Classical and Quantum Information Theory
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Routing and Communications on Wireless Network as a Quantum Spin Problem

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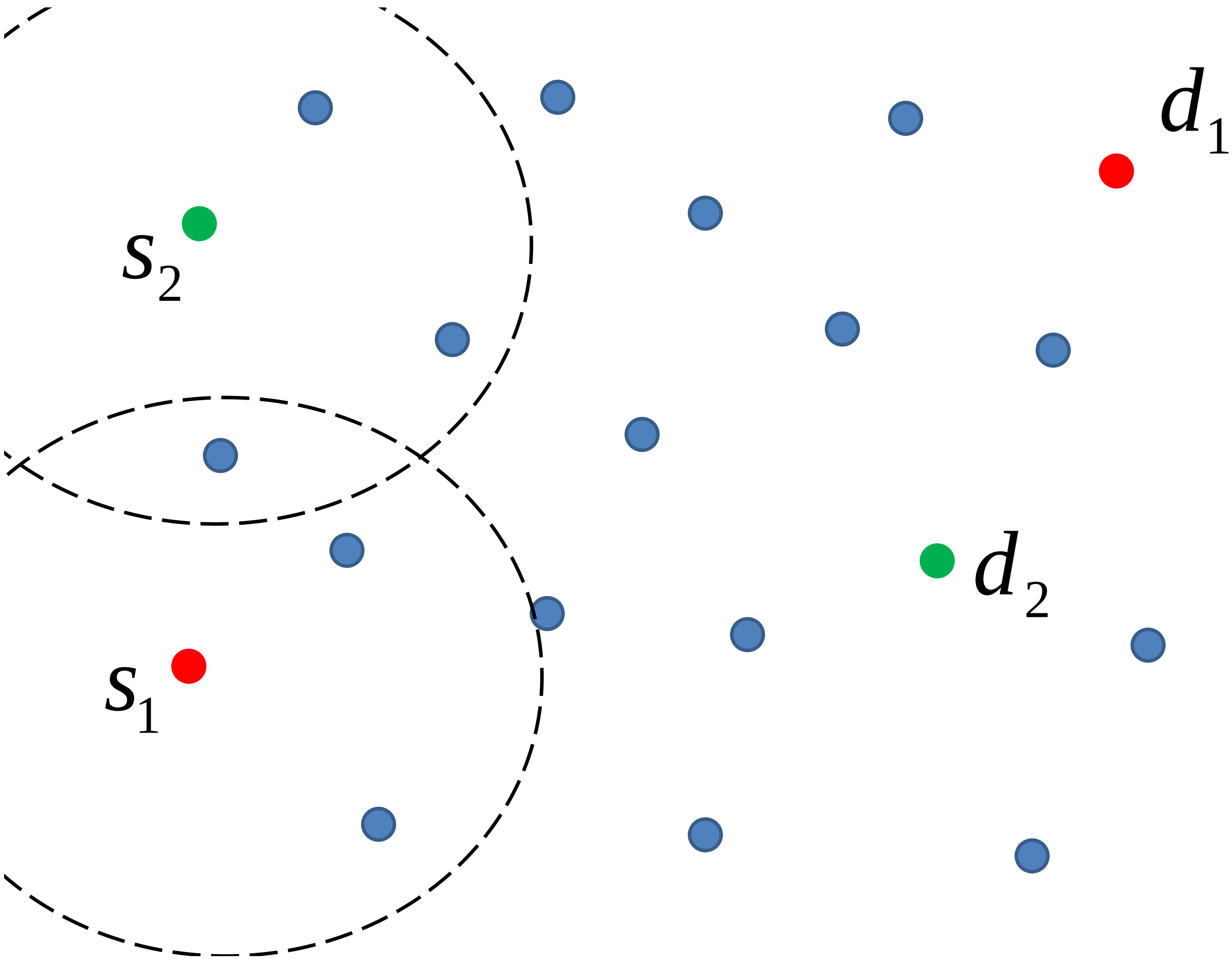
We analyze modern ad hoc networks, e.g. Wireless Sensor Networks (WSN) consisting of identical nodes capable of transmitting, receiving and sending of radio signals. We assume that there is a large number of nodes in the network. For a network operating according to fixed operational rules, we aim to develop a theoretical approach which sets a comprehensive framework and allows approximate evaluation of the network performance measured in terms of the network throughput, total consumption of energy, network delay etc. Our approach allows direct comparison with high fidelity simulations, and also with existing and future testbeds. A critical barrier to progress in the field lies in complexity of the problems and, especially, dynamical, i.e. ever changing, and stochastic nature of the network setting.



d_1
s_2
d_2
s_1

- Node is a mini-computer + radio sender/receiver
- Each node can generate packets to be delivered to other node
- A generated packet is sent as a radio signal that can be detected by neighbor nodes
- The detected signal is decoded and written to memory
- After that it is resent and erased from memory
- So, a packet delivery is a multi-hop process

- A number of neighbor nodes that can receive the signal is larger than unity

- There is a probability of information lost due to noises

- However, the reproduction number should be larger than unity

- Thus the number of nodes occupied by a packet increases

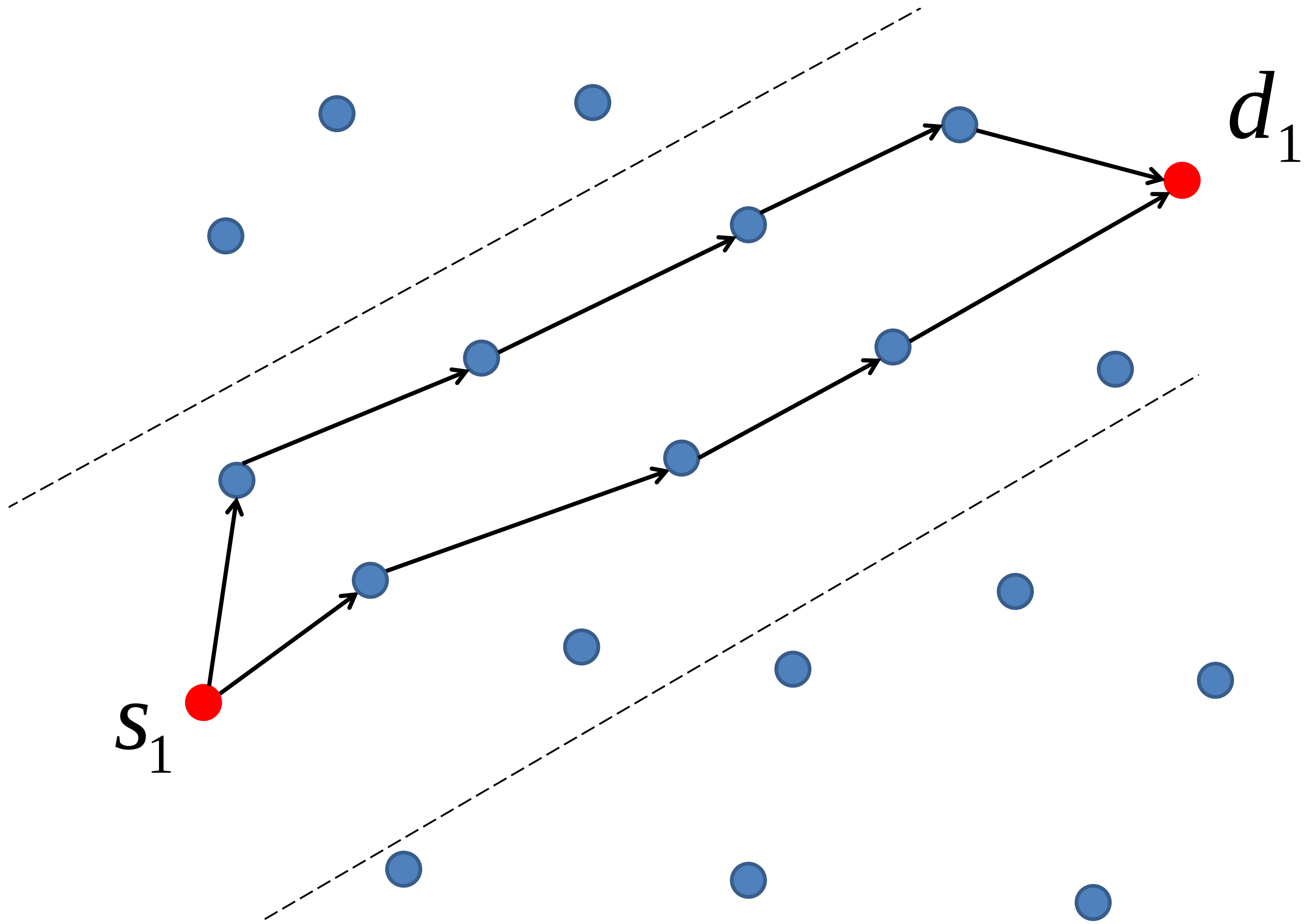
- This effects diminishes the network throughput since the waiting time before resending a packet depends on the node memory occupation

- Some additional rules are needed to achieve a compact cloud (spot) of nodes occupied by a packet propagating from source to destination

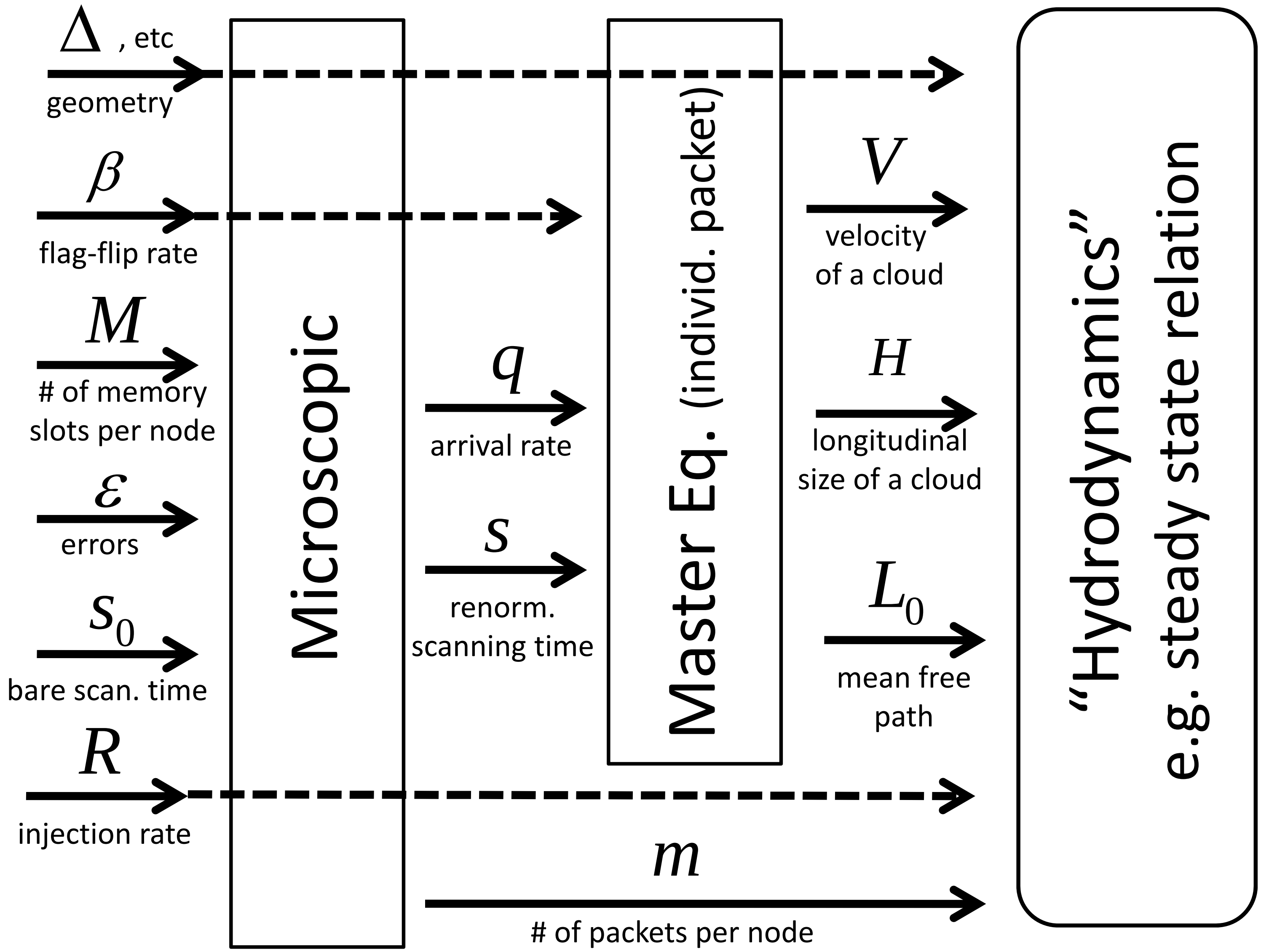
- Forward rule: a packet is written in the memory only if the current node is closer to the destination node then the last sender.

- Corridor Rule: a packet is written in the memory only if the current node lies in the corridor connecting the source and the destination.

- It implies that the geographical information about any node is available and that the positions of the source and the destination are contained in the message header.



d_1
s_1



Δ , etc
geometry
β
flag-flip rate
M
# of memory slots per node
ε
errors
s_0
bare scan. time
R
injection rate
Microscopic
q
arrival rate
s
renorm. scanning time
m
# of packets per node
Master Eq. (individ. packet)
V
velocity of a cloud
H
longitudinal size of a cloud
L_0
mean free path
"Hydrodynamics"
e.g. steady state relation

We adopt Doi technique for the master equation level of the network description. If at some instance t node j of the network contains the packet one says that the node is in the $|+\rangle$ state, while the state would be $|-\rangle$ if the node does not have the packet. Then, any "pure" state of the entire network will be denoted by $|\mu\rangle$, where μ stands for the set of $+$ and $-$ states at all the nodes of the network. If the state $|\mu\rangle$ is realized with the probability $P(\mu)$ one says that the entire network is in the state

$$|s\rangle = \sum_{\mu} P(\mu)|\mu\rangle, \quad \sum_{\mu} P(\mu) = 1, \tag{1}$$

where the last condition reflects that the total probability is equal to unity.

It is convenient to think about $|+\rangle$ and $|-\rangle$ as of quantum spin states of a node, i.e. they can be considered as bra vectors transformed by the Pauli matrixes

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2)$$

at each node of the network. Then the normalization condition, i.e. conservation of probability, reads

$$\langle 0| \exp\left(\sum \sigma_j^-\right) |s\rangle = 1, \quad (3)$$

where $|0\rangle \equiv |-, \cdots, -\rangle$ is the vacuum state with all spins down (the packet is absent in the network), and j numerates nodes of the network. Of course

$$\exp\left(\sum \sigma_j^-\right) = \prod_j (1 + \sigma_j^-). \quad (4)$$

In these notations the master equation becomes

$$\partial_t |s\rangle = -\mathcal{H}|s\rangle \,, \tag{5}$$

where $\widehat{\mathcal{H}}$ is the Hamiltonian (operator) dependent on the operational rules of the network determining transitions between different states and stated in terms of the spin operators. The Hamiltonian is real and non-Hermitian. The normalization condition (3) results in the condition

$$\langle 0| \exp\left(\sum \sigma_j^-\right) \mathcal{H} = 0, \tag{6}$$

to be satisfied for any Hamiltonian constructed from the master equation. An example

$$\mathcal{H} = \sigma^- \sigma^+ - \sigma^+. \tag{7}$$

Let us now introduce an average of an operator a defined as

$$\langle a \rangle \equiv \langle 0 | \exp\left(\sum \sigma_j^-\right) a | s \rangle. \tag{8}$$

A motivation for the definition is clarified if we consider a diagonal operator, then

$$\langle a \rangle = \sum_\mu P_\mu a_\mu, \tag{9}$$

that is the standard statistical average. It will be convenient for us to use the designation $\varphi_j = \langle \sigma_j^- \rangle$. Then using simple algebra one finds $\langle \sigma_j^z \rangle = -1 + 2\varphi_j$ and $\langle \sigma_j^+ \rangle = 1 - \varphi_j$. Calculating a time derivative of $\langle a \rangle$, one arrives at the Heisenberg equation

$$\partial_t \langle a \rangle = \langle [\mathcal{H}, a] \rangle, \tag{10}$$

where we used Eqs. (5,6).

Now we are going to construct the 'Hamiltonian' responsible for the marked packet transmission. We assume that the packet sent by a node i can be received by a node j with some probability q_{ij} depending on the separation between the nodes i and j. A dependence of q_{ij} on the separation reflects real transmission conditions. We assume that q_{ij} has a finite correlation radius. The rule gives the Hamiltonian $\mathcal{H} = \sum_i \mathcal{H}_i$, where

$$\mathcal{H}_i = \sigma_i^+ \sigma_i^- - \prod_j [1 + q_{ij}\sigma_j^+ - q_{ij}\sigma_j^- \sigma_j^+] \sigma_i^- . \tag{11}$$

One can directly check that the normalization condition (6) is satisfied for the Hamiltonian (11). Our Corridor and Forward rules can be converted into conditions on q_{ij}: they are non-zero provided the node j lies inside the corridor and is closer to the destination node than the node i.

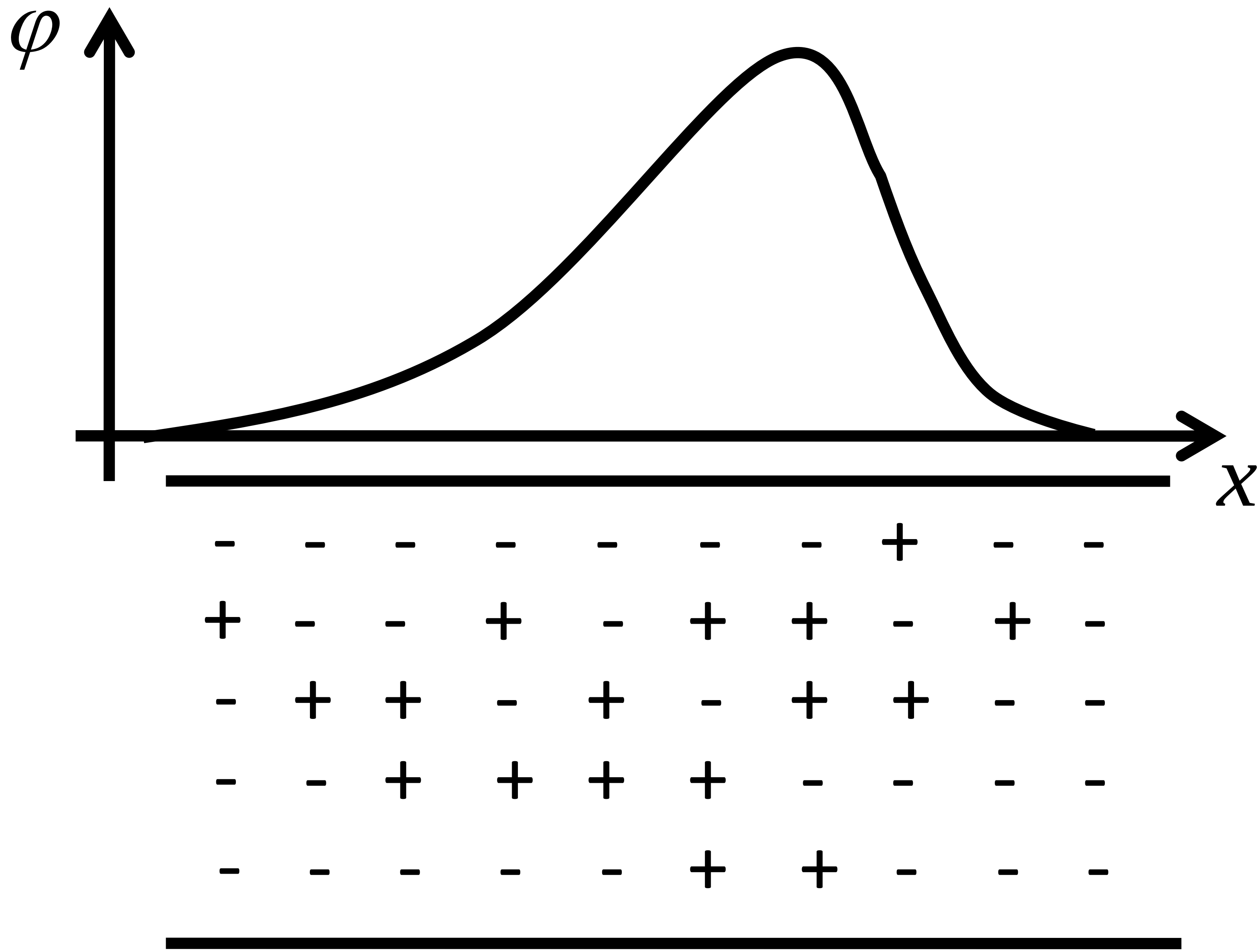
Now we calculate in accordance with Eq. (10)

$$\partial_t \varphi_i = -\varphi_i + \sum_j q_{ji} \langle \sigma_i^+ \sigma_j^- \rangle. \tag{12}$$

The quantity φ_i can be interpreted as average occupation number of the node i. Therefore the equation (12) describes a stochastic evolution of a cloud (spot) of nodes occupied by our marked packet. The spot moves through the corridor from source to destination. The equation (12) can be a starting point for a mean field approximation if we substitute $\langle \sigma_i^+ \sigma_j^- \rangle \to \langle \sigma_i^+ \rangle \langle \sigma_j^- \rangle$. Then we obtain the 'classical' (mean field) equation

$$\partial_t \varphi_i = -\varphi_i + \sum_j q_{ji} \varphi_j (1 - \varphi_i). \tag{13}$$

The approximation is reasonable if the number of communicating nodes is large enough.


φ
x

A solution of the equation (13) describes a $1d$ spot propagation along the corridor. The φ profile has a well defined frond and a tail behind it (see Figure). Unfortunately, the tail grows as time goes. That means that the total number of nodes occupied by our marked packet increases as time goes. The property is not pleasant since it leads to diminishing throughput. Therefore we should introduce additional rules preventing the occupation number increase and making it constant (in average). For the purpose we introduce flags on the modes that can be $+$ or $-$, and attached to our marked packet. If the flag is $+$, then the packet can be received by the node, otherwise the receipt is forbidden. Initially, the flag is $+$, and it is converted to $-$ with some probability if the node is receiving the packet. The rule kills the extended tail in φ_i.

Let us formalize the flag dynamics. For the purpose one can introduce the Pauli matrices $\boldsymbol{\tau}_i$ defined on the flag space. Then the above rule is converted into the Hamiltonian

$$\mathcal{H}_i = \sigma_i^+ \sigma_i^- - \prod_j [1 + \tau_j^+ \tau_j^- (q_{ij}\sigma_j^+ - q_{ij}\sigma_j^- \sigma_j^+)]\sigma_i^-, \tag{14}$$

$$\mathcal{H} = \sum_i \mathcal{H}_i - \beta \sum_i (\tau_i^- - \tau_i^+ \tau_i^-)\sigma_i^-, \tag{15}$$

where β determines the probability of the flag flip. Now we derive from Eqs. (10,14,15) the following equations

$$\partial_t \varphi_i = -\varphi_i + \sum_j q_{ji} \langle \sigma_i^+ \sigma_j^- \tau_i^+ \tau_i^- \rangle, \tag{16}$$

$$\partial_t \langle \tau_i^+ \tau_i^- \rangle = -\beta \langle \tau_i^- \sigma_i^- \rangle. \tag{17}$$

Again, it is reasonable to examine the system in the mean field approximation. Decomposing the right hand sides of the equations (16,17) we find

$$\partial_t \varphi_i = -\varphi_i + \sum_j q_{ji} \varphi_j (1 - \varphi_i) \phi_i, \tag{18}$$

$$\partial_t \phi_i = -\beta \varphi_i \phi_i, \tag{19}$$

where $\phi_i = \langle \tau_i^- \rangle$. Thus, the effectiveness of the transmission goes dawn as the number of packets passed through the node increases. That ensures an extension of the tail. It is clear that the quantity β determines the spot length (size in the propagation direction). The length diminishes as β increases. The spot thickness is determined by the corridor width. Thus, these two quantities determine an average number of the node occupied that (asymptotically) does not depend on the spot position.

Let us consider a model where the corridor is divided into a number of slices and that the nodes do communicate between neighbor slices only. Then we obtain from Eq. (18)

$$\partial_t \varphi_{1,\alpha} = -\varphi_{1,\alpha} + \sum_\beta q_{\beta\alpha} \varphi_{0,\beta}(1 - \varphi_{1,\alpha})\phi_{1,\alpha}, \tag{20}$$

where the Greece indices numerate nodes inside a slide, we consider two subsequent slices 0 and 1, and $q_{\alpha\beta}$ are corresponding pair probabilities. The equation (20) can be further simplified if we assume that all the probabilities are the same for all node pairs from the subsequent slices. Then the fields φ and ϕ do not depend on α and we obtain ($Q = qn$)

$$\partial_t \varphi_1 = -\varphi_1 + Q\varphi_0(1 - \varphi_1)\phi_1. \tag{21}$$

The signal can propagate provided $Q > 1$.

Let us consider a particular case where $Q-1 \ll 1$. Then we can pass to the continuous limit where the equation for φ is transformed to

$$\partial_t \varphi + \partial_x \varphi = \epsilon\varphi - \varphi^2 + (1/2)\partial_x^2 \varphi, \tag{22}$$

where $\epsilon = Q\phi - 1$ and we kept main terms of the expansion over ϵ, φ and a characteristic length. The equation (22) describes a field φ propagating with the unit velocity to the right. Near the front $\phi = 1$. If β in Eq. (17) is small, ϕ gradually decays behind the front. Then φ is adiabatically adjusted to the value ϵ and we obtain from Eq. (17) $\partial_t \phi = -\beta\epsilon\phi$. Asymptotically, at large t the field ϕ passes to $1/Q$ and we obtain $\varphi \approx \epsilon \sim (Q-1)\exp(-\beta t)$ where t is time from the front passing or, equivalently, distance to the running front. Thus, β determines the signal length. The case confirms our general expectations.

One could think about a generalization of our solution for a more complicated case where the node interaction q_{ij} is distance-dependent that makes the corridor cross-section non-homogeneous and involves a number of slices into game. We don't expect something qualitative different in this case comparing to our simple model since the only essential ingredient here is the corridor homogeneity. Namely, we believe that in this case a soliton-like solution exists that describes a propagation of the spot from the source to the destination with the occupation number of nodes that depends on q_{ij} and β.

In the above mean field approximation a probability to deliver the packet to the destination node is equal to unity since we deal with a soliton-like solution that propagates uniformly along the corridor. However, such soliton-like solution can be destroyed due to some fluctuations. We can imagine two types of such fluctuations, 'classical' and 'quantum'. 'Classical' fluctuations are related to the network loading. Say, the packet density can be so high that it blocks the packet transmission. 'Quantum' fluctuations are related to processes where nodes occupied by the marked packet disappear. That happens, say, if all transmission processes are unsuccessful at a time.

Our nearest problem to be solved is to estimate a probability of 'quantum' destruction of the propagating spot. If N nodes are occupied by the marked packet before the spot disappearing then a probability of the event can be estimated as $(1-q)^{nN}$ where n is a number of potential recipients and the transmission probability q is assumed to be the same for all recipients. (A generalization for q_{ij} is obvious.) The problem is to estimate N in the expression. Clearly, it should be less than the average occupation number of nodes in the spot since $(1-q)^{nN}$ strongly depends on N and some preliminary process is needed to reduce N. Formally, to estimate the process we should find an instanton leading from the soliton-like state to vacuum. That gives a principal contribution to L_0.

The next step could be a consideration of an disordered network. One could imagine that the disorder produces a number of ‘bottle necks’ along the corridor thus decreasing essentially L_0. Another type of quenched disorder is related to fluctuations of the probabilities q_{ij} explained by peculiarities of the nodes and of landscape. Probably, this type of the disorder produce similar effects creating some ‘bottle necks’. And, finally, all the effects should be combined together.

Summary

1. We demonstrated that it is possible to formulate simple rules for the signal transmission through the network ensuring compact spot propagation.

2. We formulated an effective spin Hamiltonian describing the marked packet propagation and formulated the mean field approximation.

3. We established phenomena leading the description out of the mean field approximation and determining its mean free path.