 \in \mathbf{Z}$

for all *simple* $\mathbf{X}_i$

Further Evidence

UMC	Restrictions	Invariant	Complexity	$\mathfrak{B}_n$ Image
$\mathcal{C}(\mathfrak{sl}_2, q)$	$5 \leq \ell \neq 6$	$V_L(q^2)$	$\#P$-hard **no FPRAS?**	dense
$\mathcal{C}(\mathfrak{sl}_n, q)$, $3 \leq n$	$n+2 \leq \ell$, $\ell \neq 6$	$P_L'(q, n)$	$\#P$-hard **no FPRAS?**	infinite not dense
$\mathcal{C}(\mathfrak{so}_{2n+1}, q)$, $2 \leq n$	ℓ even, $2n+2 \leq \ell$, $\ell \neq 4n$	$F_L(q^{2n}, q)$	$\#P$-hard **no FPRAS?**	dense
$\mathcal{C}(\mathfrak{sp}_{2n}, q)$, $2 \leq n$	ℓ even, $2n+6 \leq \ell$, $\ell \neq 4n+2$	$F_L(q^{-2n-1}, q)$	$\#P$-hard **no FPRAS?**	dense
$\mathcal{C}(\mathfrak{so}_{2n}, q)$, $3 \leq n$	$2n+2 \leq \ell$, $\ell \neq 4n-2$	$F_L(q^{2n-1}, q)$	$\#P$-hard **no FPRAS?**	dense
$\mathcal{C}(\mathfrak{so}_4, q)$	$7 \leq \ell$	$(-1)^{c-1}[V_L(-q^{-2})]^2$	$\#P$-hard **no FPRAS?**	infinite not dense
$\mathcal{C}(\mathfrak{sl}_2, q)$	$\ell = 3$	$(-1)^{c-1}$	FP	finite abelian
$\mathcal{C}(\mathfrak{sl}_2, q)$	$\ell = 4$	$(-\sqrt{2})^{c-1}(-1)^{\mathrm{Arf}(L)}$ or 0	FP	finite
$\mathcal{C}(\mathfrak{sl}_n, q)$	$\ell = 6$	$\pm(i)^{c-1}(i\sqrt{3})^{d_3}$	FP	finite
$\mathcal{C}(\mathfrak{sp}_4, q)$	$\ell = 10$	$\pm(\sqrt{5})^{d_5}$	FP	finite
$\mathcal{C}(\mathfrak{sl}_n, q)$	$\ell = n+1$	$e^{\pi i K(L)/n}$	FP	finite abelian
$\mathcal{C}(\mathfrak{so}_p, q)$, $3 \leq p$ prime	X spin rep., $\ell = 2p$	$\pm(\sqrt{p})^{d_p}$	FP	finite
$\mathrm{Rep}(DG)$	G finite	$H_L(G)$	**FPRAS?**	finite

Thanks!