Nonequilibrium Statistics of Geophysical Flows From Cumulant Expansions and Flow Equations

Brad Marston

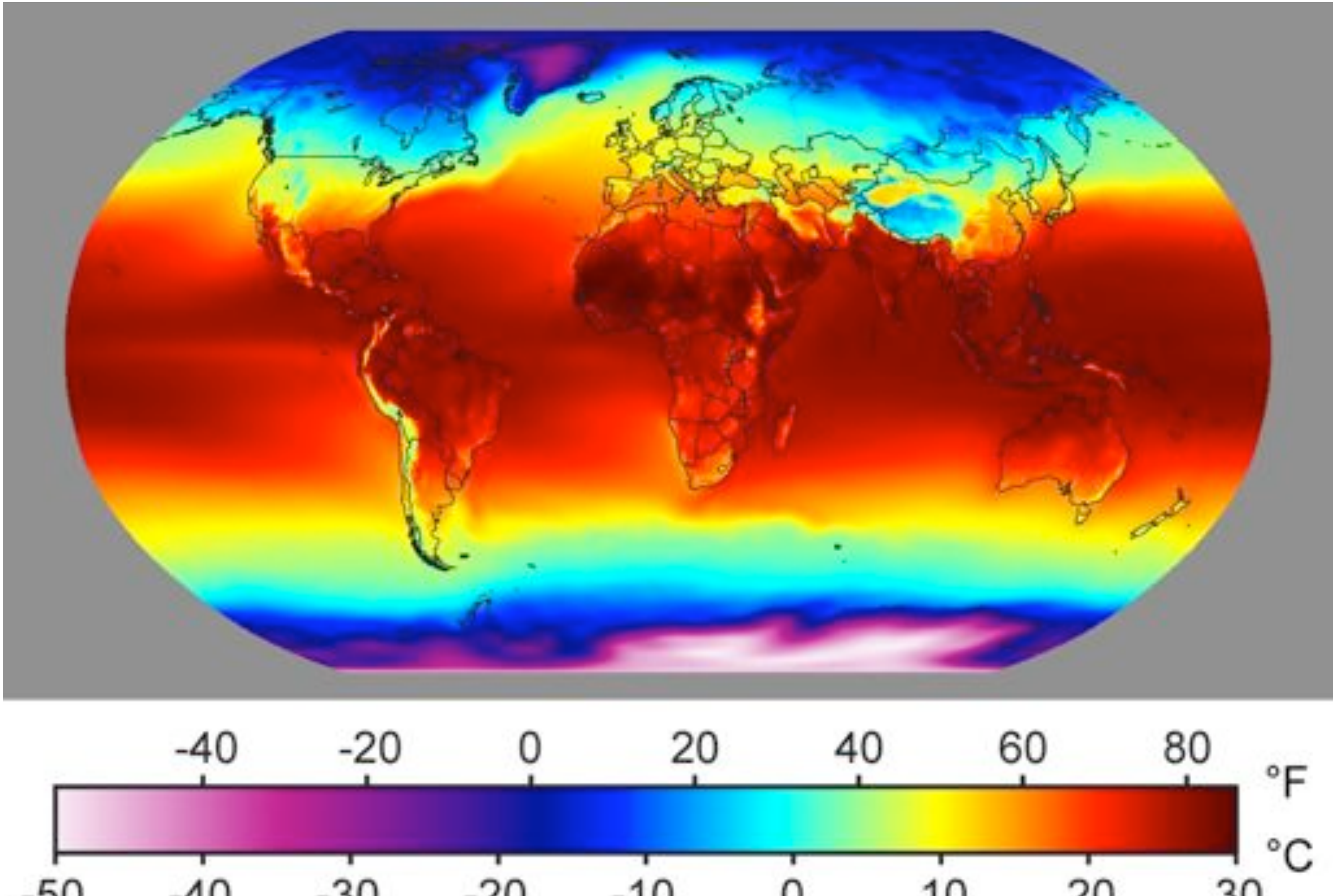
Brown University

Santa Fe

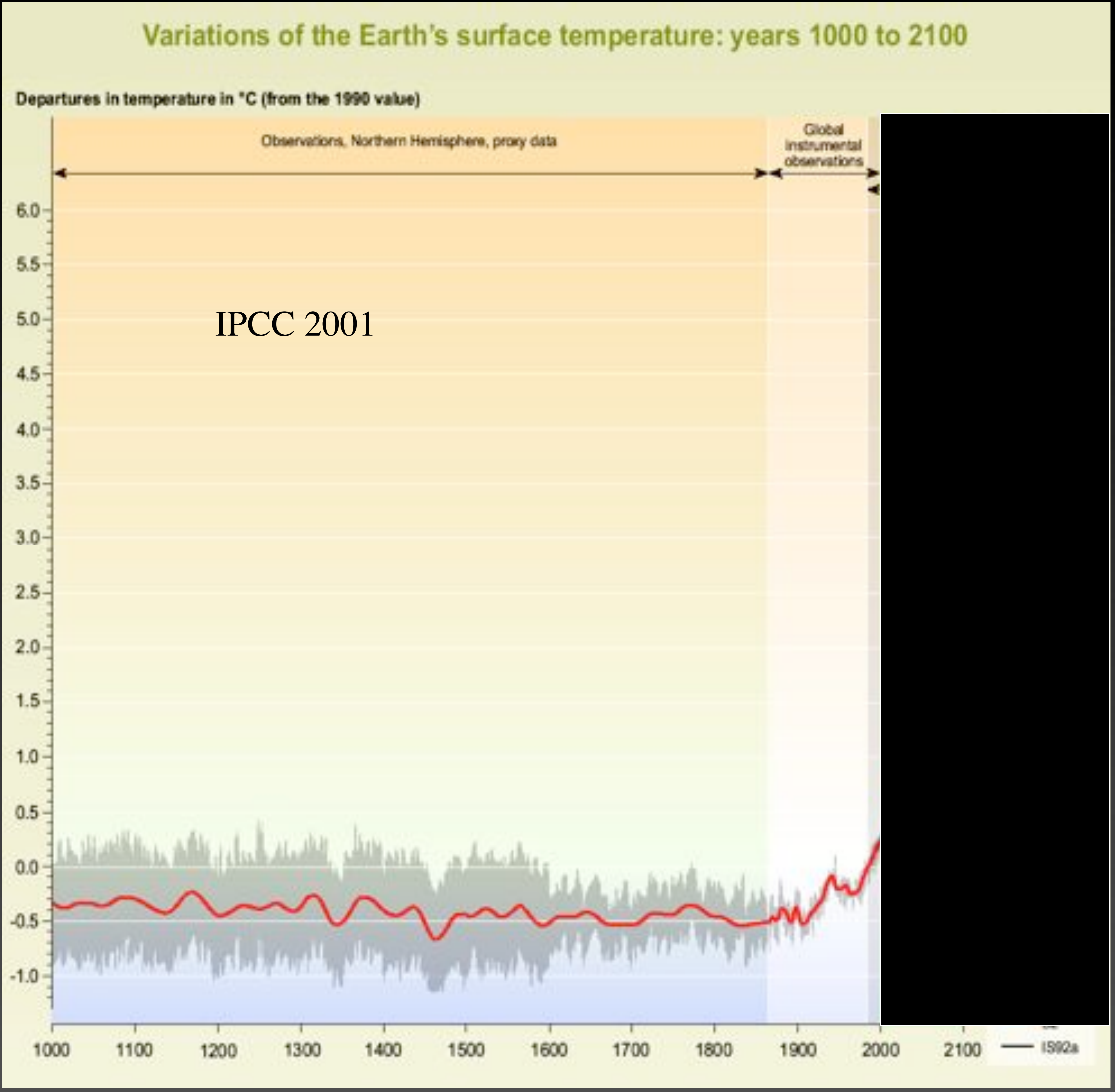
March 25, 2008

Outline

- Climate, Climate Change, and Weather
- Nonequilibrium Statistical Physics:
  - Hopf Functional Approach
  - Cumulant Expansions
  - Flow Equations
- Geophysical Fluid Dynamics:
  - Coriolis Force
  - Stratification
- Test Case: A Point Jet

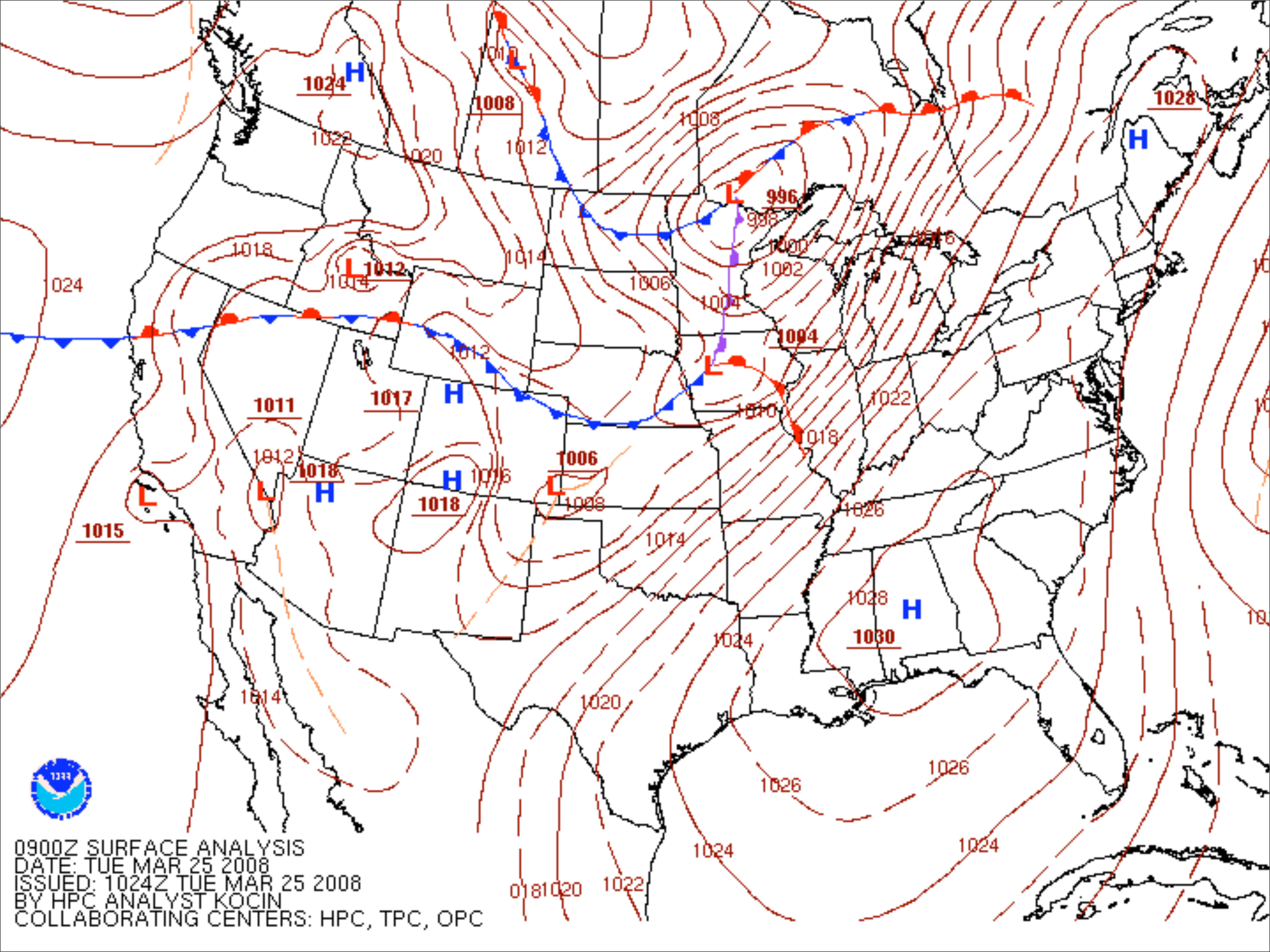


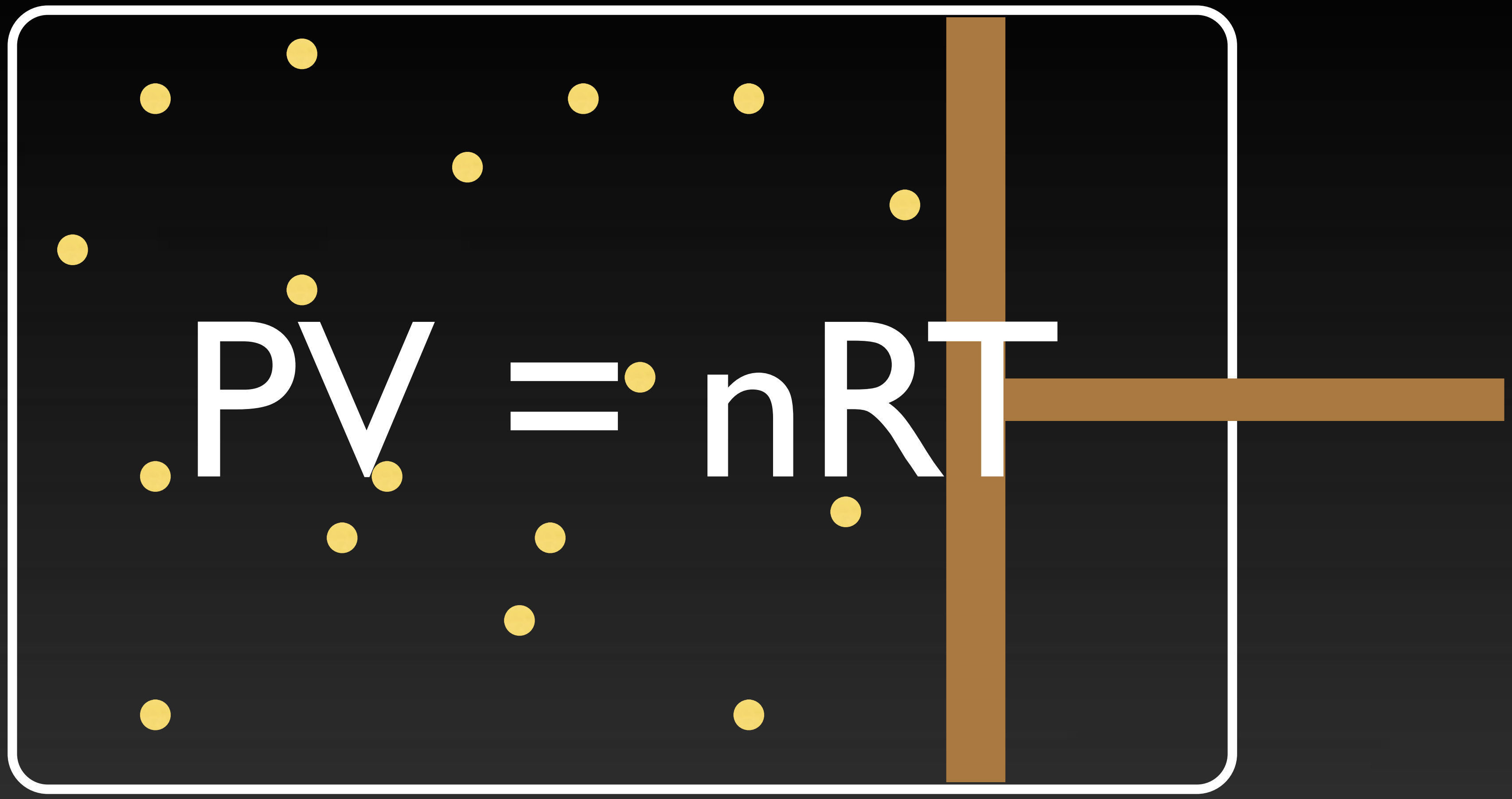
Annual Mean Temperature


Variations of the Earth's surface temperature: years 1000 to 2100
Departures in temperature in °C (from the 1990 value)
Observations, Northern Hemisphere, proxy data
Global instrumental observations
IPCC 2001
6.0
5.5
5.0
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
-0.5
-1.0
1000
1100
1200
1300
1400
1500
1600
1700
1800
1900
2000
2100
IS92a


"More than any other theoretical procedure, numerical integration is also subject to the criticism that it yields little insight into the problem. The computed numbers are not only processed like data but they look like data, and a study of them may be no more enlightening than a study of real meteorological observations. An alternative procedure which does not suffer this disadvantage consists of deriving a new system of equations whose unknowns are the statistics themselves. This procedure can be very effective for problems where the original equations are linear, but, in the case of non-linear equations, the new system will inevitably contain more unknowns than equations, and can therefore not be solved, unless additional postulates are introduced."

Edward Lorenz, The Nature and Theory of the General Circulation (1967)


0900Z SURFACE ANALYSIS
DATE: TUE MAR 25 2008
ISSUED: 1024Z TUE MAR 25 2008
BY HPC ANALYST KOCIN
COLLABORATING CENTERS: HPC, TPC, OPC



$$PV = nRT$$

Thermodynamics vs. Statistical Mechanics

Equilibrium vs. Out-of-Equilibrium

Hopf Functional Approach

U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov*

$$\frac{dx}{dt} = x^2 \qquad \Psi(t,u) \equiv e^{iux(t)}$$

$$i\frac{\partial}{\partial t}\Psi = u\frac{\partial^2}{\partial u^2}\Psi \qquad i\frac{\partial}{\partial t}\overline{\Psi} = u\frac{\partial^2}{\partial u^2}\overline{\Psi}$$

$$\begin{aligned}
\hat{H}\overline{\Psi}_0 &= 0 \\
\overline{\Psi}_0(u) &= \exp\{iu\langle x\rangle - \frac{1}{2!}u^2(\langle x^2\rangle - \langle x\rangle^2) + \ldots\} \\
\langle x\rangle &= -i\frac{\partial\overline{\Psi}_0(u)}{\partial u}\bigg|_{u=0}
\end{aligned}$$

General Equations of Motion

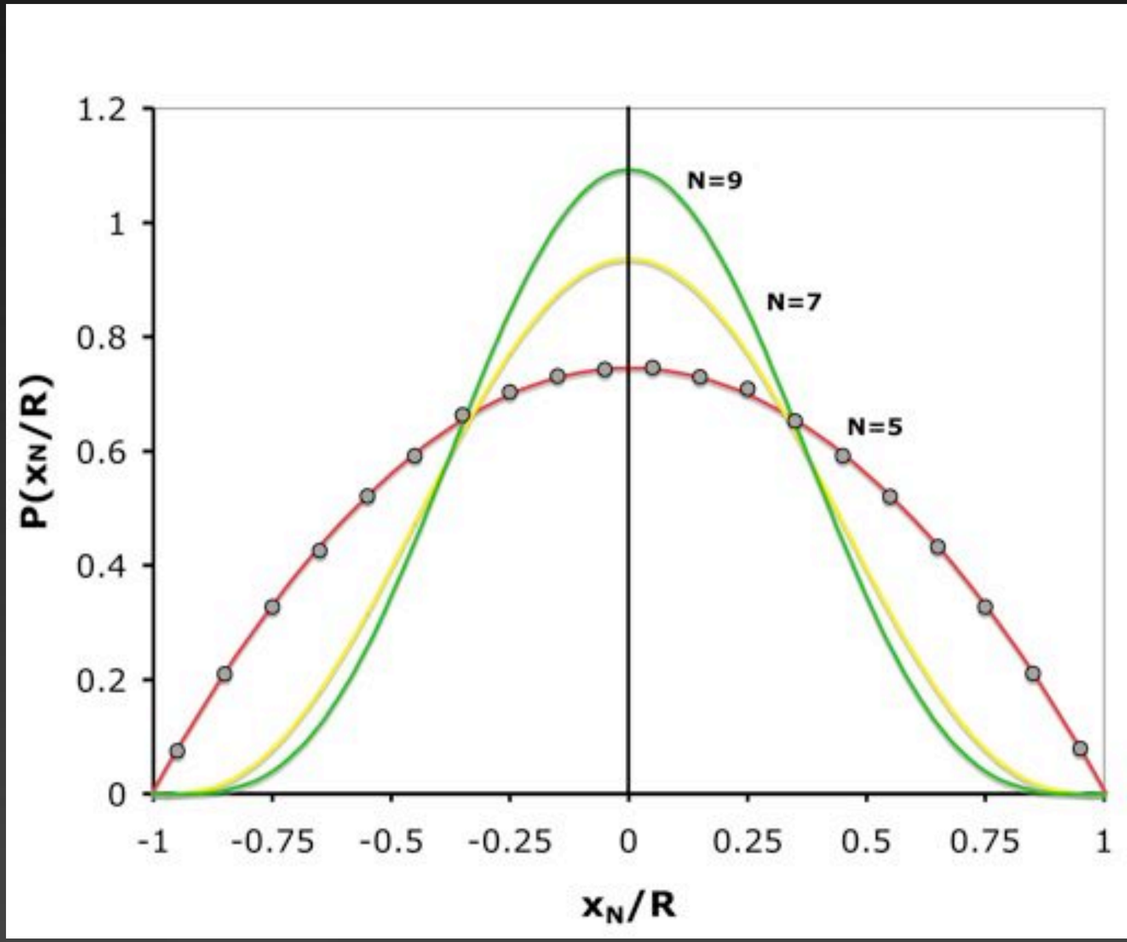
$$\frac{dx_i}{dt} = A_{ij}x_j + B_{ijk}x_jx_k$$

$$\hat{H} \quad = \quad iA_{ij}u_i\frac{\partial}{\partial u_j} + B_{ijk}u_i\frac{\partial^2}{\partial u_j \partial u_k}$$

Exact Solution For Orszag-McLaughlin Dynamics

$$\frac{dx_i}{dt} = x_{i+1}x_{i+2} + x_{i-1}x_{i-2} - 2x_{i+1}x_{i-1}$$

$$\begin{aligned} \Psi_0(\vec{u}) \quad &= \quad \frac{3\sqrt{\pi}}{4}\,(R|\vec{u}|/2)^{-3/2} \\ &\times \quad J_{3/2}(R|\vec{u}|) \end{aligned}$$



Ookie Ma and JBM: “Exact Equal Time Statistics of Orszag-McLaughlin Dynamics By The Hopf Characteristic Functional Approach,” J. Stat. Mech. Th. Exp. p10007 (2005)

Flow Equation Approach

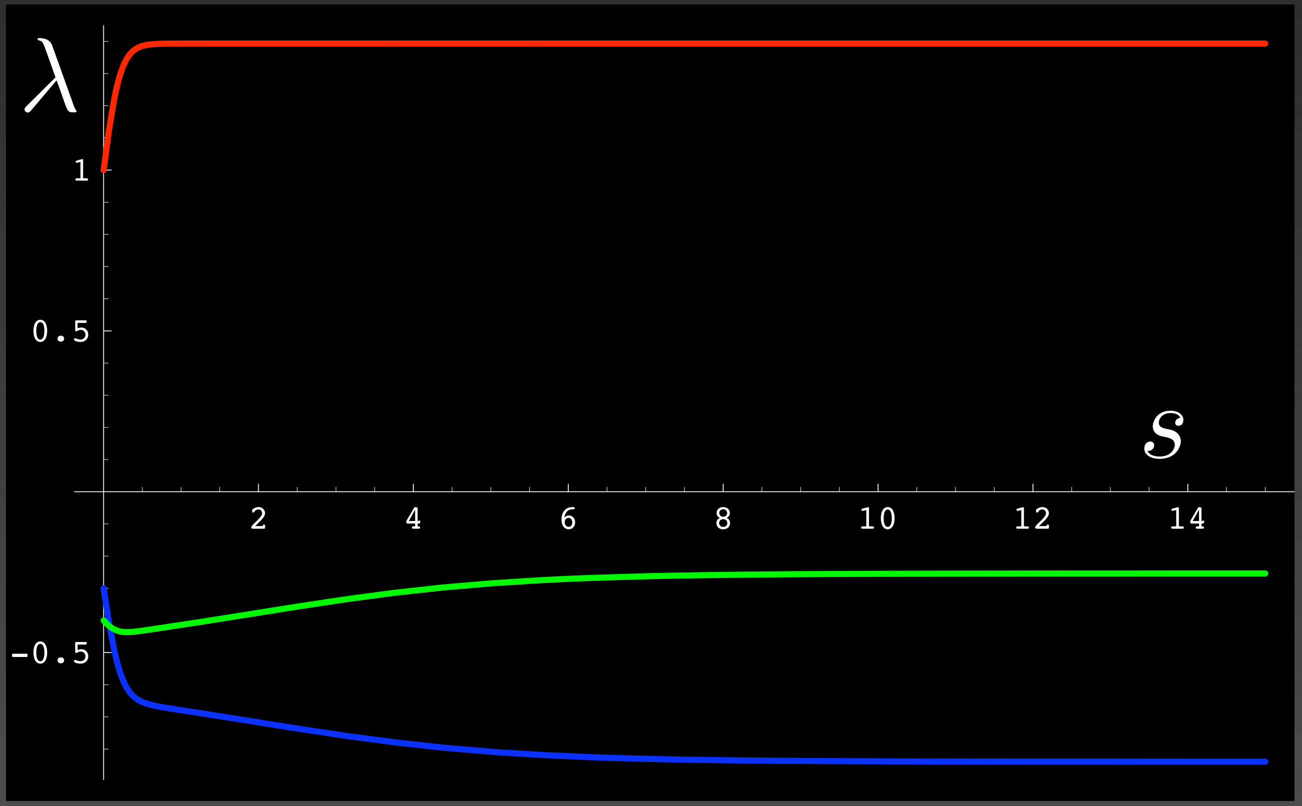
S. D. Glazek and K. G. Wilson, PRD **48**, 5863 (1993); F. Wegner, Ann. Phys. **3**, 77 (1994).

$$\hat{H}(s) = \hat{H}_0(s) + \hat{H}'(s)$$

$$\hat{G}(s) = [\hat{H}_0(s),\ \hat{H}(s)]$$

$$\frac{d}{ds}\,\hat{H}(s) = [\hat{G}(s),\ \hat{H}(s)] \qquad \frac{d}{ds}\,\hat{O}(s) = [\hat{G}(s),\ \hat{O}(s)]$$

$$H(0) = \begin{pmatrix} 1.0 & 0.9 & 0.5 \\ 0.7 & -0.3 & 0.0 \\ 0.2 & -0.3 & -0.4 \end{pmatrix}$$



$$H_0(s) = \begin{pmatrix} \lambda_1(s) & 0 & 0 \\ 0 & \lambda_2(s) & 0 \\ 0 & 0 & \lambda_3(s) \end{pmatrix}$$

Closure

$$\left[u_i \frac{\partial^2}{\partial u_j \partial u_k},\ u_\ell \frac{\partial^2}{\partial u_m \partial u_n} \right] = \delta_{j\ell}\ u_i \frac{\partial^3}{\partial u_k \partial u_m \partial u_n} + \ldots$$

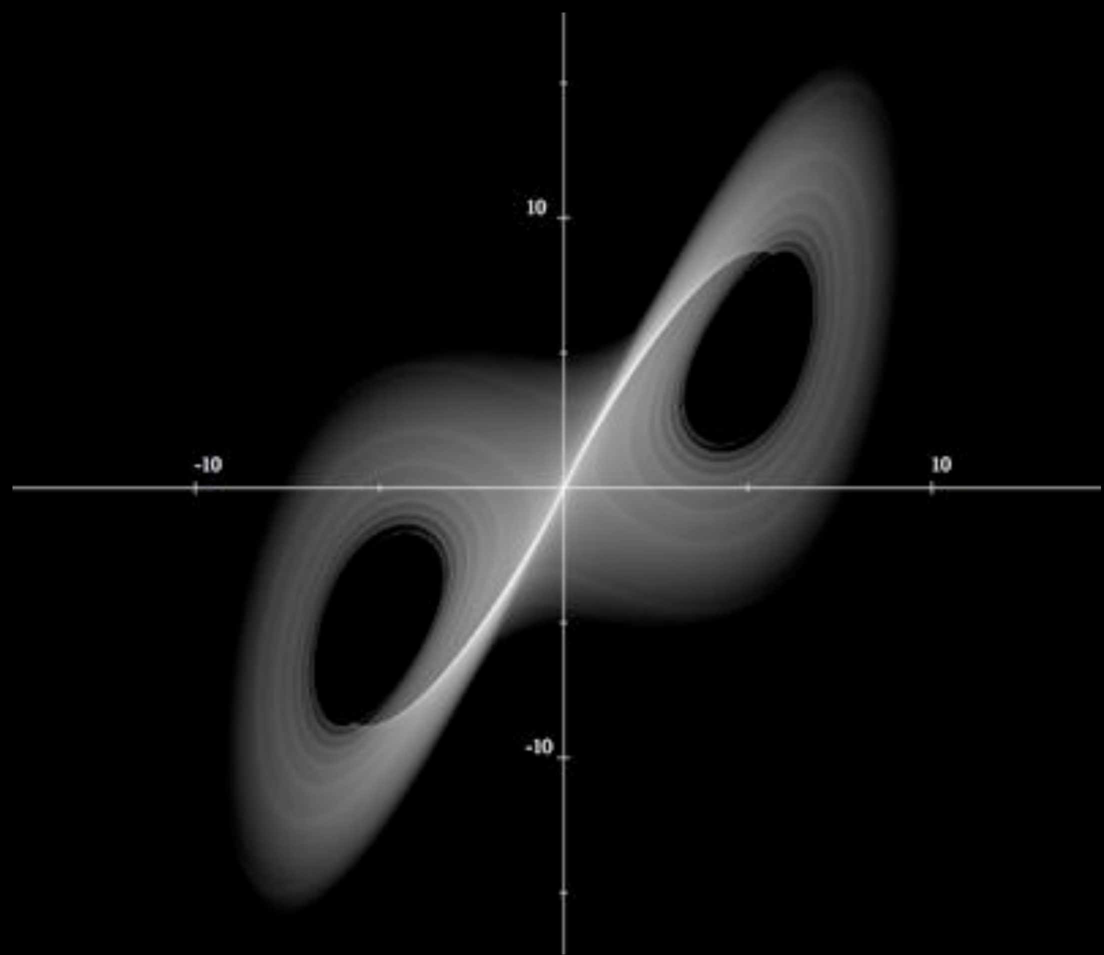
more derivatives w.r.t. u

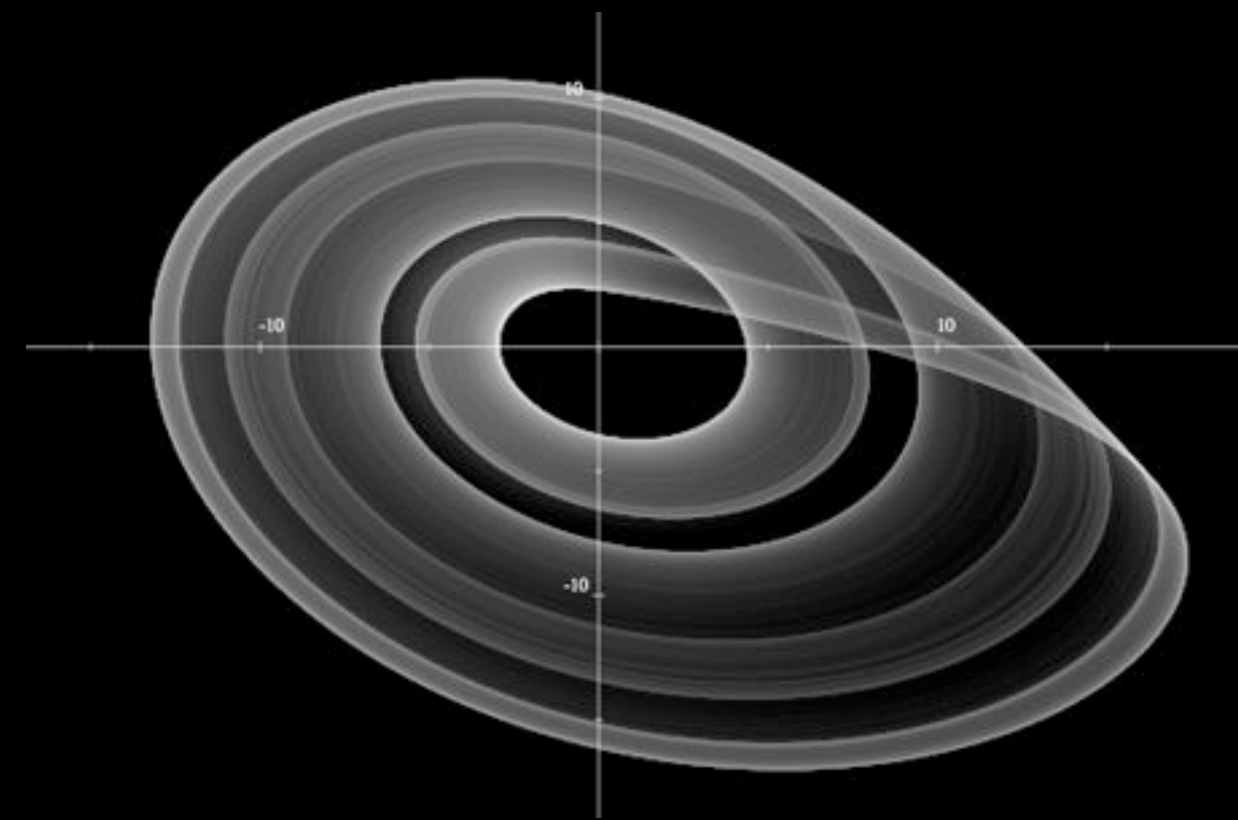
more powers of u

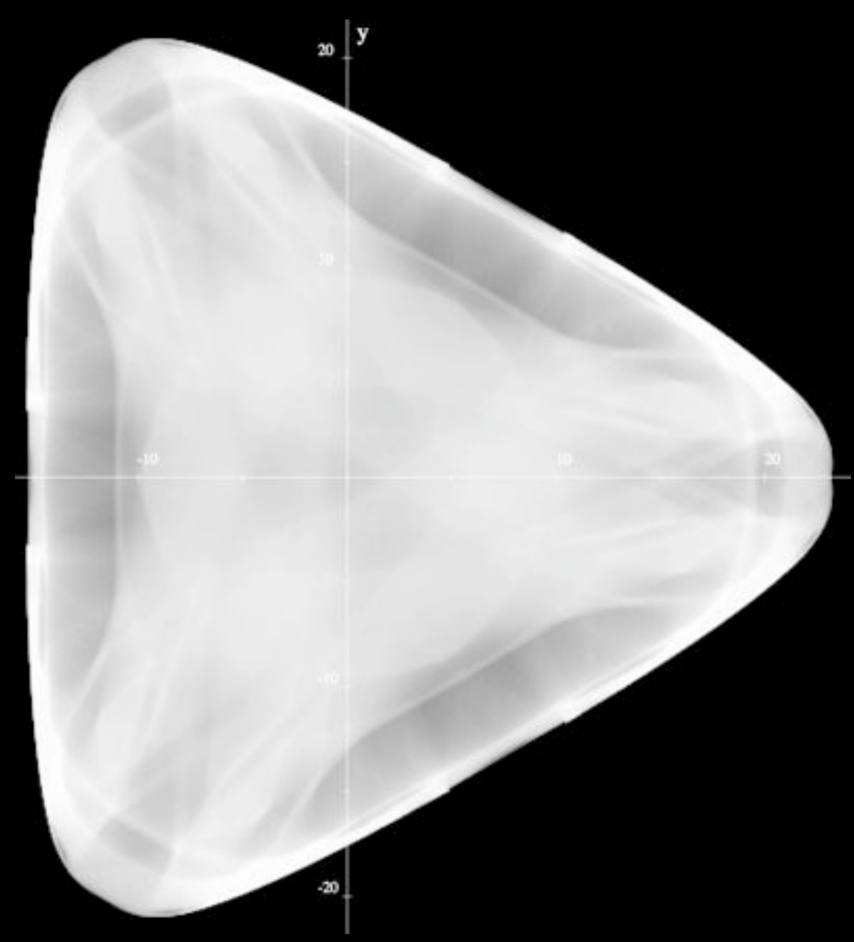
$$\begin{aligned}
\hat{A} \quad = \quad & a + \boxed{a_i \frac{\partial}{\partial u_i} + a_{ij} \frac{\partial^2}{\partial u_i \partial u_j}} + \boxed{a_{ijk} \frac{\partial^3}{\partial u_i \partial u_j \partial u_k}} + \ldots \\
+ \quad & a_i^{(1)} u_i + \boxed{a_{ij}^{(1)} u_i \frac{\partial}{\partial u_j} + a_{ijk}^{(1)} u_i \frac{\partial^2}{\partial u_j \partial u_k}} + \ldots \\
+ \quad & a_{ij}^{(2)} u_i u_j + \boxed{a_{ijk}^{(2)} u_i u_j \frac{\partial}{\partial u_k}} + \ldots \\
+ \quad & \boxed{a_{ijk}^{(3)} u_i u_j u_k} + \ldots \\
+ \quad & \ldots
\end{aligned}$$

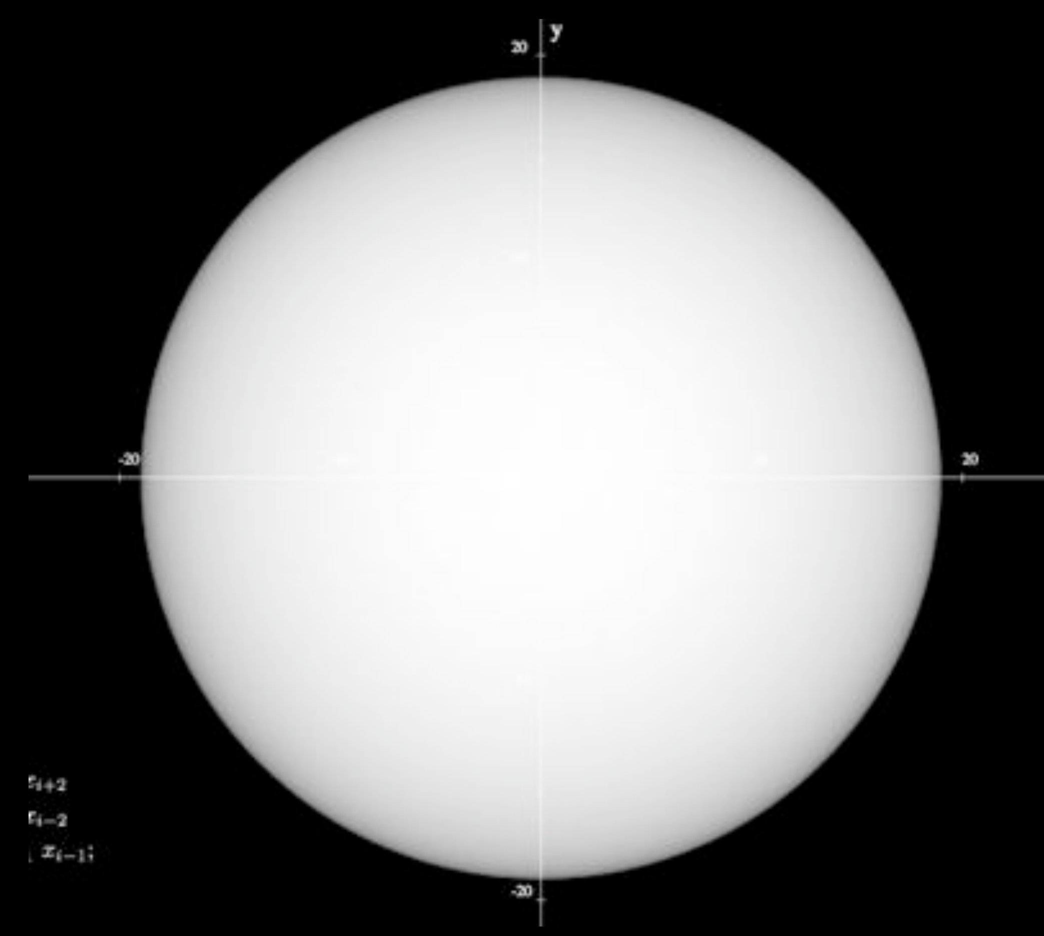
Direct Numerical Simulations vs. Statistical Approaches

Observable	DNS	Cumulant	Hopf / Flow
$\langle z \rangle$	22.7	23.4	22.4
$\langle y^2 \rangle$	42.6	32.9	40.2

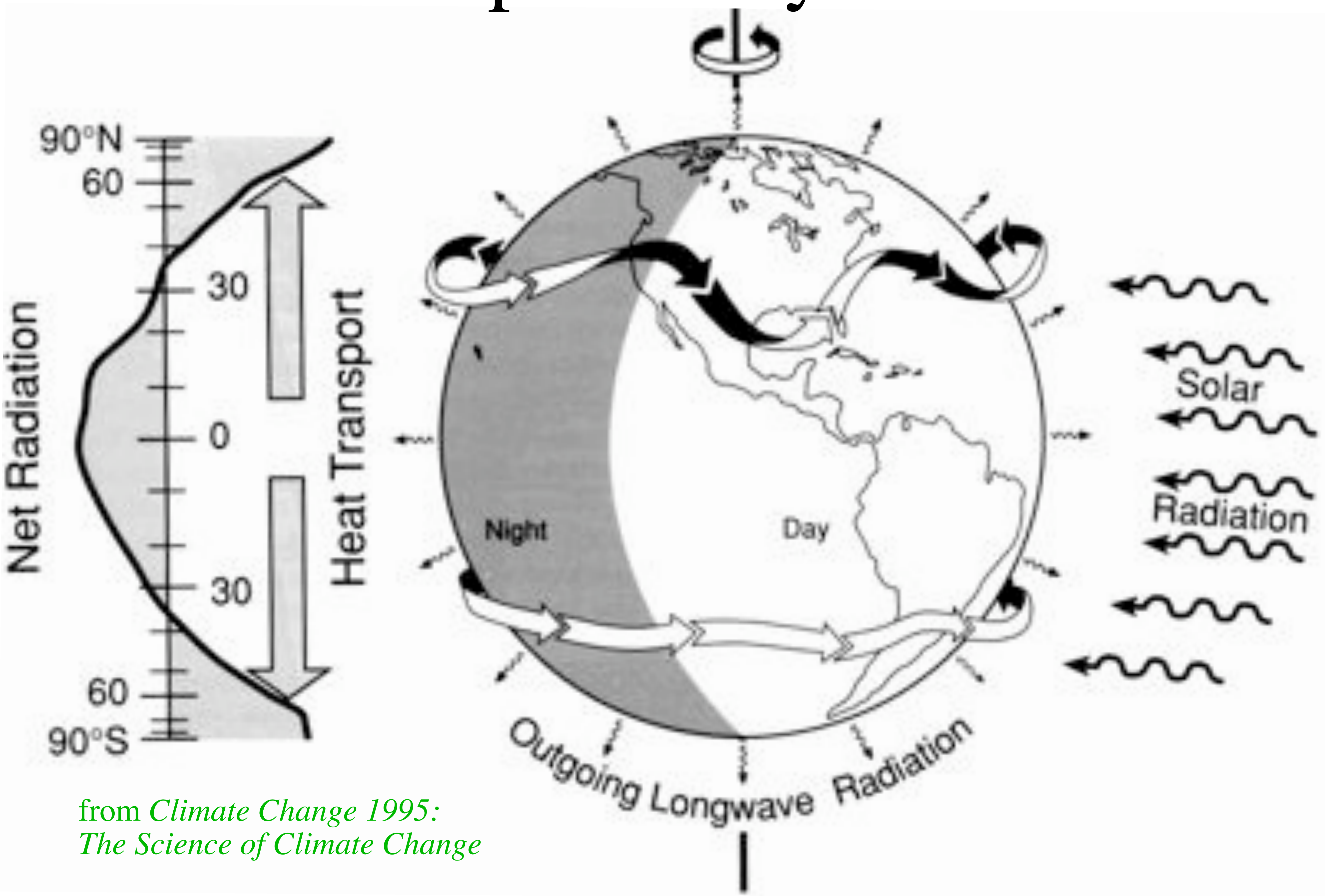

10
-10
-10
10


10
-10
10
-10


y
20
-20


y
20
-20
20
-20

Atmospheric Dynamics



from *Climate Change 1995: The Science of Climate Change*

Single Layer Models

$$\frac{D\omega}{Dt} = 0$$

Vorticity

$$\omega = \hat{r} \cdot (\vec{\nabla} \times \vec{v})$$

$$\frac{\partial \omega}{\partial t} + \vec{v} \cdot \vec{\nabla} \omega = 0$$

$$\frac{\partial \omega}{\partial t} + J(\psi, \omega) = 0$$

$$\begin{aligned} \vec{v} &= \hat{r} \times \vec{\nabla} \psi \\ \vec{\nabla} \cdot \vec{v} &= 0 \\ \omega &= \nabla^2 \psi \end{aligned}$$

$$J(\psi, \omega) \equiv \frac{\partial \psi}{\partial x} \frac{\partial \omega}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \omega}{\partial x}$$

Freely Decaying Turbulence on Sphere

Coriolis Force

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0$$

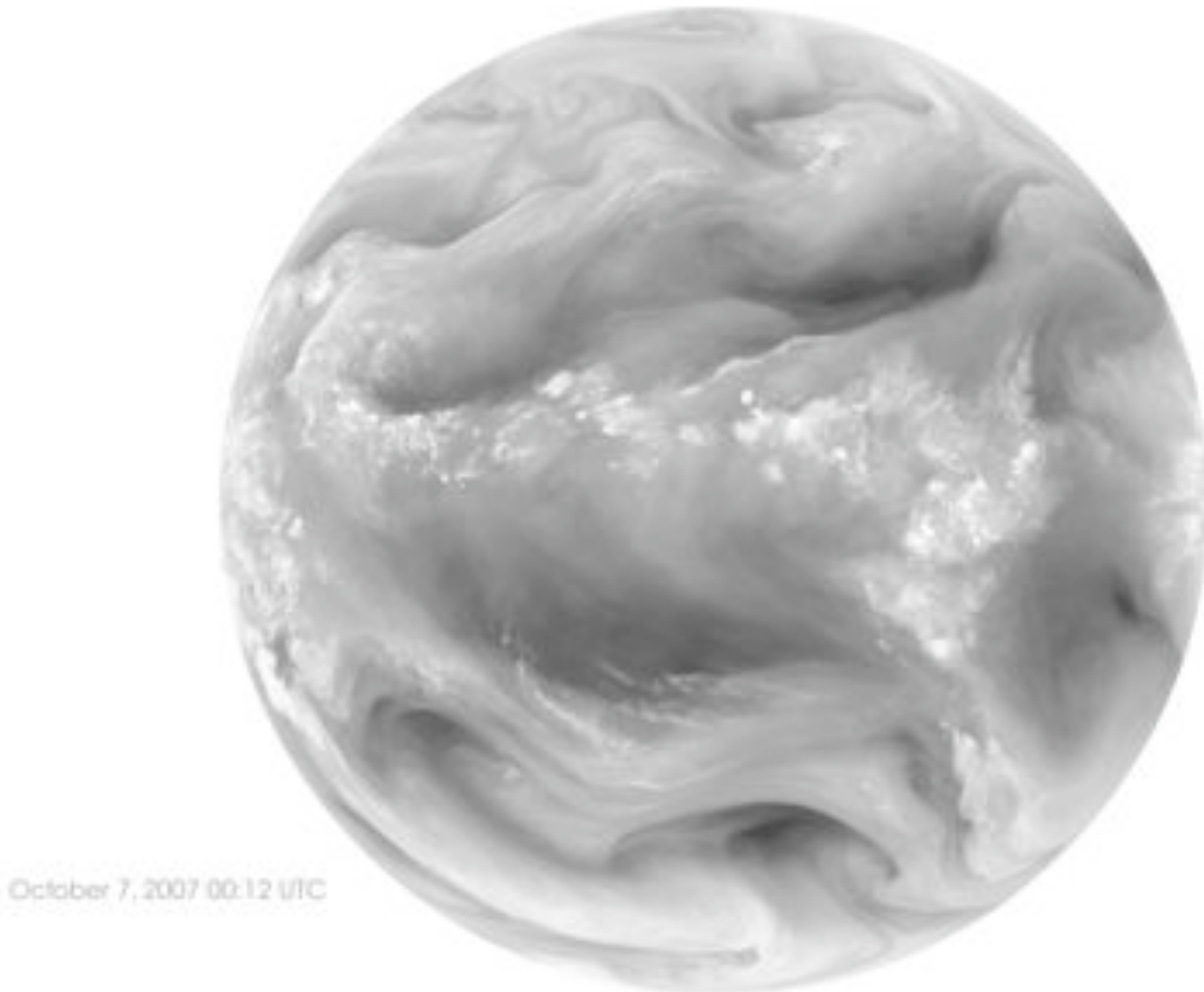
Relative vorticity

$$\begin{aligned} q &= \omega + f \\ &= \nabla^2 \psi + f \end{aligned}$$

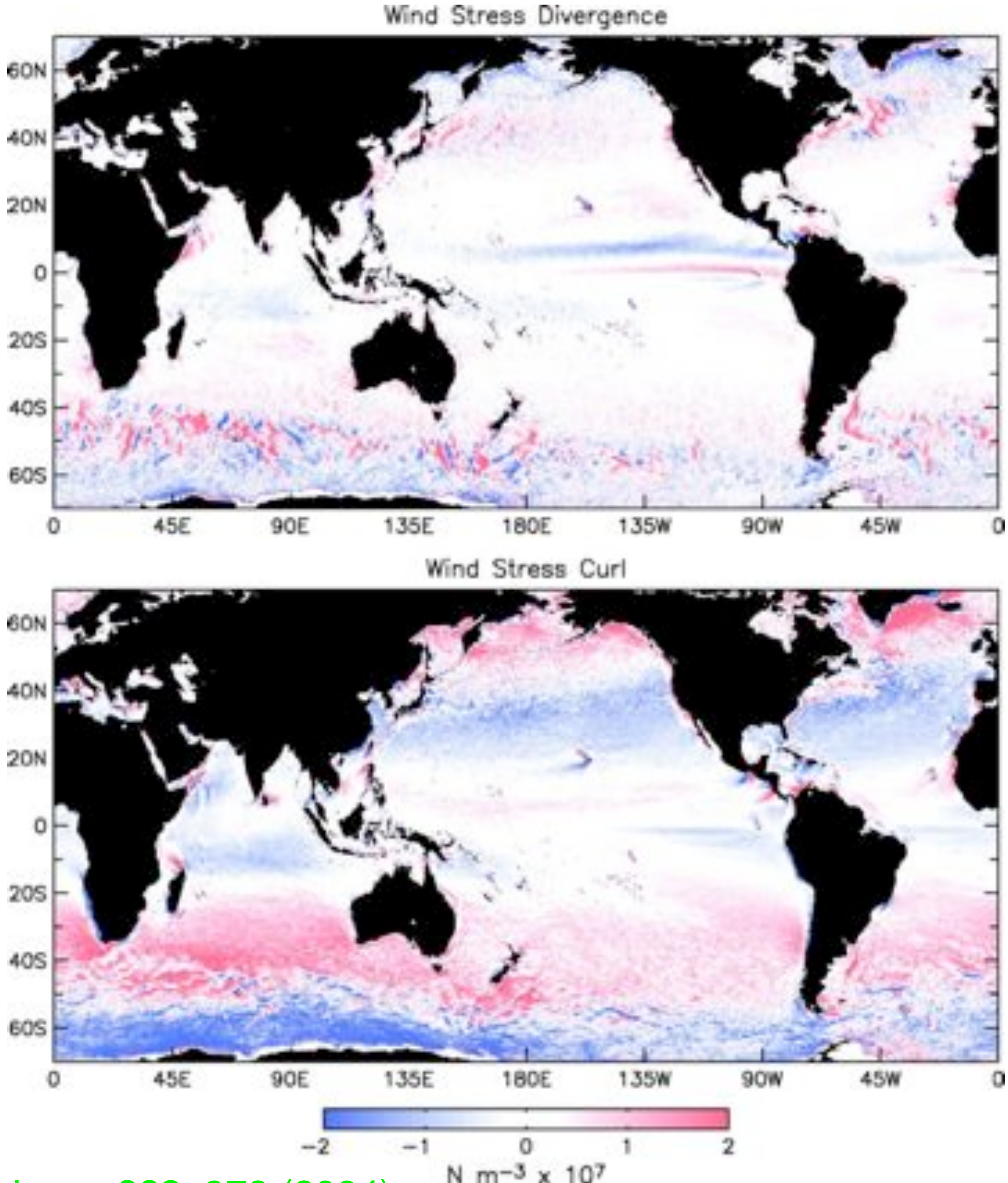
Coriolis term

Absolute vorticity

$$f = 2\Omega \sin(\phi)$$


October 7, 2007 00:12 UTC

Coriolis Force

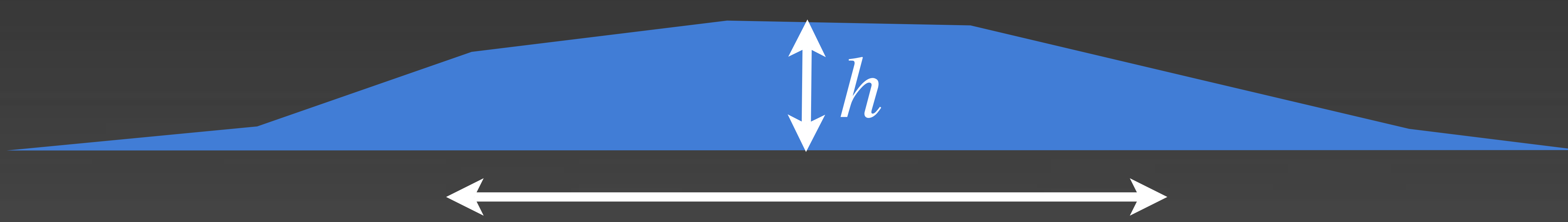


Chelton *et al.*, Science **303**, 978 (2004)

Stratification

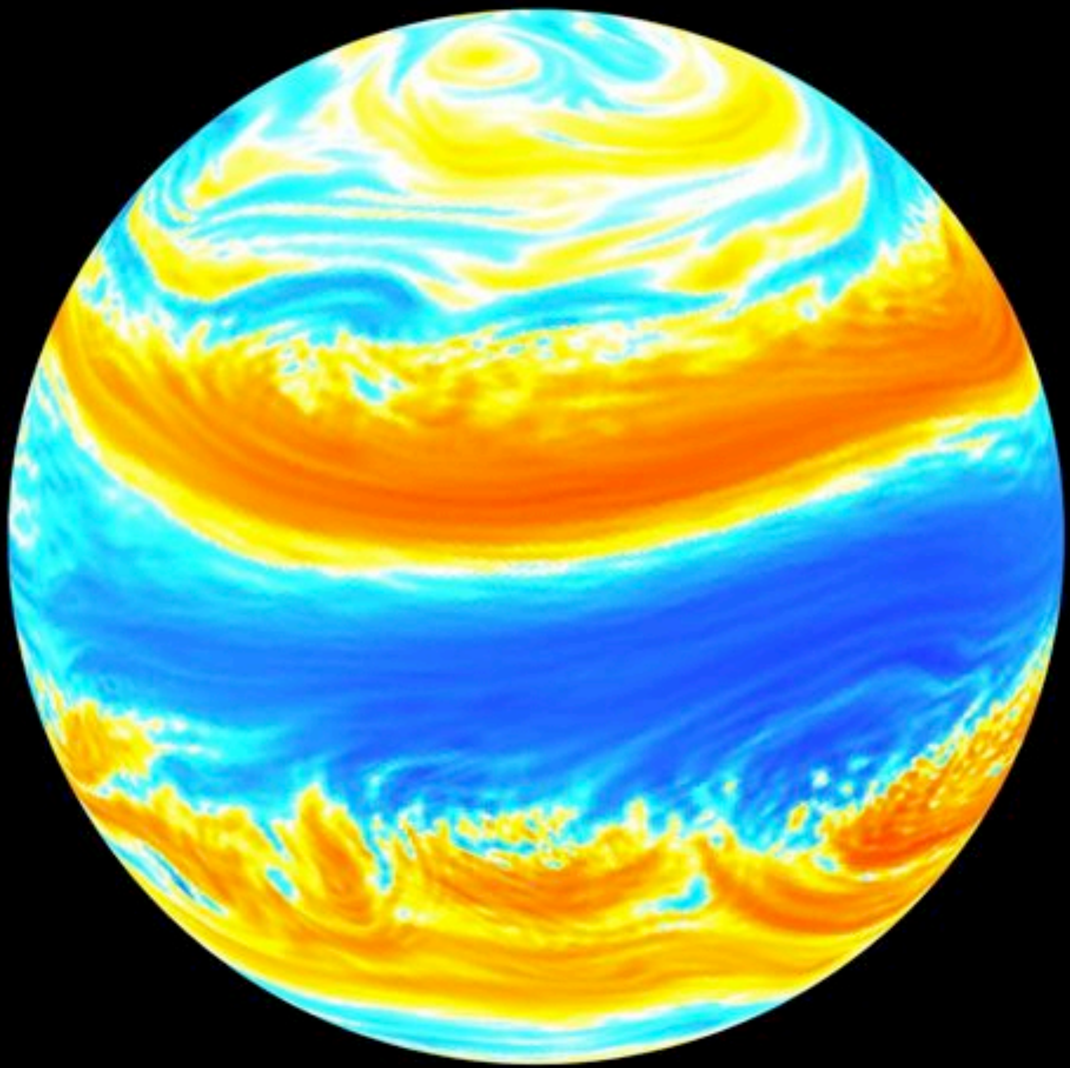
$$q = \nabla^2 \psi + f - \frac{\psi}{\ell_R^2}$$

$$\ell_R^2 = \frac{gh}{f^2}$$

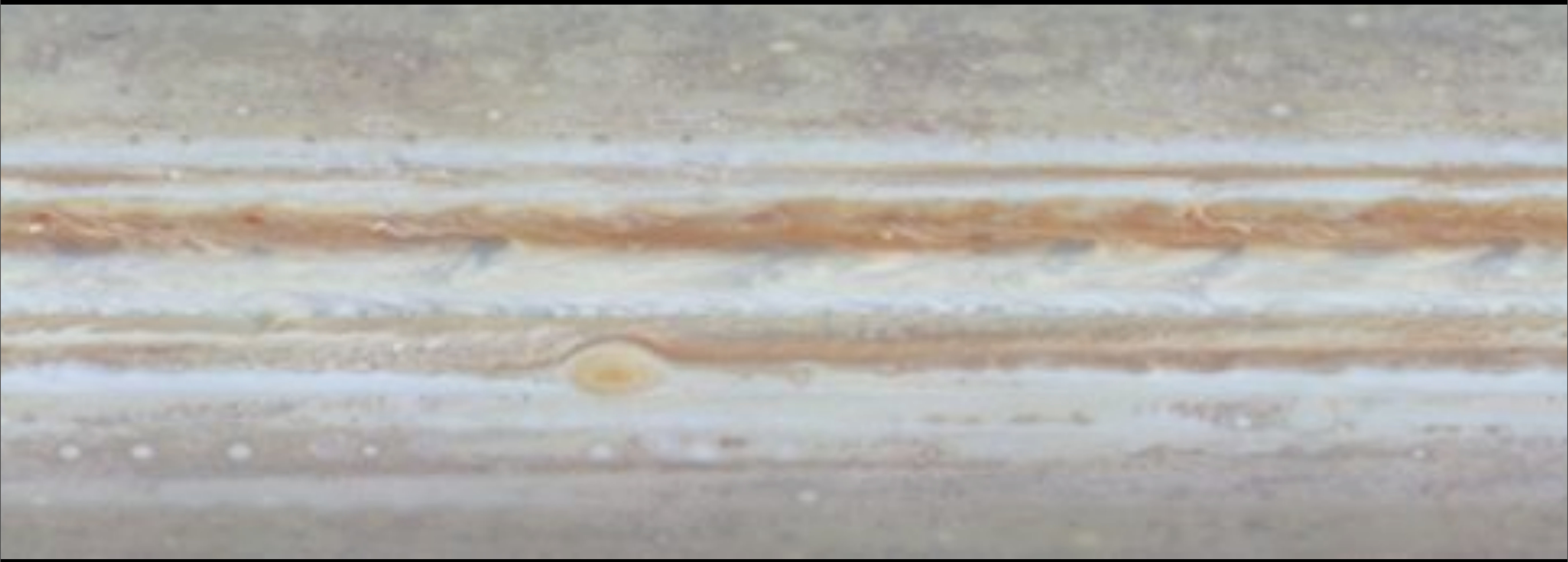


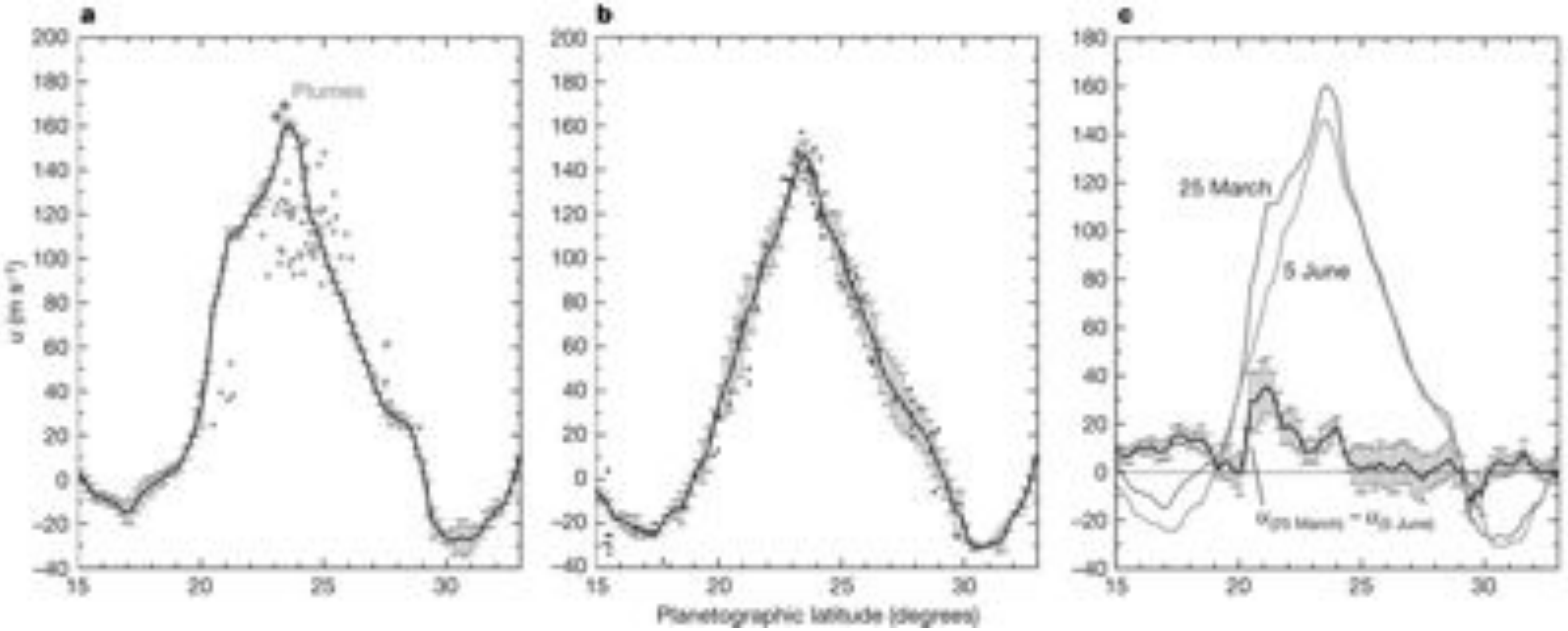
$$\ell_R = \mathcal{O}(1,000 \text{ km})$$

Stratification Sets Synoptic Length Scale

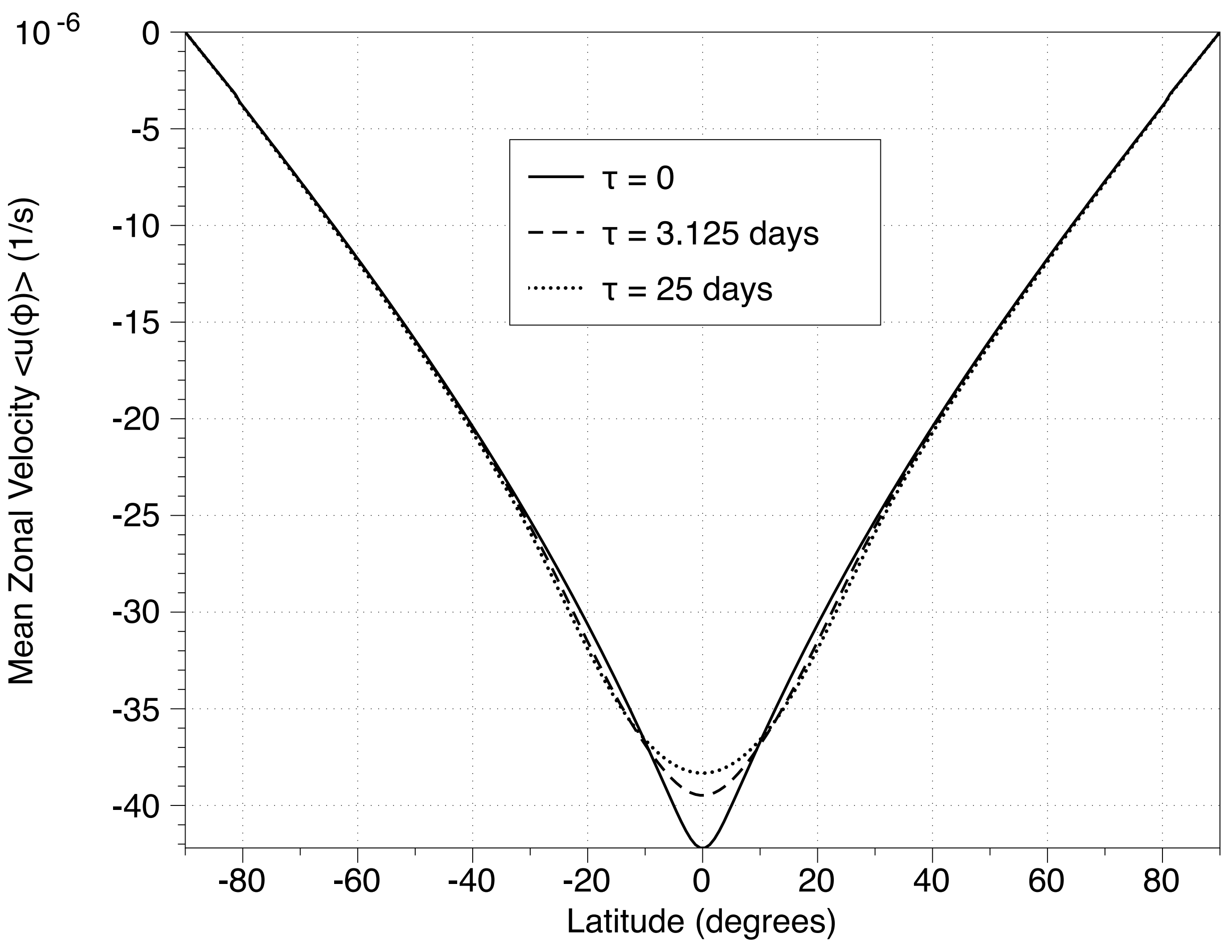


Test Case: A Point Jet





A. Sanchez-Lavega *et al.* Nature **451**, 437 (2008)

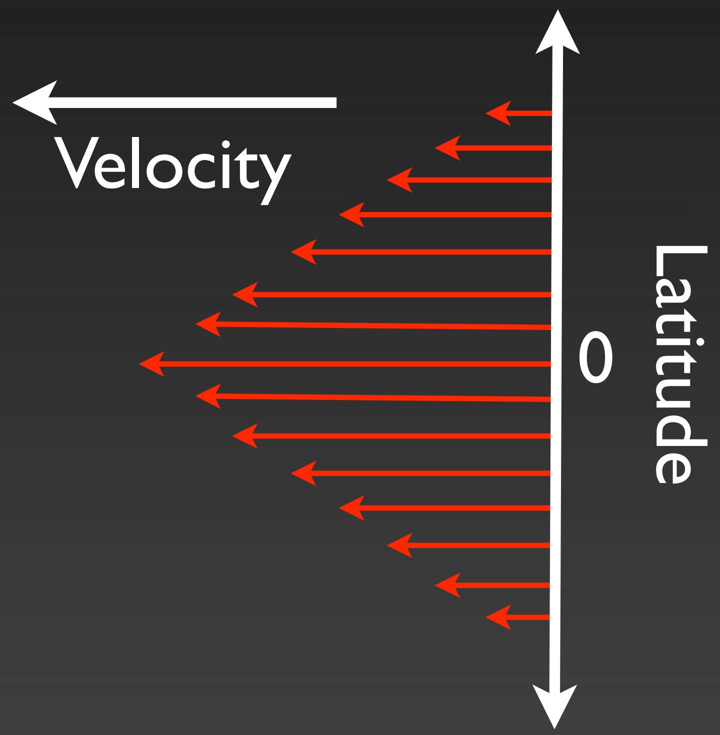


$$\frac{\partial q}{\partial t} + J[\psi, q] = \frac{q_{\mathrm{jet}} - q}{\tau}$$

Forced-Damped Equatorial Barotropic Jet

$$\frac{\partial \omega}{\partial t} + \vec{v} \cdot \vec{\nabla}(\omega + f) = \boxed{(\omega_0 - \omega)/\tau}$$

$$\omega_0 = -0.6\Omega \ \mathrm{sgn}(\phi)$$



"Point" jet

Schoeberl and Lindzen, J. Atmos. Sci. **41**, 1368 (1984); Shepherd, J. Fluid. Mech. **196**, 91 (1988)

Limiting Cases With No Fluctuations

$$\frac{\partial \omega}{\partial t} + \vec{v} \cdot \vec{\nabla}(\omega + f) = (\omega_0 - \omega)/\tau$$

$$\tau \to 0 : \quad \omega \to \omega_0$$

$$\tau \to \infty : \quad \omega \to \text{equilibrium jet}$$

[Turkington *et al.*, PNAS**99**, 12346 (2001); Weichman, PRE**73**, 036313 (2006)]

Largest fluctuations at intermediate relaxation times

Cumulant Expansion

$$\langle \omega(\phi, \lambda) \rangle = c_1(\phi)$$ Azimuthal symmetry

$$\langle \omega(\phi, \lambda)\ \omega(\phi', \lambda') \rangle_C = c_2(\phi, \phi', \lambda - \lambda')$$

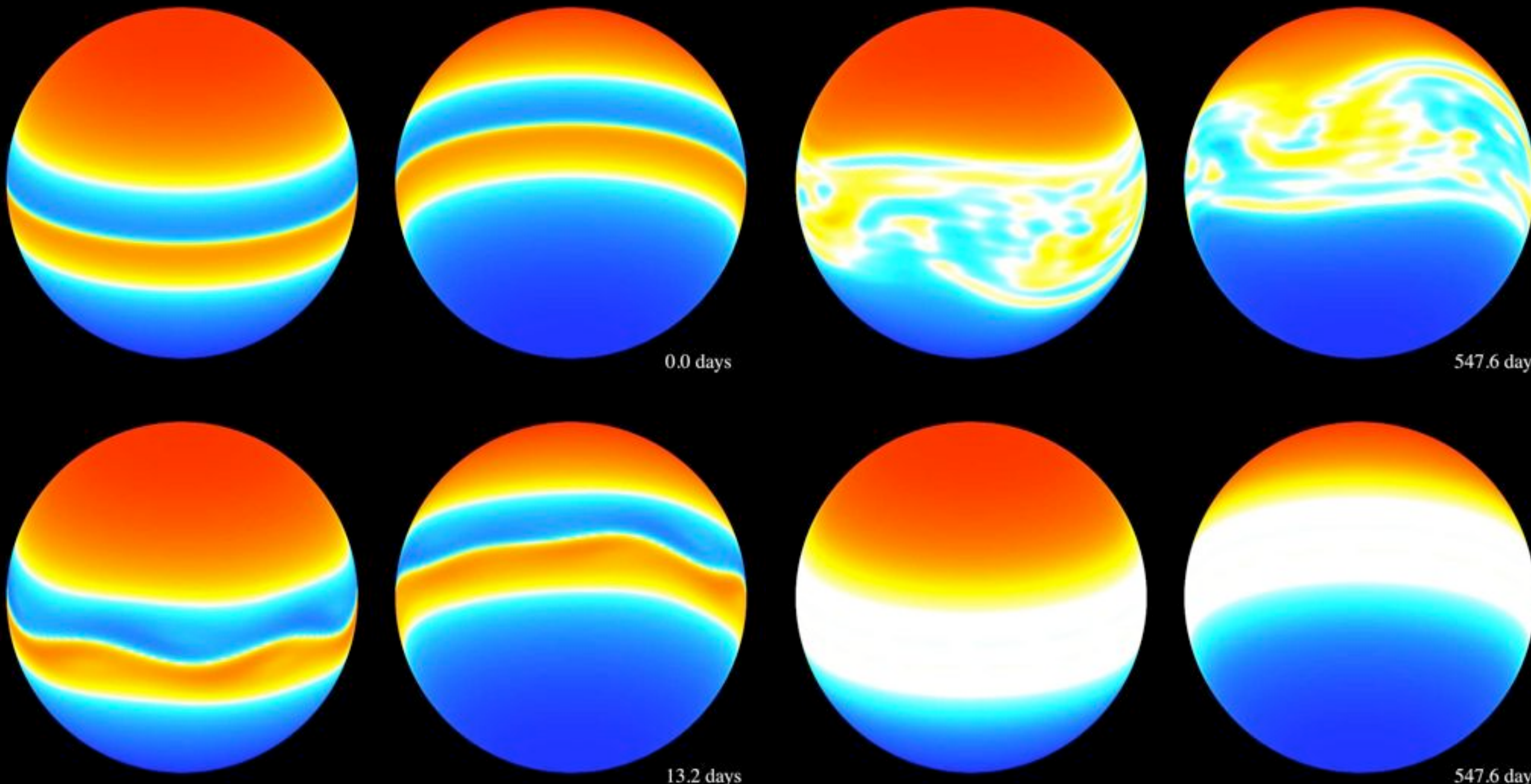
$$\langle \omega\ \omega' \rangle_C \equiv \langle \omega\ \omega' \rangle - \langle \omega \rangle \langle \omega' \rangle$$

$$\langle \omega\ \omega'\ \omega'' \rangle_C = 0, \text{ etc.}$$ Closure

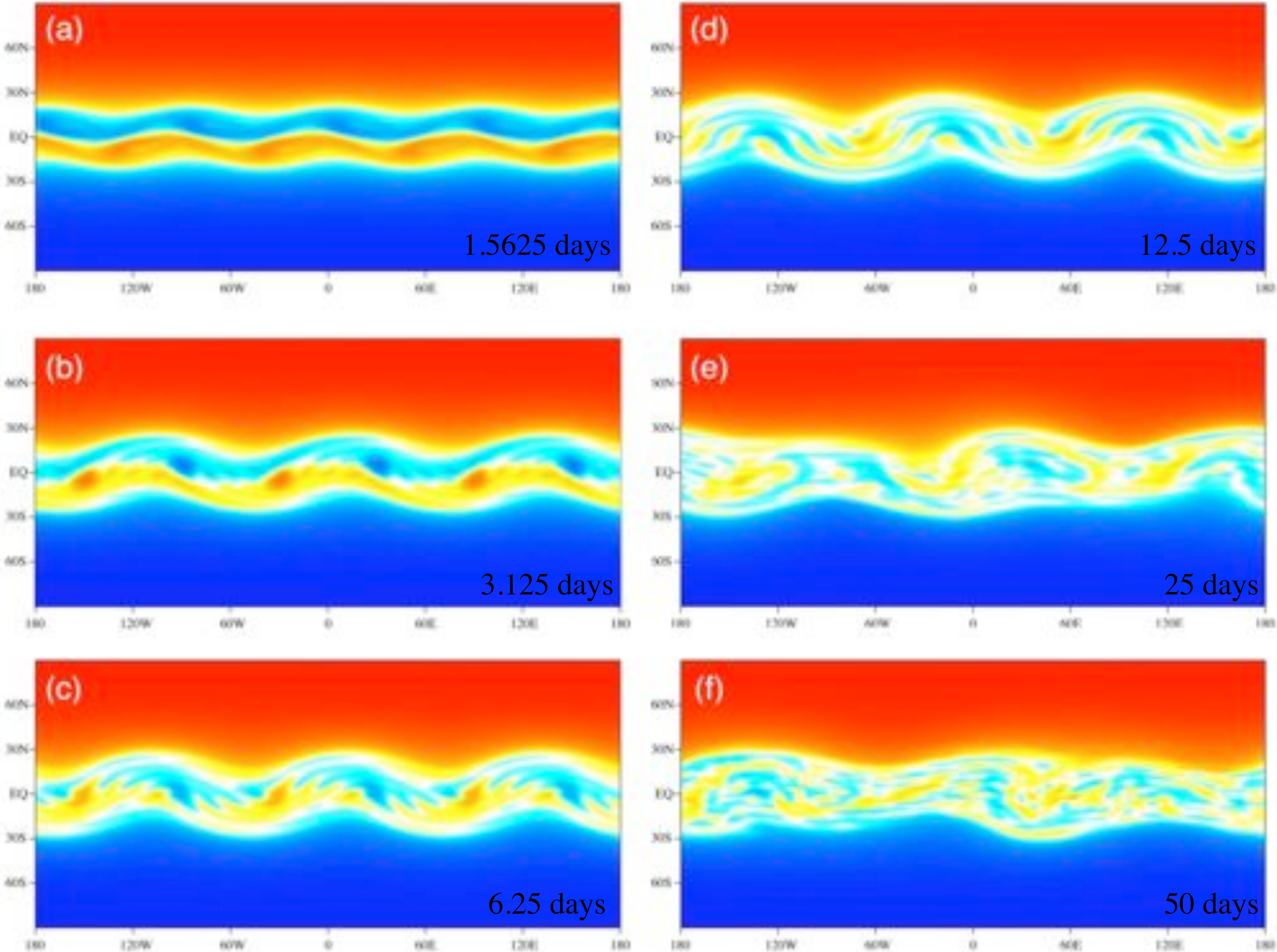
$$\langle \omega \omega \rangle_C \geq 0$$ Positivity

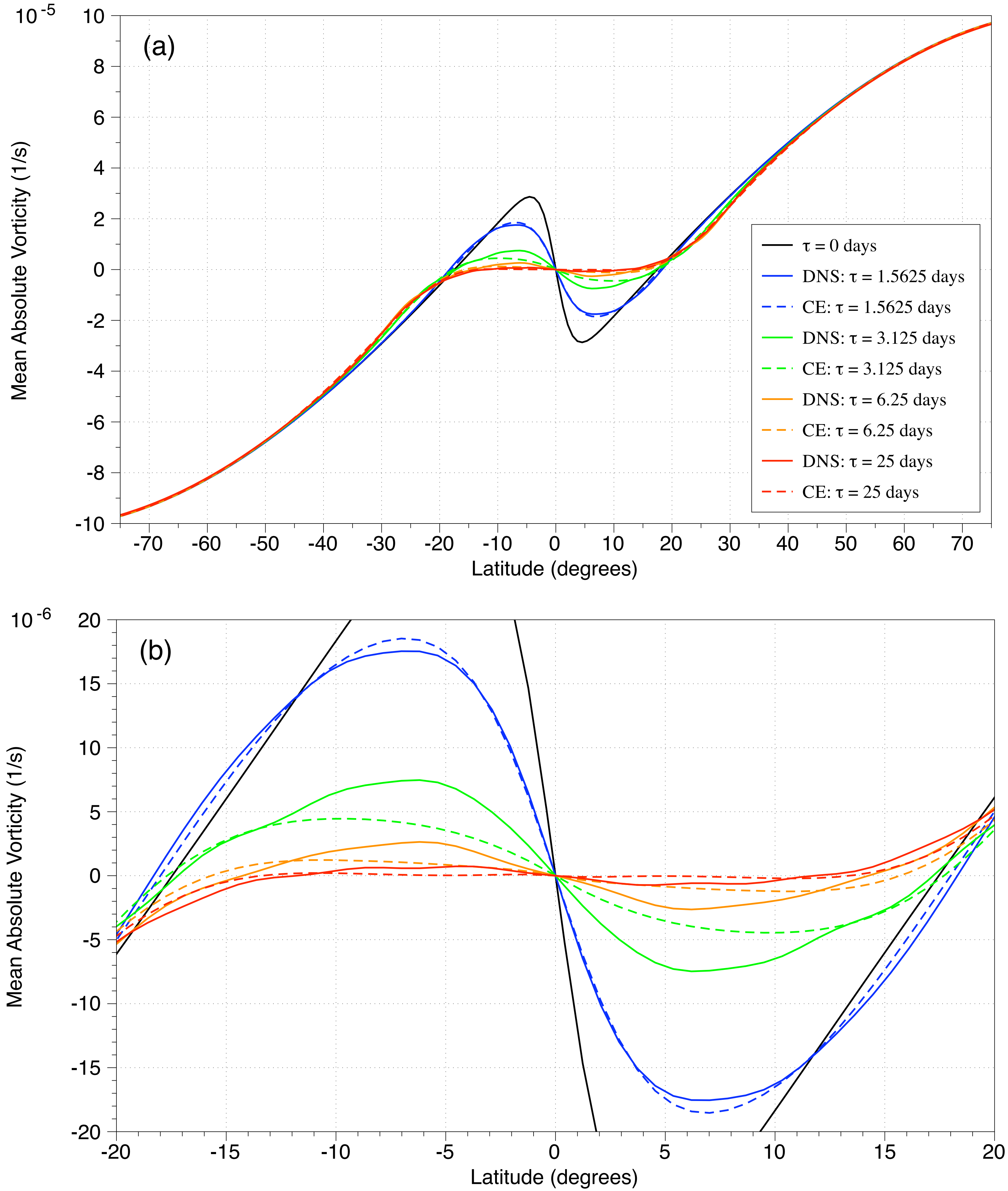
Direct Numerical Simulation of Jet

jet relaxation time = 25 days

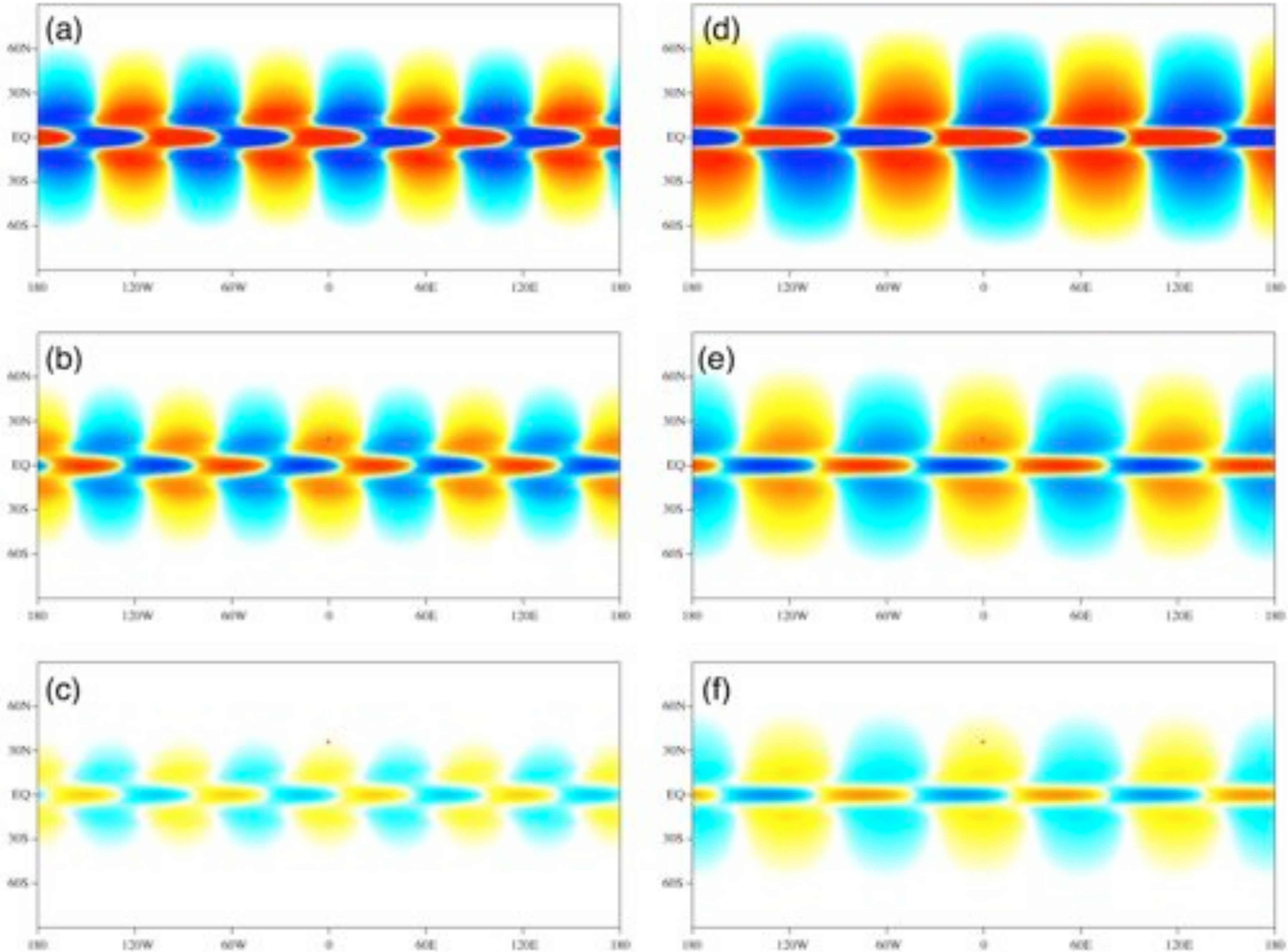


JBM, E. Conover, and T. Schneider, arXiv:0705.0011, J. Atmos. Sci. (in press)

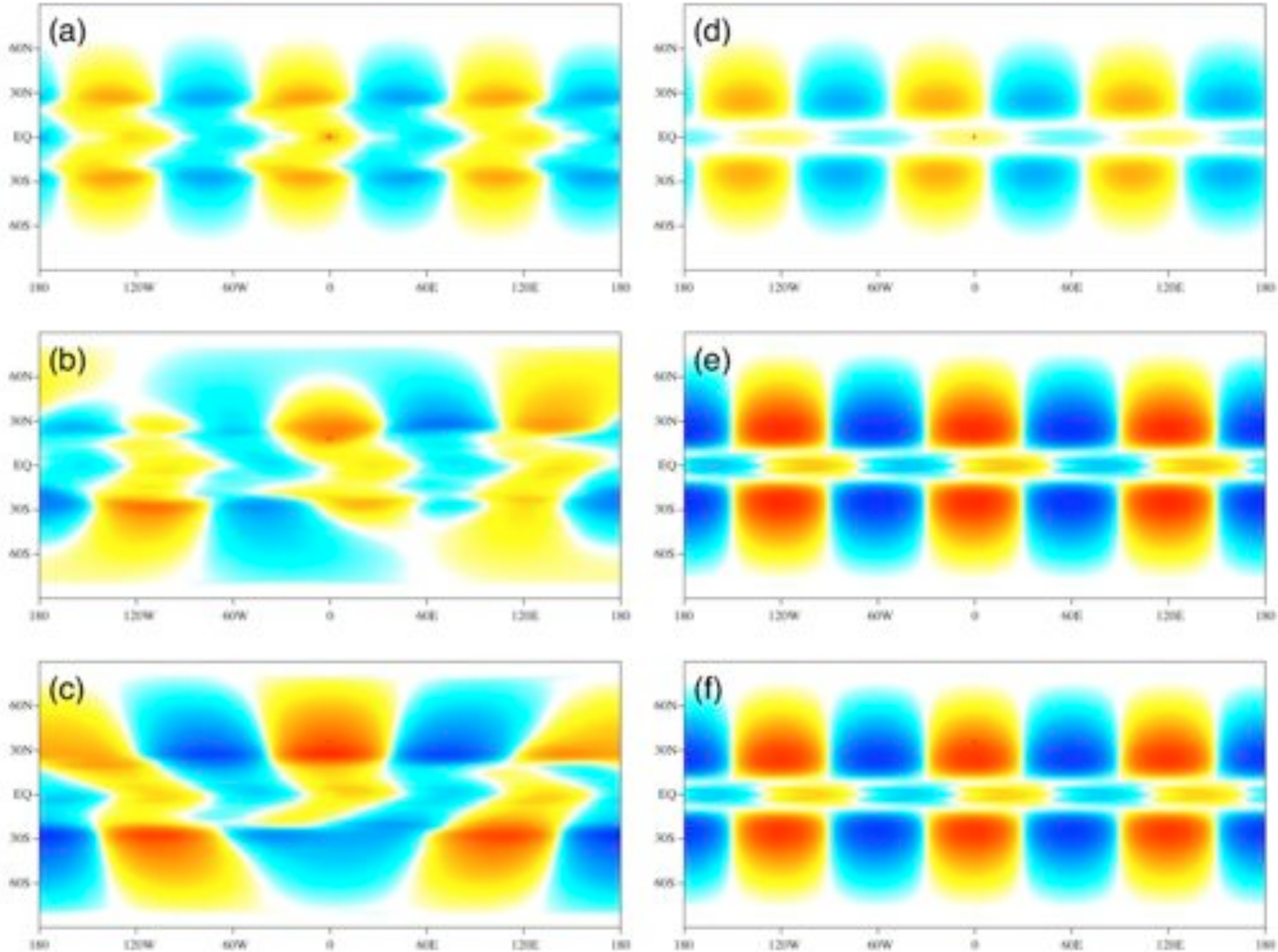

(a)
1.5625 days
(b)
3.125 days
(c)
6.25 days
(d)
12.5 days
(e)
25 days
(f)
50 days

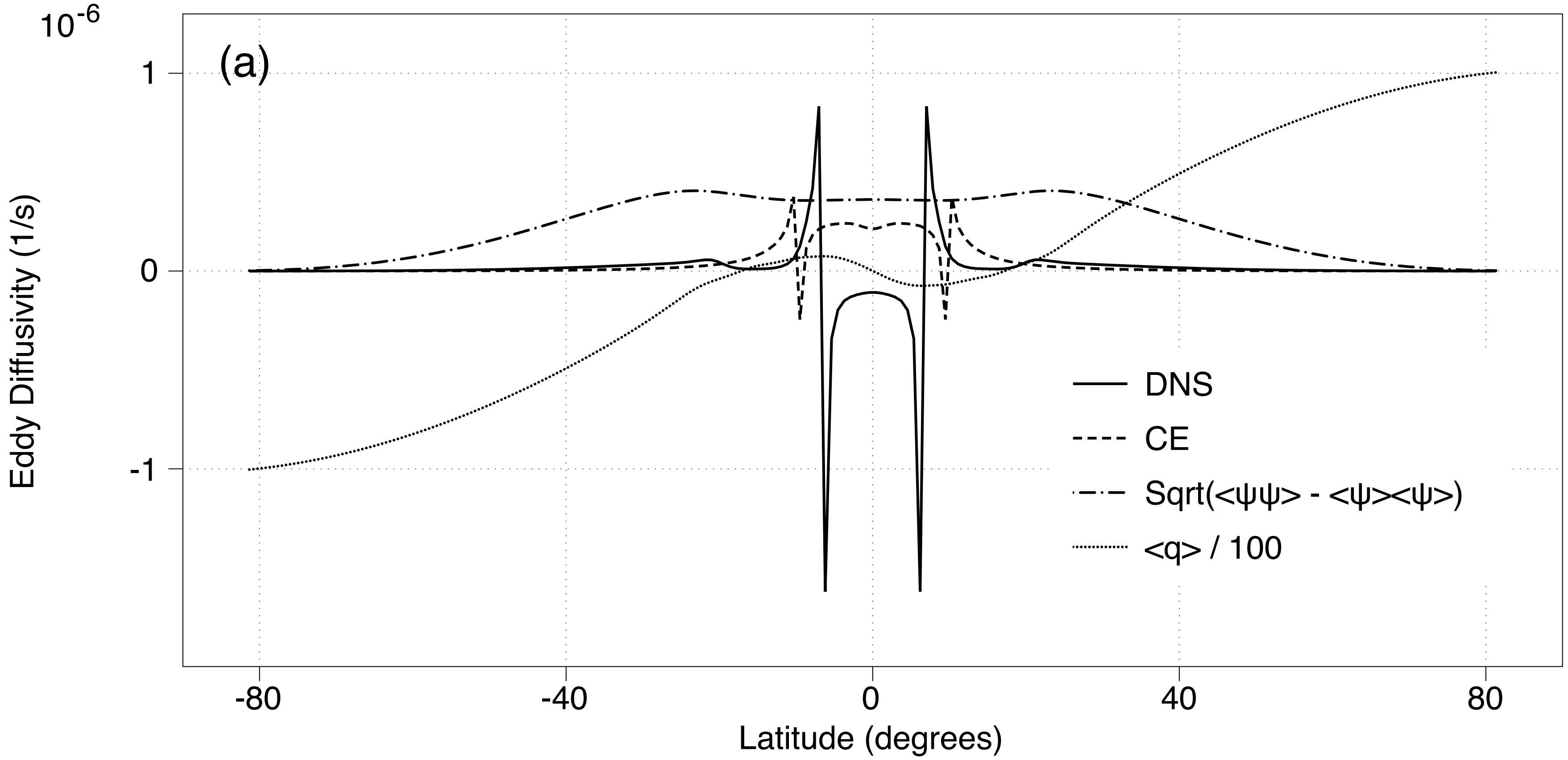

(a)
Mean Absolute Vorticity (1/s)
10 -5
Latitude (degrees)
τ = 0 days
DNS: τ = 1.5625 days
CE: τ = 1.5625 days
DNS: τ = 3.125 days
CE: τ = 3.125 days
DNS: τ = 6.25 days
CE: τ = 6.25 days
DNS: τ = 25 days
CE: τ = 25 days
(b)
Mean Absolute Vorticity (1/s)
10 -6
Latitude (degrees)

2nd Cumulant = 2-point Correlation Function

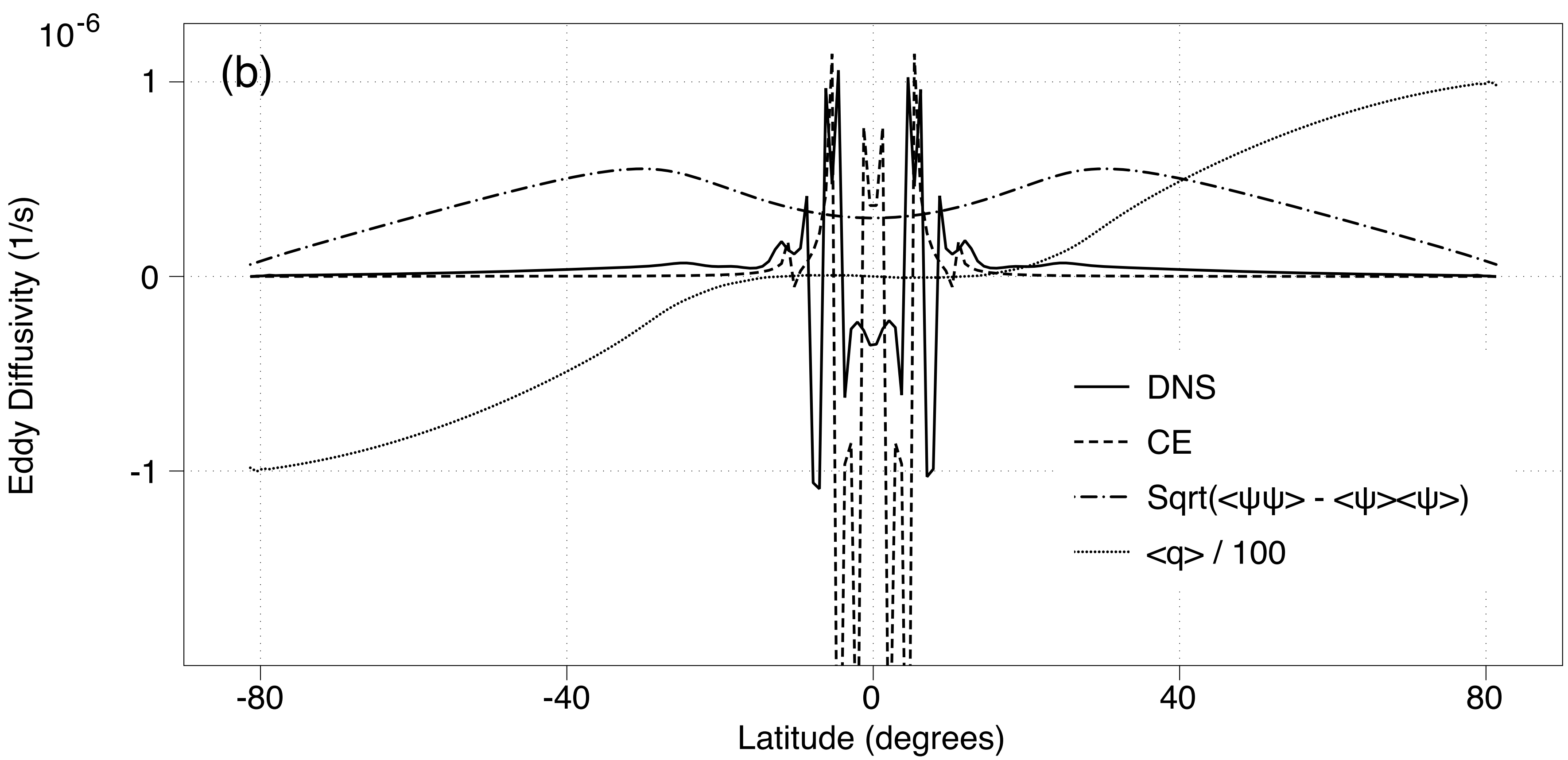


25 days





10^-6
(a)
1
0
-1
Eddy Diffusivity (1/s)
DNS
CE
Sqrt(<ψψ> - <ψ><ψ>)
<q> / 100
-80
-40
0
40
80
Latitude (degrees)



10^-6
(b)
1
0
-1
Eddy Diffusivity (1/s)
DNS
CE
Sqrt(<ψψ> - <ψ><ψ>)
<q> / 100
-80
-40
0
40
80
Latitude (degrees)

O’Gorman and Schneider, Geophys. Res. Lett. **34**, L22801 (2007)

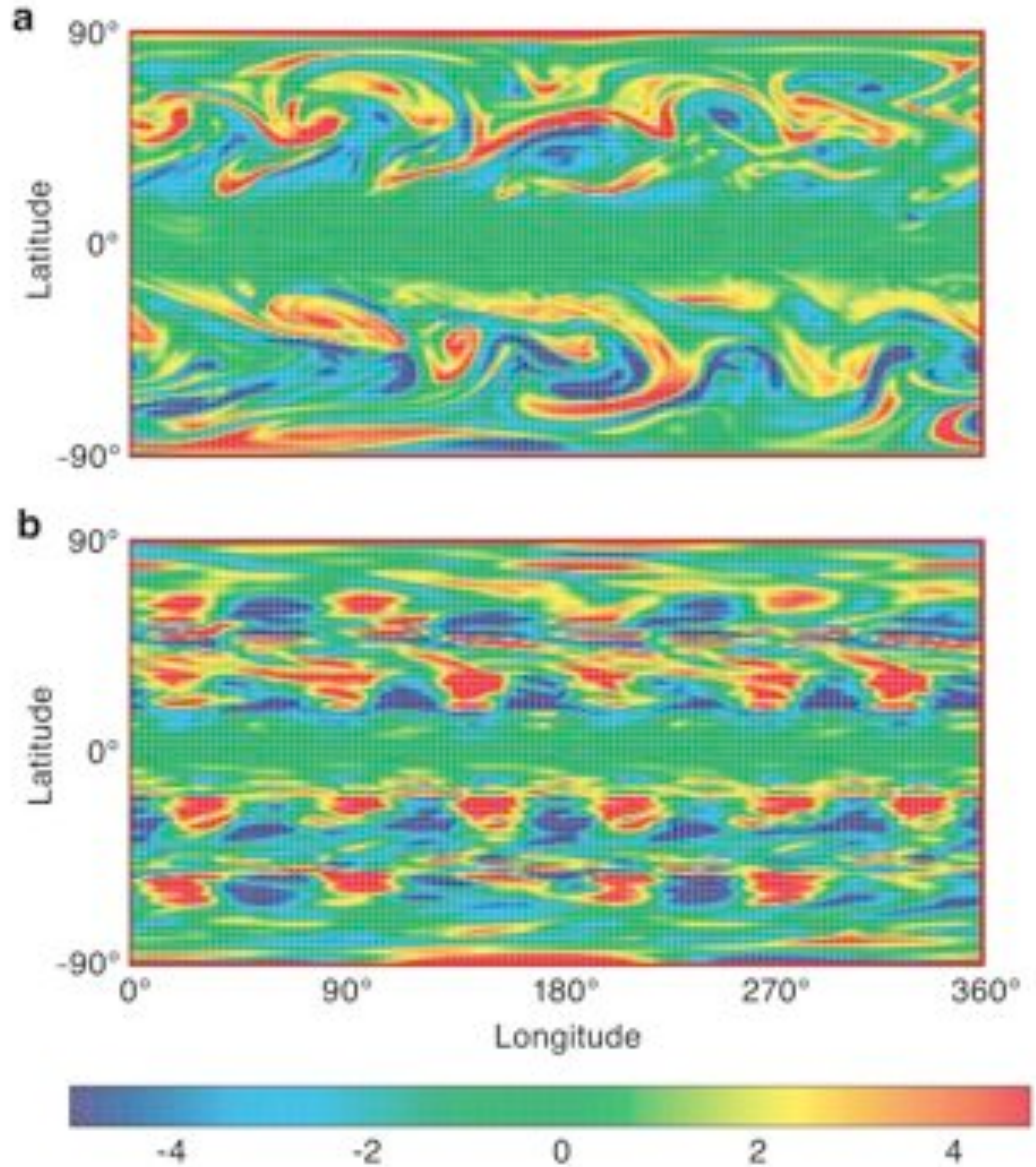


Figure 1. Typical instantaneous vorticity fields (10^{-5} s^{-1}) in (a) the full simulation and (b) the simulation without eddy-eddy interactions. The horizontal surface shown is in the mid-troposphere at $\sigma = 0.5$. The fields are shown at times after the simulations have reached statistically steady states.

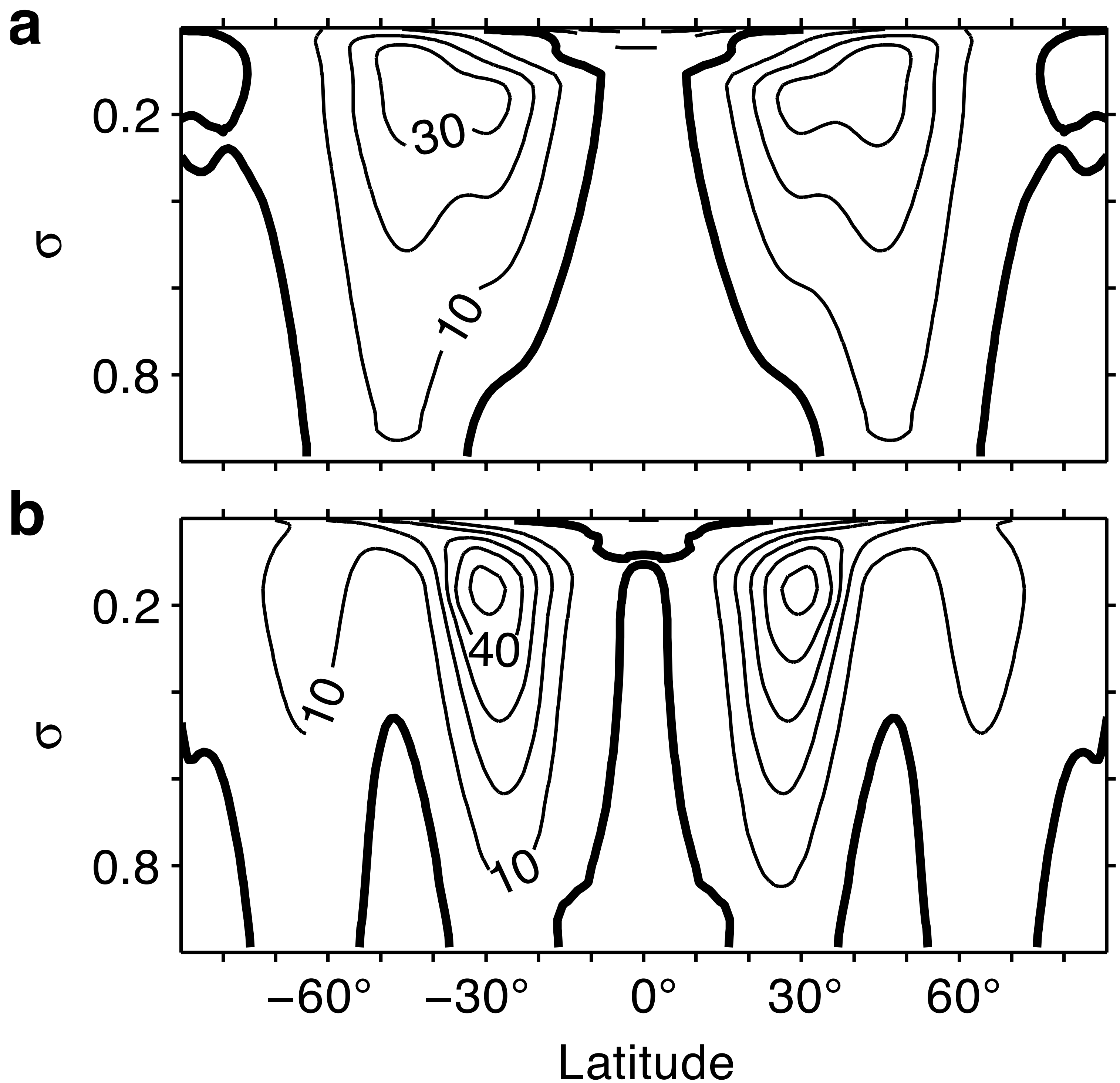
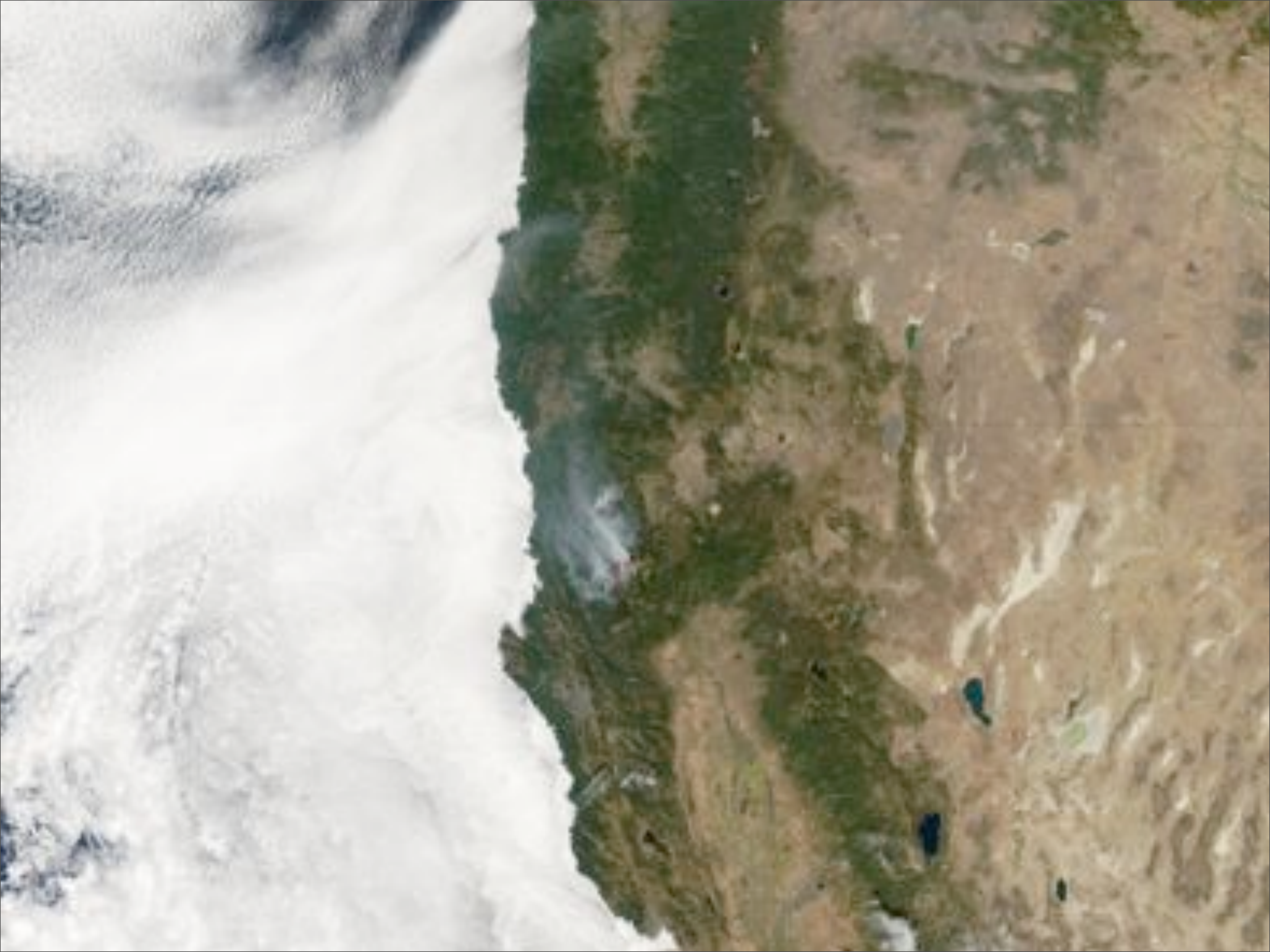


Figure 3. Mean eastward wind (m s^{-1}) in the meridional plane in (a) the full simulation and (b) the simulation without eddy-eddy interactions. The mean is a zonal, time, and interhemispheric average with mass weighting. The thick solid lines are the zero-wind lines.

Advantages of Statistical Approach

- Deeper understanding possible.
- Possibility to treat all processes statistically, not just the subgridscale ones.
- Inhomogeneous geophysical flows with mean shear flows are less nonlinear than isotropic turbulence -- progress possible.
- Faster: Time-independent fixed point.
- Faster: Statistics vary slowly in space.


Lake Mead



What Can Theoretical Thinking Contribute?

Aspen Center for Physics

Summer 2005 Workshop
Novel Approaches To Climate
Funding: NSF, BP Research, & ICAM

John Harte's long-term ecosystem heating experiment at the Rocky Mountain Biological Laboratory near Aspen.

Kavli Institute for Theoretical Physics

Physics of Climate Change
April 28 -- July 11, 2008
Frontiers of Climate Science
& Engineering the Earth
May 6 -- 10, 2008

Co-organizers:
J. Carlson, G. Falkovich, J. Harte,
J. B. Marston, and R. Pierrehumbert