Information propagation in interacting spin systems with disorder

Tobias J. Osborne

Royal Holloway, University of London

with Jens Eisert and Christian Burrell

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Royal Holloway
University of London

Introduction

- Quantum spin systems
- Information propagation in quantum spin systems (the Lieb-Robinson bound)
- (At least) 3 regimes:
  - Localisation
  - Diffusion
  - Quantum zeno
- Fault tolerance in quantum computers

Introduction: what is a quantum spin system?

(n spins)

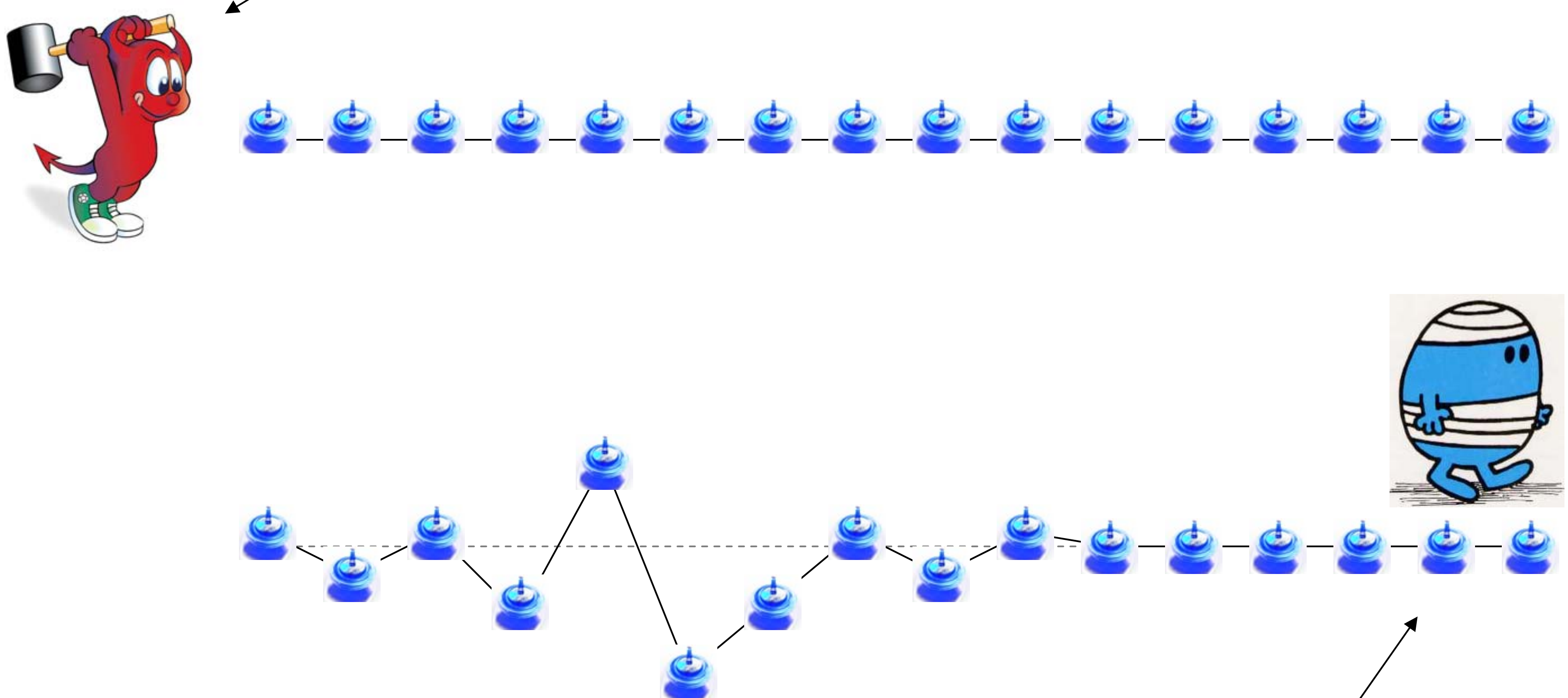
$h_j \equiv \mathbb{I}_{1\cdots j-1} \otimes h_j \otimes \mathbb{I}_{j+2\cdots n}$

$\mathbb{C}^d$

Hilbert space:	$\mathcal{H} = \bigotimes_{j=1}^{n} \mathbb{C}^d$
Hamiltonian:	$H(t) = \sum_{j=1}^{n-2} h_j(t)$
Normalisation:	$\Vert h_j(t) \Vert = O(1)$

Problem: information propagation

localised disturbance at A at $t = 0$



what is felt here at B after time $t = T$?

How to quantify information propagation?

We typically study information propagation using connected time-dependent correlation functions:

$$\langle\psi|U^\dagger(t)BU(t)A|\psi\rangle - \langle\psi|U^\dagger(t)BU(t)|\psi\rangle\langle\psi|A|\psi\rangle$$

where A and B are local observables and

$$U(t) = \mathcal{T}e^{i\int_0^t H(s)ds}$$

(Initial state is usually a product state or ground state.)

Dynamics of correlations

But, for technical reasons it's more convenient to study the *Lieb-Robinson commutator*:

$$\boxed{C_A(j,t) = \sup_{\|B\|=1} \|[A(t), B]\|}$$

where $A(t) = U^\dagger(t) A U(t)$ and $U(t) = \mathcal{T} e^{i\int_0^t H(s)ds}$ and B is restricted to act nontrivially only on site j. (i.e. $\mathrm{supp}(B) = \{j\}$)

Dynamics of correlations

Physically the LR commutator measures worst case (coming from the supremum and the norm) influence of an operation on site j after a time t as measured by an observable A.

The Lieb-Robinson bound

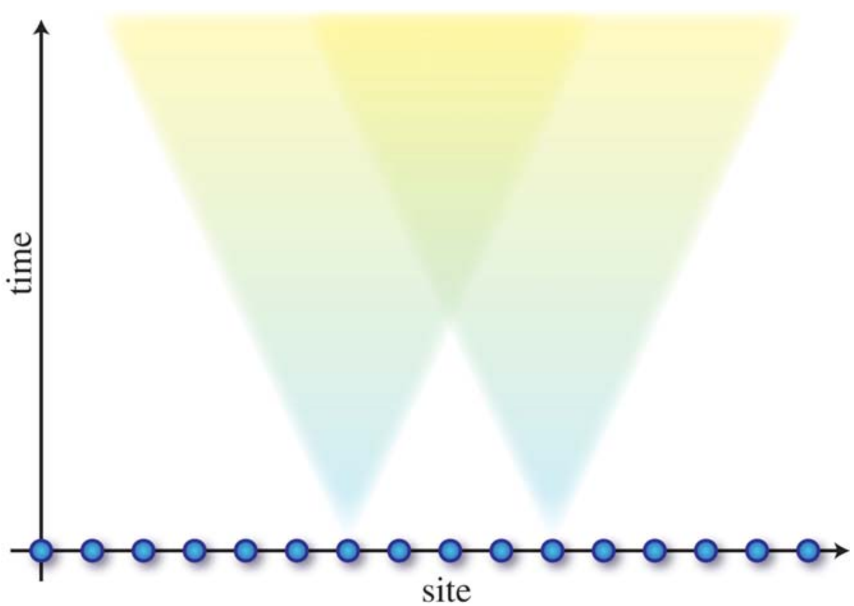
Proposition 1 (Lieb and Robinson, 1972): For any low dimensional spin system the following bound holds:

$$\|[A(t),B]\| \leq c e^{-v d(A,B)+k|t|},$$

where v, k, c are constants, and A and B are local operators.

Discussion

The LR bound says that two-point dynamical correlations are exponentially suppressed outside of an effective "light cone" with an effective "speed of light" set by $||h||$.



Discussion

We say that, if the bound is essentially saturated, information propagates *ballistically* through the system.

Can one expect better bounds? For generic translation invariant systems the answer is *no*! (This is the so called “LR wall” encountered in time-dependent DMRG.)

But do there exist non-TI systems with better bounds?

Disorder

We now focus on LR bounds for (statically and dynamically) *disordered systems*. Intuitively expect better bounds because possibility of:

- *Anderson localisation*; and

- the *quantum Zeno effect*.

Disorder

In our results we'll identify 3 types of behaviour:

1. Ballistic (standard LR);

2. Diffusive (new); and

3. Localised regimes (new)

Disorder

Our hamiltonians will be of the form

$$H(t) = \sum_{j=1}^{n-2} h_j(t) + \sum_{j=1}^{n} \xi_x^j(t)\sigma_j^x + \xi_y^j(t)\sigma_j^y + \xi_z^j(t)\sigma_j^z$$

where $\xi_\alpha^j(t)$ are either i.i.d. random variables (time independent) or (derivatives of) Wiener processes (time dependent)

2. Time-dependent disorder (diffusive regime)

We consider the disordered XY model

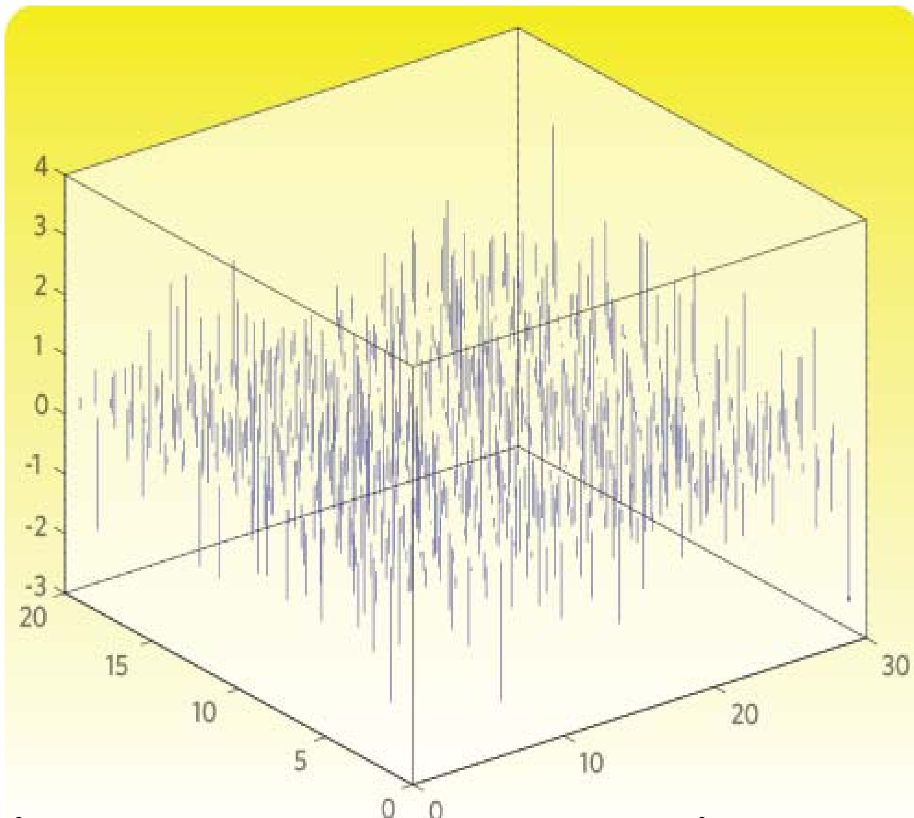
$$H_{XY}(t,\boldsymbol{\xi}(t)) = \sum_{j=1}^{n-2} \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \sum_{j=1}^{n} \xi^j(t)\sigma_j^z$$

where $\xi^j(t) = dW^j(t)$, and $W^j(t)$ is a brownian motion. We define

$$U(t,\boldsymbol{\xi}(t)) = \mathcal{T} e^{i\int_0^t H_{XY}(s,\boldsymbol{\xi}(s))ds}$$

2. Time-dependent disorder

$\xi^j(t)$



site time

2. Time-dependent disorder

Averaging over the Wiener processes (the disorder), using Itô's rule, we obtain the following master equation

$$\frac{d\rho(t)}{dt} = i\sum_{j=1}^{n-2}[h_j(t), \rho(t)] - 2\gamma\sum_{j=1}^{n}[\sigma_j^z, [\sigma_j^z, \rho(t)]]$$

where

$$\rho(t) = \mathbb{E}_{\xi(t)}[U(t, \boldsymbol{\xi}(t))\rho(0)U^\dagger(t, \boldsymbol{\xi}(t))]$$

is the state of the system after time t *averaged over the disorder.*

2. Time-dependent disorder

In the Heisenberg picture we therefore have that the dynamics of *averaged* observables satisfy

$$\frac{dA(t)}{dt} = i\sum_{j=1}^{n-2}[A(t), h_j(t)] - 2\gamma\sum_{j=1}^{n}[[A(t), \sigma_j^z], \sigma_j^z]$$

Intuition: the disorder acts as a continuous weak measurement hence the Zeno effect should suppress information propagation.

2. Time-independent disorder

Proposition 2 (Burrell, Eisert, and Osborne, (2008)). Let A be a local observable. Then for $H_{XY}(t, \boldsymbol{\xi}(t))$ the LR commutator of the *averaged* observable satisfies the bound

$$\|[\mathbb{E}_{\boldsymbol{\xi}(t)}[A(t)], B]\| \leq c \frac{|t|}{d(A,B)^2}$$

This is diffusive behaviour.

2. Time-dependent disorder

Conjecture 1. Consider the hamiltonian

$$H(t) = \sum_{i=1}^{n-2} h_j + \sum_{i=1}^{n} \xi^j(t)\sigma_j^z$$

where $\xi^j(t) = dW^j(t)$, and $W^j(t)$ is now a continuous-time martingale, and h_j is general. Then

$$\|[\mathbb{E}_{\xi(t)}[A(t)], B]\| \leq c\frac{|t|}{d(A,B)^2}$$

2.1. Interlude: time-independent disorder is *much* stronger

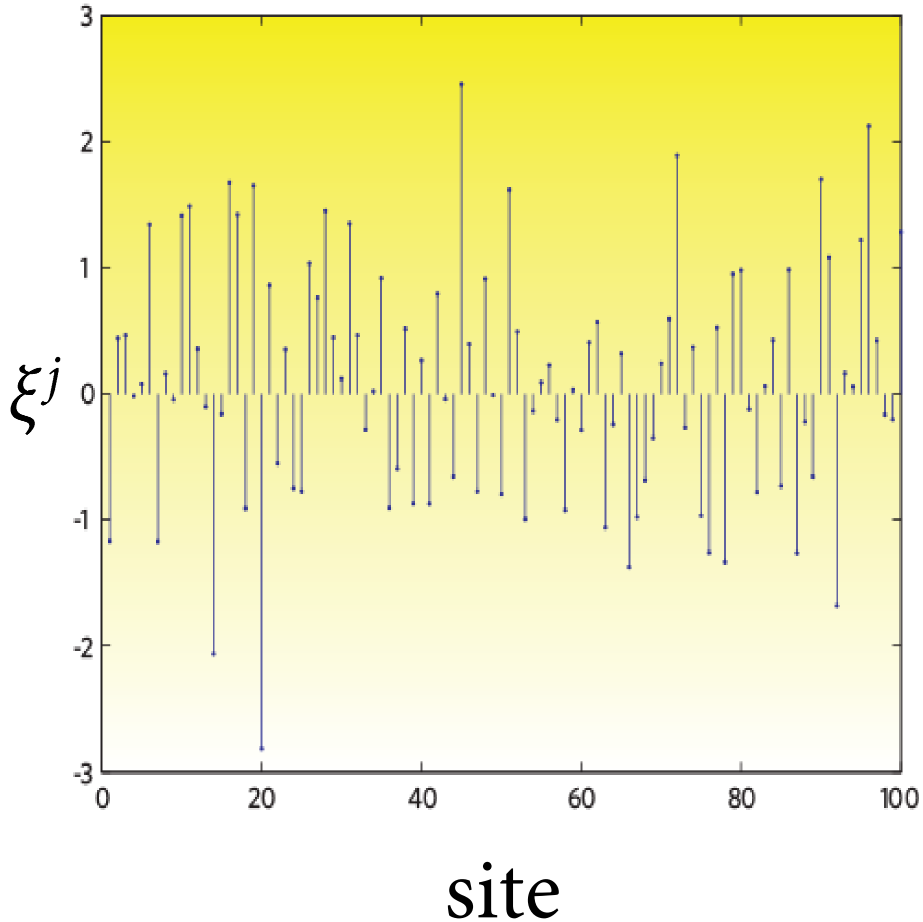
We consider the disordered XY model

$$H = \sum_{j=1}^{n-2} \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \sum_{j=1}^{n} \xi^j \sigma_j^z$$

Where ξ^j are chosen according to, eg., Gaussian distribution. *Quenched disorder*.

2.1. Time-independent disorder

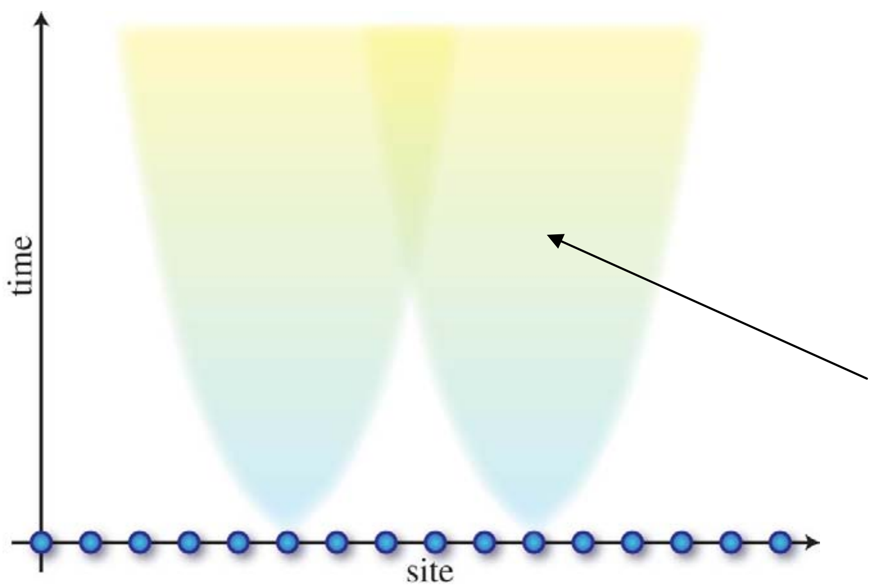
Gaussian distributed field



2.1. Time-independent disorder

Proposition 3 (Burrell and Osborne, (2007)). Let A be a local operator, then for almost all ξ_j the disordered XY model satisfies "*information localisation*"

$$\|[A(t), B]\| \leq c|t|e^{-kd(A,B)}$$



"logarithmic light cone"

2.1. Time-independent disorder

Conjecture 2. Consider the hamiltonian

$$H = \sum_{j=1}^{n-2} h_j + \sum_{j=1}^{n} \xi^j \sigma_j^z$$

where ξ_j are i.i.d. RV's, then for almost all ξ_j

$$\|[A(t), B]\| \leq c|t|e^{-kd(A,B)}$$

[Maybe need the hamiltonian to be

$$H = \sum_{j=1}^{n-2} h_j + \sum_{j=1}^{n} \xi_x^j \sigma_j^x + \xi_y^j \sigma_j^y + \xi_z^j \sigma_j^z \quad ?]$$

3. Time-dependent disorder: ballistic to localised crossover

We consider now a *general* hamiltonian+noise

$$H(t) = \sum_{j=1}^{n-2} h_j(t) + \gamma \sum_{j=1}^{n} \xi_x^j(t)\sigma_j^x + \xi_y^j(t)\sigma_j^y + \xi_z^j(t)\sigma_j^z \qquad (*)$$

Averaging over the Wiener processes (the disorder), using Itô's rule, we obtain the following master equation

$$\frac{d\rho(t)}{dt} = i\sum_{j=1}^{n-2}[h_j(t),\rho(t)] - 2\gamma\sum_{j=1}^{n}\sum_{\alpha=x,y,z}[\sigma_j^\alpha,[\sigma_j^\alpha,\rho(t)]]$$

3. Time-dependent disorder

In the Heisenberg picture we therefore have

$$\frac{dA(t)}{dt} = i\sum_{j=1}^{n-2}[A(t), h_j(t)] - 2\gamma\sum_{j=1}^{n}\sum_{\alpha=x,y,z}[[A(t),\sigma_j^\alpha],\sigma_j^\alpha]$$

Again intuition: the disorder acts as a continuous weak measurement hence the Zeno effect should suppress information propagation.

3. LR for time-dep. disorder

Proposition 4: for the model (*) if $\|h_j(t)\| < O(\gamma)$ then

$$\|[\mathbb{E}_{\xi(t)}[A(t)], B]\| \le ce^{-kd(A,B)}$$

where A is a local operator with $O(1)$ support. If $\|h_j(t)\| \ge O(\gamma)$ then information can, in principle, propagate ballistically.

(We drop the expectation symbol from now on.)

Proof

Proof: set up LR commutator (assume $A(0)$ lives on site 1 and B lives on site j):

$$C_A(j,t) = \sup_{\|B\|=1} \|[A(t), B]\|$$

Now, we work out $C_A(j, t)$ a little time later:

$$\begin{aligned} C_A(j, t+\epsilon) \le (1 - 8\epsilon\gamma n) C_A(j,t) + \epsilon \|[A(t), [H_0, B]]\| \\ + \epsilon\gamma \|[8nA(t) - \mathcal{F}(A(t)), B]\| \end{aligned}$$

where

$$\mathcal{F}(M) = \sum_{j=1}^{n} \sum_{\alpha=x,y,z} [[M, \sigma_j^\alpha], \sigma_j^\alpha] \quad \text{and} \quad H_0 = \sum_{j=1}^{n-2} h_j(t)$$

Proof

The superoperator $\mathcal{F}(M)$ is a sum of projections onto the linear space of traceless operators:

$$8nM - \mathcal{F}(M) = 2\sum_{j=1}^{n}\sum_{\alpha=0}^{3}\sigma_j^\alpha M \sigma_j^\alpha$$

so that the last term on LHS becomes

$$\begin{aligned}\epsilon\gamma\|[8nA(t) - \mathcal{F}(A(t)), B]\| &\le 2\epsilon\gamma\sum_{k\neq j}\sum_{\alpha_k=0}^{3}\|[A(t), \sigma_k^\alpha B \sigma_k^\alpha]\| \\ &= 8\epsilon\gamma(n-1)C_A(j,t)\end{aligned}$$

(Our upper bound will hold only a.e. but this is irrelevant for integrating the ODE.)

Proof continued

Putting this together with

$$\|[A(t),[H_0,B]]\| \leq \kappa(C_A(j-1,t)+C_A(j,t)+C_A(j+1,t))$$

Gives us

$$\frac{C_A(j,t+\epsilon)-C_A(j,t)}{\epsilon} \leq -8\gamma C_A(j,t)+$$
$$\kappa(C_A(j-1,t)+C_A(j,t)+C_A(j+1,t))$$

Taking limsup gives us:

$$\frac{\mathcal{D}C_A(j,t)}{\mathcal{D}t} \leq -8\gamma\Delta_j(t)+\kappa(C_A(j-1,t)+C_A(j,t)+C_A(j+1,t))$$

Proof continued

Write as a matrix equation

$$\frac{\mathcal{D}\mathbf{C}(t)}{\mathcal{D}t} \leq \mathbf{MC}(t)$$

where

$$\mathbf{M} = \begin{pmatrix} \kappa - 8\gamma & \kappa & \cdots & 0 \\ \kappa & \kappa - 8\gamma & \kappa & \cdots \\ 0 & \kappa & \kappa - 8\gamma & \kappa \\ & & & \ddots \end{pmatrix} = (\kappa - 8\gamma)\mathbb{I} + \kappa\mathcal{T}$$

Proof continued

Integrating the equality case (or iterating the integral equation):

$$\mathbf{C}(t) = e^{(\kappa-8\gamma)t} e^{t\kappa\mathcal{T}} \mathbf{C}(0)$$

where

$$C_A(j,0) \le \delta_{1,j} \| A(0) \|$$

After tedious algebra to bound taylor series we find that if $\gamma \ge \kappa + O(1)$ then

$$C_A(j,t) \le c e^{-\omega j}$$

Proof continued

If this condition isn't satisfied then information can, in principle, propagate ballistically: the evolution of a 1D quantum computer can maintain coherence for long time scales under threshold.

Our results can be seen as a continuous-time analogue of D. Aharonov, Phys. Rev. A 62, 062311 (2000) on fault tolerance in quantum computers. Our bound applies to all local observables.

Summary

- Disorder can improve bounds on information propagation in quantum spin chains

- Time-independent case exploits Anderson localisation

- Time-dependent case exploits quantum Zeno effect.

- Ballistic, diffusive, and localised regimes can occur.