Quantum Graphical Models and Belief Propagation

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Outline

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1. Graphical models
2. Belief propagation
3. Quantum graphical models
4. Quantum belief propagation
5. Examples
   - Quantum turbo-codes
   - Many-body simulations

Graphical models

- Bayesian networks (artificial intelligence).
- Factor graphs (image recognition).
- Tanner graphs (coding theory).
- Markov networks (statistical physics).
- etc.

Common features:

- A (sparse) graph $G = (V, E)$.
- Random variables u, each associated with a vertex $u \in V$.
- An efficiently specifiable distribution $P(V) = P(u_1, u_2, \ldots)$.
- Edges $e = (u, v)$ encode some kind of dependency relation in P.

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Conditional independence

Let A, B, and C be three random variables with distribution $P(A, B, C)$.

We say that A and C are independent given B if

- Conditional mutual information vanishes $I(A : C|B) = 0$.
- $P(A, B, C) = P(A)P(B|A)P(C|B)$ which suggests $A \to B \to C$.
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Defining the mutual distribution $P(A : B) = \frac{P(A,B)}{P(A)P(B)}$, we can characterize conditional independence by

- $P(A, B, C) = P(A)P(B)P(C)P(A : B)P(B : C)$ which does not suggest a causal relation.

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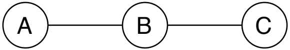

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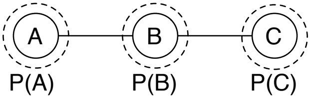

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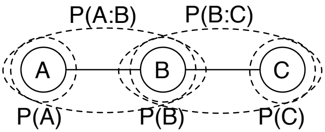

Markov random fields

Given a graph $G = (V, E)$ and a distribution $P(V)$, the pair $(G, P(V))$ forms a Markov Random Field iff:

- For all $U \subset V$, $I(U : V - U - n(U)|n(U)) = 0$.
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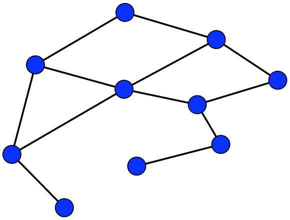

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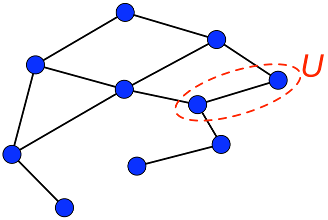

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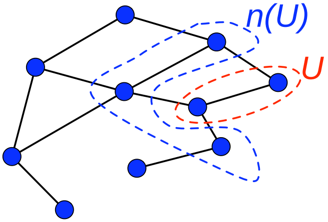

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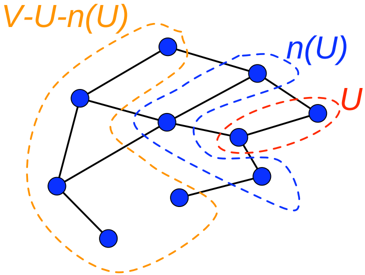

Hammersley-Clifford Theorem

Theorem (Hammersley-Clifford)

The pair $(G, P(V))$ *is a positive* $(P > 0)$ *random Markov field iff*

$$P(V) = \frac{1}{Z} \prod_{C \in \mathfrak{C}(G)} \psi(C).$$

Special case: bifactor states (pairwise RMF)

When largest clique size is 2 (2d square lattice) or when $\psi(C)$ is trivial for $|C| > 2$, MRF are of the form

$$\begin{aligned} P(V) &= \frac{1}{Z} \prod_{v \in V} \mu(v) \prod_{(u,v) \in E} \nu(u:v) \\ &= \frac{1}{Z} \exp\left\{-\beta\Big(\sum_v h_v + \sum_{\langle u,v \rangle} k_{uv}\Big)\right\}. \end{aligned}$$

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Description of the algorithm

Task (basic case)

Given a graph $G = (V, E)$ and a bifactor distribution $P(V)$ on G, compute marginals

$$P(v) = \sum_{V-v} P(V).$$

Algorithm architecture

- One processor per random variable v.
- Messages exchanged between processors related by an edge.
- Outgoing messages at v depend on local "fields" $\mu(v)$ and $\nu(u : v)$ and received messages at v.
- The marginal $P(v)$ is estimated by a belief $b(v)$ that depends on the received messages at v and the local fields.
- Exact when G is a tree and complexity = *diameter*(G).
- Good heuristic on loopy graphs.

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Belief propagation algorithm

Algorithm

- Initialization $m_{u\rightarrow v}(v) = cte$.
- Iterations $m_{u\rightarrow v}(v) \propto \sum_u \mu(u)\nu(u:v)\prod_{v'\in n(u)-v} m_{v'\rightarrow u}(u)$.
- Beliefs $b(u) \propto \mu(u)\prod_{v\in n(u)} m_{v\rightarrow u}(u)$.
- $b(u,v) \propto \mu(u)\mu(v)\nu(u:v)\prod_{w\in n(u)-v} m_{w\rightarrow u}(u)\prod_{w\in n(v)-u} m_{w\rightarrow v}(v)$.

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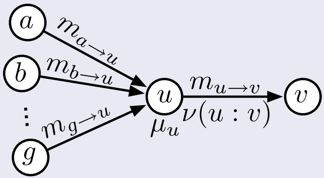

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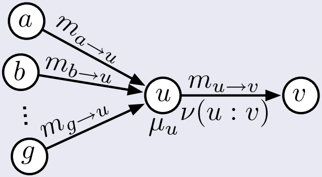

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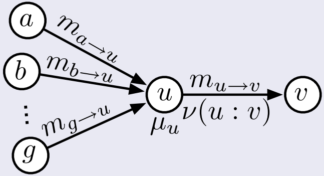

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System description

- A (sparse) graph $G = (V, E)$.
- Each vertex u is associated a quantum system (spin) u with Hilbert space $\mathcal{H}_u$.
- An efficiently specifiable quantum state ρ_V on $\mathcal{H}_V = \bigotimes_{u \in V} \mathcal{H}_u$.
- Edges $e = (u, v)$ encode some kind of dependency relation in ρ_V.

How to specify ρ_V?

- Many possible generalizations of classical bifactor states.
- They have applications in different contexts:
  - Quantum many-body.
  - Quantum error correction.

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Bifactor state: $P(V) = \frac{1}{Z}\prod_{v\in V}\mu(v)\prod_{(u,v)\in E}\nu(u:v)$.

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Problems:

- Ambiguity in order of the terms.
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Quantum generalisations

In analogy with the classical case, define

- Conditional state $\rho^{(n)}_{A|B} = \rho_B^{-1} \star^{(n)} \rho_{AB}$.
- Mutual state $\rho^{(n)}_{A:B} = (\rho_A^{-1}\rho_B^{-1}) \star^{(n)} \rho_{AB}$.

Quantum conditional independence

Given three quantum systems A, B, and C and a joint state ρ_{ABC}, we say that A and C are independent given B if $I(A:C|B) = 0$ which implies:

- $\rho_{ABC} = \rho_A \star^{(n)} \rho_{B|A}^{(n)} \star^{(n)} \rho_{C|B}^{(n)}$ which suggests $A \to B \to C$.
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These conditions differ for different values of n and differ between each other.

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Theorem

For $n = \infty$, all conditions are equivalent and imply conditional independence.

Theorem

For $n = 1$, the first two conditions are equivalent and imply conditional independence.

Theorem (Quantum Hammersley-Clifford)

If (ρ_V, G) is a positive quantum Markov network, then

$$\rho_V = \bigodot_{C \in \mathfrak{C}(G)} \sigma_C = \exp\left\{ -\beta \sum_{C \in \mathfrak{C}(G)} h_C \right\}.$$

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1. Graphical models
2. Belief propagation
3. Quantum graphical models
4. **Quantum belief propagation**
5. Examples
   - Quantum turbo-codes
   - Many-body simulations

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M. Hastings '07

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Let $G = (V, E)$ be a graph and let

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be a bifactor state on G.

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If G is a tree and (G, ρ_V) is a quantum Markov random field, then the beliefs b_u converge to the correct marginals $\rho_u = Tr_{V-u}\{\rho_V\}$ in a time proportional to depth(G).

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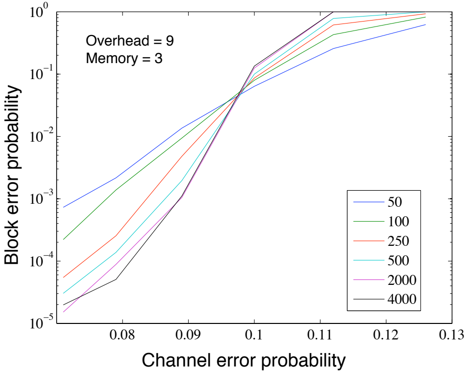

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- Error probability decreases as number of encoded qubits increases.
- Error-free "phase transition" at 0.1.
- With finite size, 10^{-4} threshold around $\epsilon = 0.08$.

Best performance to date at this rate.

Poulin, Tillich, and Ollivier'07.

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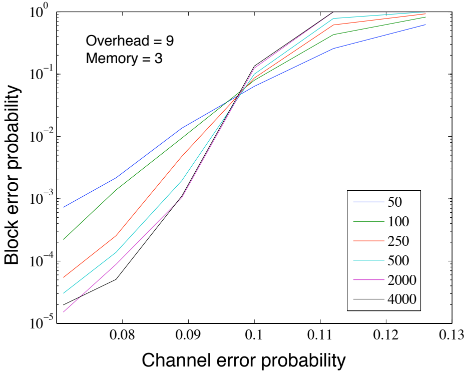

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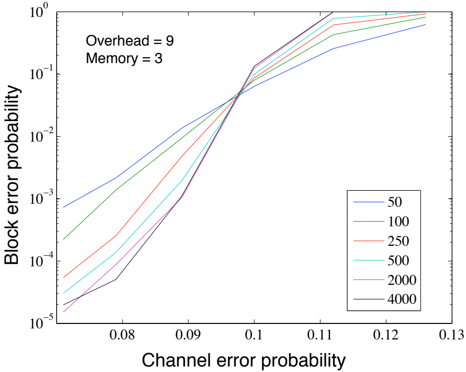

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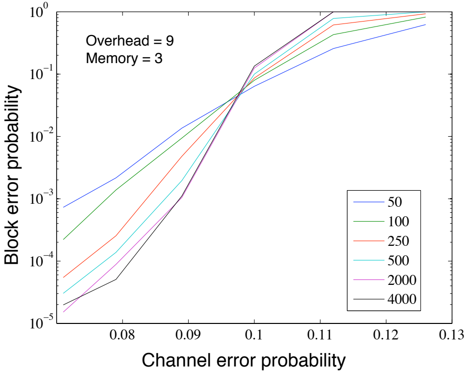

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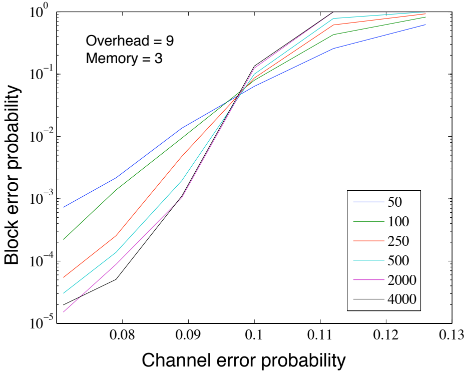

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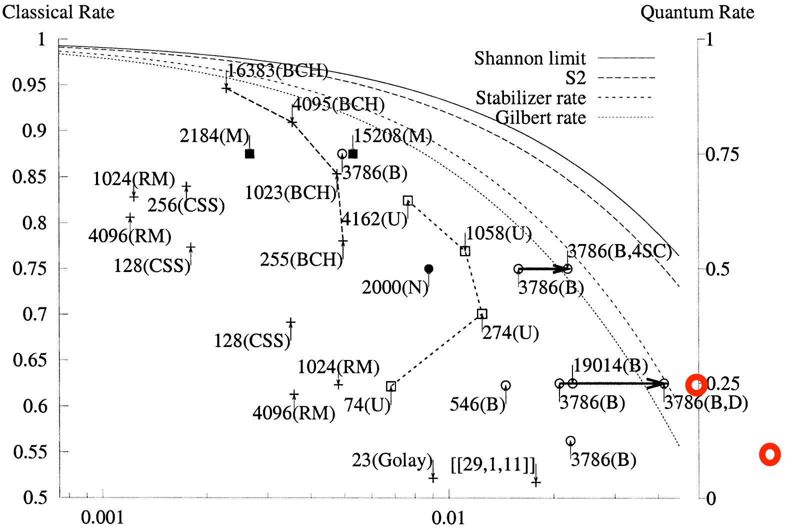

MacKay, Mitchison, McFadden, IEEE'04.

Outline

One dimensional classical system

Consider the 1d classical system with hamiltonian $H = \sum_i h_i + \sum_{\langle ij \rangle} J_{ij}$.

Its Gibbs distribution is ($\mu(i) = e^{-\beta h_i}$ and $\nu(i,j) = e^{-\beta J_{ij}}$)

$$\begin{aligned}\rho(i_1, i_2, \ldots) &= \frac{1}{Z} e^{-\beta H(i_1, i_2, \ldots)} \\ &= \frac{1}{Z} \mu(i_1)\nu(i_1, i_2)\mu(i_2)\nu(i_2, i_3)\mu(i_3)\ldots\end{aligned}$$

So the partition function can be evaluated step by step:

$$m_{1\to 2}(i_2) = \sum_{i_1} \mu(i_1)\nu(i_1, i_2)$$

$$m_{2\to 3}(i_3) = \sum_{i_2} m_{i_1 \to i_2}(i_2)\mu(i_2)\nu(i_2, i_3)$$

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$$\vdots$$

$$Z = \sum_{i_N} m_{i_{N-1}\to i_N}\mu(i_N)$$

One dimensional classical system

Consider the 1d classical system with hamiltonian $H = \sum_i h_i + \sum_{\langle ij \rangle} J_{ij}$. Its Gibbs distribution is ($\mu(i) = e^{-\beta h_i}$ and $\nu(i,j) = e^{-\beta J_{ij}}$)

$$\begin{aligned}\rho(i_1, i_2, \ldots) &= \frac{1}{Z} e^{-\beta H(i_1, i_2, \ldots)} \\ &= \frac{1}{Z} \mu(i_1)\nu(i_1, i_2)\mu(i_2)\nu(i_2, i_3)\mu(i_3)\ldots\end{aligned}$$

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$$\rho_V = \frac{1}{Z} e^{-\beta H} = \frac{1}{Z} \mu_{i_1} \odot \nu_{i_1:i_2} \odot \mu_{i_2} \odot \nu_{i_2:i_3} \odot \mu_{i_3} \ldots$$

Bottleneck for computing Z:

$$Tr_A\{\mu_A \odot \nu_{A:B} \odot \mu_B \odot \nu_{B:C} \odot \mu_C\} \neq Tr_A\{\mu_A \odot \nu_{A:B}\} \odot \mu_B \odot \nu_{B:C} \odot \mu_C$$

But it is equal when $I(A:C|B) = 0$.

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Effective thermal hamiltonian

$$H = \sum_{i=-\infty}^{\infty} h_i + J_{i,i+1} \qquad \rho = \frac{1}{Z}\exp\{-\beta H\}$$

··· -5 -4 -3 -2 -1 0 1 2 3 4 5 ···

Effective Thermal Hamiltonian = $\sum_{i=1}^{\infty} h_i + J_{i,i+1} + V_1 + V_2 + V_3 + V_4 \ldots$

Effective thermal hamiltonian

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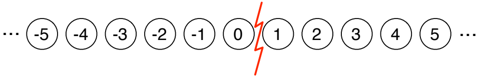

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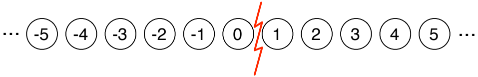

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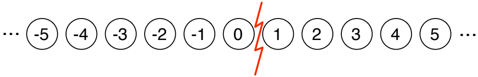

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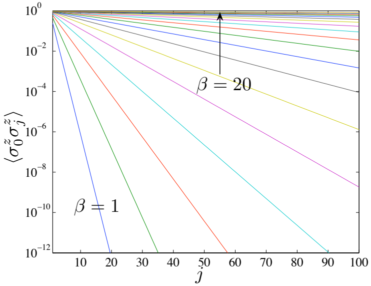

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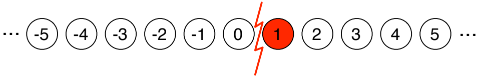

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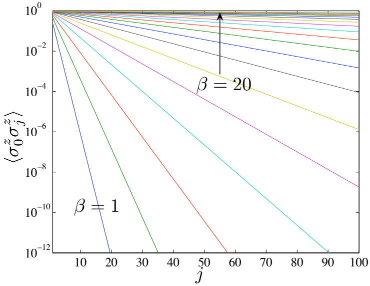

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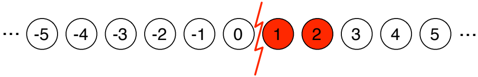

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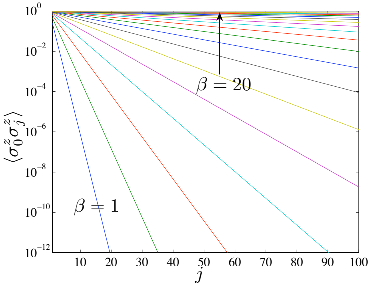

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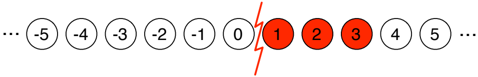

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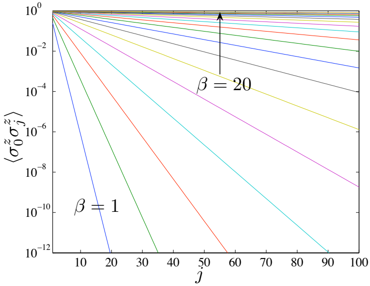

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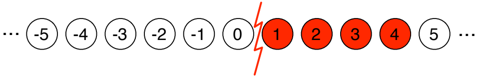

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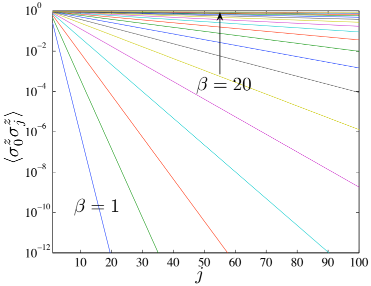

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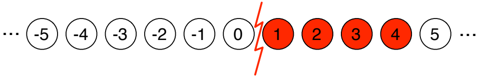

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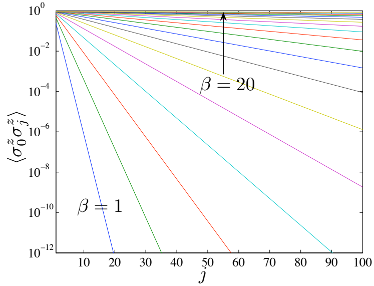

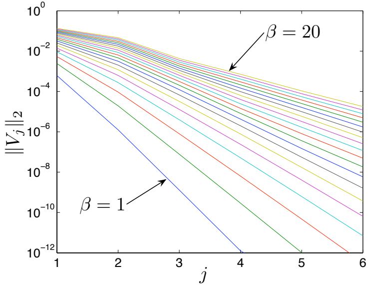

One dimensional quantum system

$$\sigma_{1-4} = e^{-\beta(h_1+h_2+h_3+h_4+J_{12}+J_{23}+J_{34})}$$

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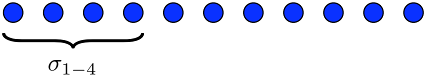

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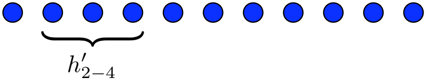

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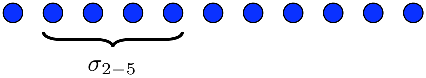

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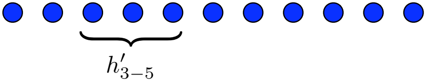

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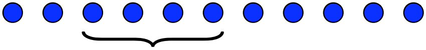

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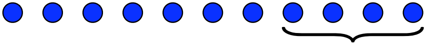

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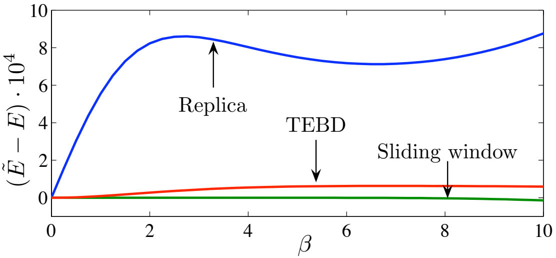
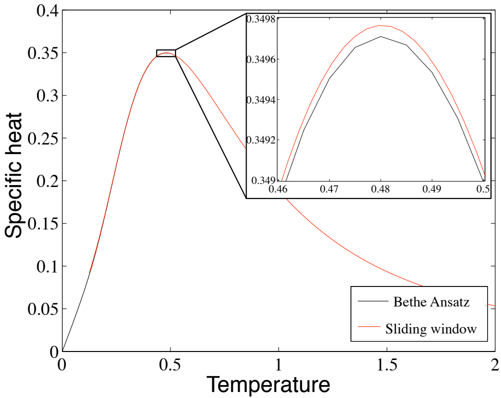
Critical 1d Ising model



Bilgin and Poulin ’07.

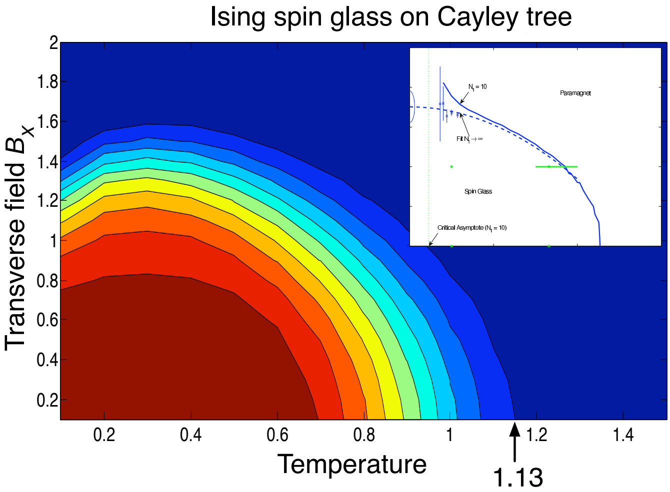
- Replica: Trotter decomposition $N_\tau = 10$ (bifactor $\star^{(10)}$).
- TEBD: Time-evolving block decimation (DMRG $\chi = 150$).
- Sliding window $\ell = 6$ (bifactor $\odot$).

1D anti-ferromagnetic Heisenberg model



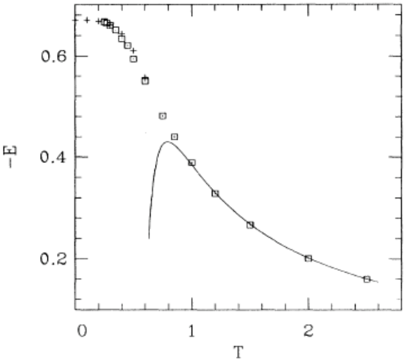
- Bethe Ansatz: exact (A.Klümper and D. C. Johnston, PRL'00).
- Sliding window $\ell = 9$ (bifactor $\odot$).

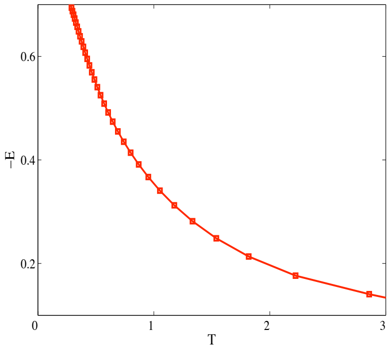
Phase diagram



Laumann, Scardicchio, and Sondhi '07, Bilgin and Poulin.

2D anti-ferromagnetic Heisenberg model





- Quantum Monte Calrlo: M.S. Makivić and H.-Q. Ding PRB'91.
- 10th-order J/T expansion.
- Quantum Belief propagation, window size 7.

- Belief propagation operating on graphical models is a powerful, highly parrallelizable, heuristic for all sorts of inference problems.
- Many of these properties carry over to the quantum realm:
  - Half Hammersley-Clifford Theorem (Markov $\Rightarrow$ Gibbs).
  - Good heuristic for iterative decoding of sparse and quantum turbo codes.
  - Good heuristic for many-body systems on graphs with no small loops.

See poster by Ersen Bilgin for more details

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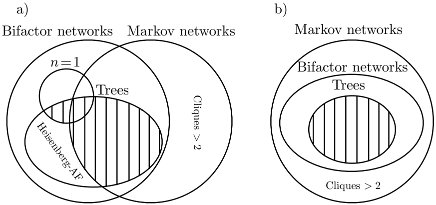

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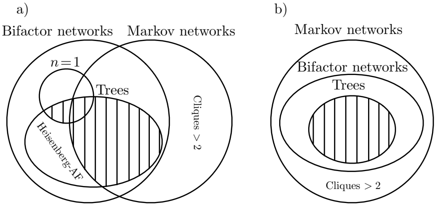

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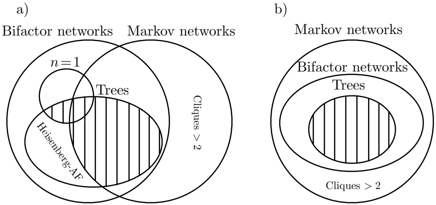

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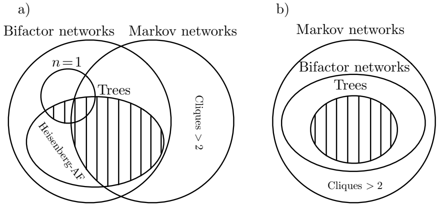

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