Measures of Entanglement in Dynamical Systems

Density matrix of block of spins in the ground state of a Hamiltonian

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Abstract

To build quantum computer or any other quantum device we need to maintain quantum control. Entanglement is an important recourse for quantum control. Many-body physics is used to study measures of entanglement in ground state of different dynamical systems.

Binary system $\{A\&B\}$, which is in a pure state: a unique wave function denoted by $|\Psi^{A,B}\rangle$. it will be a unique ground state of a dynamical model: interacting spins, Bose gas or Hubbard model.

$$|\Psi^{A,B}\rangle = \sum_{j=1}^{d} \quad |\Psi_j^A\rangle \otimes |\Psi_j^B\rangle \, , \quad d > 1$$

Measure of entanglement. Von Neumann entropy of the subsystem

$$S(\rho_A) = -Tr_A\,(\rho_A \ln \rho_A)$$

$$\rho_A = Tr_B \left(|\Psi^{A,B}\rangle \langle \Psi^{A,B}| \right)$$

John Preskill, Matthew Hastings, Jens Eisert, Frank Verstraete, John Cardy

Another measure is Rényi entropy of a subsystem:

$$S_{\alpha}(\rho_A) = \frac{1}{1-\alpha} \ln Tr_A(\rho_A^{\alpha})$$

where α is a parameter.

spectrum of density matrix is also interesting, even eigenvectors are important [measurement].

If whole binary system $A\&B$ is in a mixed states [thermodynamics]. Mutual entropy:

$$I_{\{A,B\}} = S_A + S_B - S_{A\cup B}$$

Another measure of entanglement of mixed states was suggested by Asher Peres. We can describe the mixed binary system by a density matrix $\rho_{A,B}$. Although $\rho_{A,B}$ is a positive matrix, the partially transposed $\rho_{A,B}^{T_A}$ does not to have be positive.

$$\mathcal{N}_{A,B} = |\sum \text{negative eigenvalues of } \rho_{A,B}^{T_A}|$$

is called negativity.

1 AKLT Spin Chain

Implementation of AKLT in optical lattices was proposed by I.Cirac as well as the use of AKLT model for *universal quantum computation* .

Simplest version consists of a chain of spin-1's in the bulk, and two spin-1/2 on the boundary. If we denote by $\vec{S}_k$ the vector of spin-1 operators and by $\vec{s}_b$ spin-1/2 operators at boundaries then the Hamiltonian of the system is

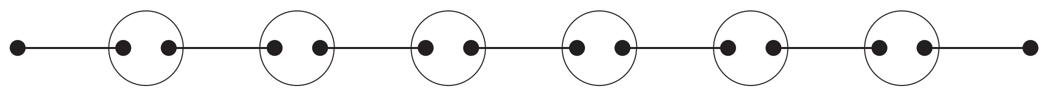
$$\mathcal{H}_{AKLT} = \sum_{k=-N}^{N} \left(\vec{S}_k \vec{S}_{k+1} + \frac{1}{3}(\vec{S}_k \vec{S}_{k+1})^2 + \frac{2}{3} \right) + \pi_{-N,-N+1} + \pi_{N,N+1}$$

$\frac{1}{2}\vec{S}_k \vec{S}_{k+1} + \frac{1}{6}(\vec{S}_k \vec{S}_{k+1})^2 + \frac{1}{3}$ is a projector on a state of spin $\mathbf{2}$.

The terms π describe interaction of boundary spin 1/2 and next spin 1; π is a projector on a state with spin 3/2:

$$\pi_{N,N+1} = \frac{2}{3}\left(1 + \vec{s}_{N+1} \vec{S}_N\right)$$

The ground state $|GS\rangle_{\text{lattice}}$ is unique and there is a gap [Haldane gap].



a dot is spin-$\frac{1}{2}$ and circles mean symmetrization [it makes spin-1 at each lattice site]. Solid lines mean anti-symmetrization [this prevents two neighboring spins form forming spin 2].

Correlation function are

$$\left(\frac{3}{4}\right) < \vec{S}_x \vec{S}_1 > = (-1/3)^x = p(x)$$

Continuos block of $\boldsymbol{x}$ consecutive spins. The density matrix is

$$\boldsymbol{\rho(x) = Tr}_{\text{out}} \ \left(|\boldsymbol{GS}\rangle_{\text{lattice}}\langle \boldsymbol{GS}|\right) \qquad \maltese$$

Entries are correlation functions.

In 2004 Fan, Korepin, Roychowdhury: limiting density matrix is

a **4** dimensional projector $\circledast$

$$\lim_{x\to\infty} \rho(x) = \lim_{\beta\to\infty} \left(\frac{\exp\left[-\beta \mathcal{H}_{block}\right]}{tr \exp\left[-\beta \mathcal{H}_{block}\right]} \right)$$

This Hamiltonian describes interaction inside of the block:

$$\mathcal{H}_{block} = \sum_{k=1}^{x} \left(\vec{S}_k \vec{S}_{k+1} + \frac{1}{3} (\vec{S}_k \vec{S}_{k+1})^2 + \frac{2}{3} \right) \qquad \boxplus$$

Degenerate ground state. MAIN RESULT: Von

Neumann entropy of the block is equal to Rényi entropy

$$S(\infty) = \ln 4 \qquad \bigstar$$

We also evaluated the entropy of a finite block on a finite chain:

$$S_{\mathrm{vN}}(x) = 2 - \frac{3\,(1-p(x))}{4}\log\,(1-p(x)) - \frac{1+3p(x)}{4}\log\,(1+3p(x))$$

1.1 The Spin-S AKLT Spin Chain

AKLT construction for higher integer spin S. To get unique ground state we place spin S in each bulk cite and spin $S/2$ at lattice ends [boundary spins].

Hamiltonian is

$$\mathcal{H}^{S}_{AKLT} = \sum_{j=-N}^{N} \sum_{J=S+1}^{2S} A_J P^J_{j,j+1} + \pi_{-N,-N+1} + \pi_{N,N+1}$$

$P^J_{j,j+1}$ projects the bond spin $\vec{J}_{j,j+1} = \vec{S}_j + \vec{S}_{j+1}$ on the subspace of magnitude J and the coefficient A_J is positive. The boundary terms describing interaction between spin $S/2$ and spin S are

$$\pi_{-N,-N+1} = \sum_{J=S/2+1}^{3S/2} B_J P^J_{-N,-N+1};\ \ \pi_{N,N+1} = \sum_{J=S/2+1}^{3S/2} B_J P^J_{N,N+1} \qquad B_J > 0$$

Use the Schwinger boson representation. The spin operators are represented by the Schwinger bosons as $S^+_j = a^\dagger_j b_j$, $S^-_j = b^\dagger_j a_j$, and $S^z_j = (a^\dagger_j a_j - b^\dagger_j b_j)/2$, where $a^\dagger_j$ and $b^\dagger_j$ satisfy $[a_i, a^\dagger_j] = [b_i, b^\dagger_j] = \delta_{ij}$ with the all the other commutators vanishing. One can say that $a^\dagger_j$ and $b^\dagger_j$ represent two copies of harmonic oscillator in each lattice cite.

Constraint is $a_j^\dagger a_j + b_j^\dagger b_j = 2S$ in each lattice site. The VBS state

$$|\text{VBS}\rangle = \prod_{j=-N}^{N+1} (a_j^\dagger b_{j+1}^\dagger - b_j^\dagger a_{j+1}^\dagger)^S |\text{vac}\rangle \qquad \divideontimes$$

Consider a block of the x bulk spins. Define the block Hamiltonian by

$$\mathcal{H}_{block} = \sum_{j=1}^{x} \sum_{J=S+1}^{2S} A_J P_{j,j+1}^J$$

The ground state is $(S+1)^2$ degenerate. Limiting density matrix of large block of spins is projector to the degenerate ground state of the block Hamiltonian $\between$

$$\lim_{x\to\infty} \rho(x) = \lim_{\beta\to\infty} \left(\frac{\exp\left[-\beta \mathcal{H}_{block}\right]}{tr \exp\left[-\beta \mathcal{H}_{block}\right]} \right)$$

Hosho Katsura, Takaaki Hirano, Ying Xu, Vladimir Korepin 2008. Von Neumann entropy of the block is equal to Rényi entropy

$$S(\infty) = \ln(S+1)^2 \qquad \ltimes$$

We also evaluated the entropy of a finite block for the finite chain.

We also generalized this result to $SU(n)$ Lie group, see Katsura, Hirano, Korepin arXiv:0711.3882 For vector representation

$$S(\infty) = \ln n^2 \qquad Ⓢ$$

1.2 AKLT Spin Chain on a 2D Cayley Tree

AKLT can be constructed for arbitrary graph. The value of the spin in a vertex is a half of number of nearest neighbors. Let us consider a Cayley tree (Bethe tree). Each lattice site has three neighbors, there are no loops [it is also known as Bethe tree].Each site has spin 3/2. The Hamiltonian of the AKLT model on the Cayley tree is

$$\mathcal{H}^{C}_{AKLT} = \sum_{(i,j)} P_3(\vec{S}_i + \vec{S}_j)$$

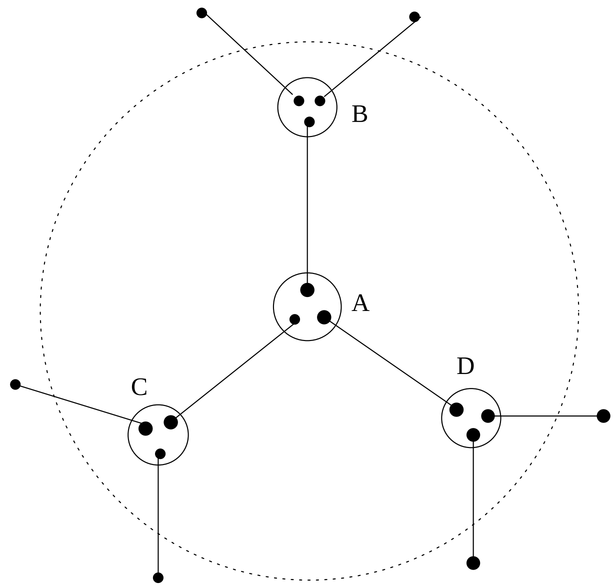


Figure 1: 4 sites Cayley tree

$\vec{S}_i$ is the spin 3/2 operator at site i and P_3 is a projector on a joint state of two neighboring spins with spin 3, and (i, j) are neighbors on the lattice. The ground state is unique VBS state. We calculated the entropy of a block of spins in a Cayley tree Heng Fan, Vladimir Korepin, Vwani Roychowdhury arXiv:quant-ph/0511150

1.3 Von Neumann Entropy of the XY Spin Chain

$$\mathcal{H}_{XY} = -\sum_{j=-\infty}^{\infty} (1+\gamma)\sigma_j^x\sigma_{j+1}^x + (1-\gamma)\sigma_j^y\sigma_{j+1}^y + h\sigma_j^z$$

where $0 < \gamma$ is the anisotropy parameter an $h > 0$ is the magnetic field. Can be experimentally implement in optical lattice. Solved by E.H. Lieb, T. Schulz, D. Mattis, E. Barouch and B.M. McCoy. The ground state is unique in general and in general there is a gap in the spectrum.

In the double scaling limit, when the size of the block is larger than 1 but much smaller than the length of the whole chain, the von Neumann entropy of the block has a limit. Three cases:

- Case IA: moderate magnetic field
  $2\sqrt{1-\gamma^2} < h < 2$
- Case IB: weak magnetic field
  $0 \leq h < 2\sqrt{1-\gamma^2}$
- Case II: strong magnetic field $h > 2$

If we define $I(k)$ as the complete elliptic integral

$$I(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

and the modulus

$$\tau_0 = I(k')/I(k), \qquad k' = \sqrt{1-k^2}$$

The limiting entropy of the block in the double scaling limit was calculated by Bai Qi Jin, Its and Korepin in 2004. I Peshel 2005.

$$S(\infty) = \frac{1}{6}\left[\ln\left(\frac{k^2}{16k'}\right) + \left(1 - \frac{k^2}{2}\right)\frac{4I(k)I(k')}{\pi}\right] + \ln 2\,, \quad k = \frac{\sqrt{(h/2)^2 + \gamma^2 - 1}}{\gamma} \quad \text{(IA)}\ , \tag{1}$$

$$S(\infty) = \frac{1}{6}\left[\ln\left(\frac{k^2}{16k'}\right) + \left(1 - \frac{k^2}{2}\right)\frac{4I(k)I(k')}{\pi}\right] + \ln 2\,, \quad k = \frac{\sqrt{1 - \gamma^2 - (h/2)^2}}{\sqrt{1 - (h/2)^2}} \quad \text{(IB)}\ , \tag{2}$$

$$S(\infty) = \frac{1}{12}\left[\ln\frac{16}{k^2 k'^2} + (k^2 - k'^2)\frac{4I(k)I(k')}{\pi}\right]\,, \quad k = \frac{\gamma}{\sqrt{(h/2)^2 + \gamma^2 - 1}} \quad \text{(II)}\ . \tag{3}$$

The entropy is constant on ellipsis in γ-h plane. The entropy has a local minimum $S(\infty) = \ln 2$ at the boundary between cases 1A and 1B, actually density matrix of large block of spins is a half of identical matrix 2X2 (in this case the ground state is doubly degenerate, and each of the ground states is a product state). The limiting entropy has absolute minimum $S(\infty) = 0$ is achieved at infinite magnetic field or at $\gamma = 0$ for $h > 2$, where the ground state becomes ferromagnetic. The entropy diverges to $+\infty$ at the phase transitions: $h = 2$ or $\gamma = 0$. Another interesting limit is reached around the point $\gamma = 0, h = 2$ which is the intersection of the two critical lines. This point belongs to both of the critical phases of the XY model so the entropy does not have an analytical expression on this point. The limit of the entropy reaching the point $(h, \gamma) = (2, 0)$ does not exist (it is direction dependent), we call this point *multi critical point* [it is also known as essential critical point]. I proved that depending on the approach to the essential critical point, the entropy can take any value between 0 and ∞, and the curves of constant entropy are ellipses and hyperbolas and they all meet at the multi critical point.

Might be interesting for quantum control, because small changes of parameters cause large changes in entanglement.

1.4 Rényi Entropy of the XY Spin Chain

In 2007 we investigated the properties of the Rényi entropy of the XY spin chain. The analysis showed that in the doubly scaling limit, Rényi entropy $S_\alpha(\rho_A) = \frac{1}{1-\alpha} \ln Tr_A(\rho_A^\alpha)$ approaches an asymptotic limit for the large block of spins:

$$S_R(\rho_A, \alpha) = \frac{1}{6}\,\frac{\alpha}{1-\alpha}\,\ln(k\,k') - \frac{1}{3}\,\frac{1}{1-\alpha}\,\ln\left(\frac{\theta_2(0,q^\alpha)\,\theta_4(0,q^\alpha)}{\theta_3^2(0,q^\alpha)}\right) + \frac{1}{3}\ln 2$$

Here, the elliptic parameter k , $k' = \sqrt{1-k^2}$, the modulus parameter q can be expressed in terms of $I(k)$ $q = \exp(-\pi I(k')/I(k))$. is the complete elliptic integral, and the theta functions $\theta_j(z, q)$ are defined by the standard series. Rényi entropy also constant on the same ellipsis as von Neumann entropy. The dependence on γ and h is similar. I proved that the limiting Rényi entropy of the XY spin chain can be expressed in terms of Klein's elliptic λ-function. Up to the trivial addition terms and multiplicative factors, and after a proper re-scaling, the Rényi entropy is an automorphic function with respect to a certain subgroup of the modular group; moreover, the subgroup depends on whether the magnetic field is above or below the critical value. The limit of large α defines the largest eigenvalue p_m of the density matrix of the block of spins $S_R(\rho_A, \alpha \to \infty) = -\ln p_m$:

$$\begin{aligned} S_\alpha(\alpha \to \infty) &= -\frac{1}{6}\ln\frac{kk'}{4} + \frac{\pi}{12}\frac{I(k')}{I(k)} + O\left(\frac{1}{\alpha}\right), \quad h > 2 \\ S_\alpha(\alpha \to \infty) &= -\frac{1}{6}\ln\frac{k'}{4k^2} + \frac{\pi}{6}\frac{I(k')}{I(k)} + O\left(\frac{1}{\alpha}\right), \quad h < 2 \end{aligned}$$

In the limit of small α the Rényi entropy has a singularity because in the limit the dimension of the Hilbert space of the block of spins goes to infinity:

$$\begin{aligned} S_\alpha(\alpha \to 0) &= \frac{1+\alpha}{\alpha}\frac{\pi}{12}\frac{I(k)}{I(k')} + O(\alpha), \qquad h > 2 \\ S_\alpha(\alpha \to 0) &= \frac{1+\alpha}{\alpha}\frac{\pi}{12}\frac{I(k)}{I(k')} + O(\alpha), \qquad h < 2 \end{aligned}$$

Critical behavior of Rényi entropy is similar to von Neunamm entropy, but the coefficients in from of the logarithms are different [α dependent]

The spectrum of limiting density matrix is different form AKLT. Infinitely many non-zero eigenvalues [form a sequence converging to zero]. The eigenvector corresponding to the largest eigenvalue is the ground state of XY on the block.

The expression essentially simplify if α is a power of 2.

2 Entanglement Entropy for Gapless Models

In the case of 1D critical models (gapless) the entropy of the subsystems scales logarithmically with the size of the block. More precisely for a block of x spins we have

$$S(n) = \frac{c}{3} \ln x, \qquad x \to \infty \qquad \blacksquare$$

where c is the central charge of the Virasoro algebra that describes the critical model. Holzhey, Larsen and Wilczek in 1994 (see also J.Cardy, V. Korepin). Examples are: Hubbard model, XX0 (or isotropic XY) spin chain, higher spin generalization of the isotropic XXX anti-ferromagnetic spin chain, Bose gas with δ-function interaction etc. Rényi entropy also scales logarithmically. In 2004 Bai Qi Jin, Korepin

$$S_\alpha(\rho_A) = \frac{1}{1-\alpha} \ln Tr_A(\rho_A^\alpha) = c\left(\frac{1+\alpha^{-1}}{6}\right) \ln x$$

2.1 Isotropic XXX Anti-Ferromagnetic Spin Chain with Arbitrary Spin

Let us consider the isotropic XXX anti-ferromagnet. For spin $\mathrm{s} = 1/2$ we can represent the Hamiltonian as:

$$\mathcal{H}^{1/2}_{XXX} = \sum_n X_n,$$
$$X_n = \vec{S}_n \vec{S}_{n+1} = S^x_n S^x_{n+1} + S^y_n S^y_{n+1} + S^z_n S^z_{n+1}\,.$$

The generalization to spin $\mathrm{s} = 1$ is :

$$\mathcal{H}^{1}_{XXX} = \sum_n \left\{ X_n - X_n^2 \right\}$$

Takhatajan and Babujian. Generalization for higher spin s is

$$\mathcal{H}^{s}_{XXX} = \sum_n F(X_n)$$

The function $F(X)$ is a polynomial of a degree $2\mathrm{s}$

$$F(X) = 2 \sum_{l=0}^{2\mathrm{s}} \sum_{k=l+1}^{2\mathrm{s}} \frac{1}{k} \prod_{\substack{j=0 \\ j \neq l}}^{2\mathrm{s}} \frac{X - y_j}{y_l - y_j}$$

Here $y_l = l(l+1)/2 - \mathrm{s}(\mathrm{s}+1)$. The model looks somewhat artificial but it is solvable by Bethe Ansatz. The model is critical [spectrum of this

Hamiltonians is gapless] I showed that the limiting entropy of a large block of x spins is weakly increases with spin:

$$S\left(n\right)=\frac{\mathrm{s}}{\mathrm{s}+1}\ln x,\quad \text{as}\quad x\rightarrow\infty \qquad ✠$$

2.2 The Hubbard model

Fermi Hubbard is important for strongly correlated electrons.

$$\mathcal{H}_{Hubbard}=-\sum_{\substack{j=1\\ \sigma=\uparrow,\downarrow}}\left(c_{j,\sigma}^{\dagger}c_{j+1,\sigma}+c_{j+1,\sigma}^{\dagger}c_{j,\sigma}\right)+u\sum_{j=1}n_{j,\uparrow}n_{j,\downarrow}$$

Here $c_{j,\sigma}^{\dagger}$ is a canonical Fermi operator on the lattice (creates of an electron) and $n_{j,\sigma}=c_{j,\sigma}^{\dagger}c_{j,\sigma}$ is an operator of number of electrons in site number j with spin σ, $u>0$. Charge and spin separates in the model. Below half filling $c_c=1$ and $c_s=1$. The entropy of the electrons is

$$S\left(n\right)=\frac{2}{3}\ln x\quad \text{as}\qquad x\rightarrow\infty$$

At half filed band charge degrees of freedom have a gap, but spin degrees of freedom are gapless, so the limiting entropy

$$S\left(n\right)=\frac{1}{3}\ln x\qquad \text{as}\qquad x\rightarrow\infty$$

2.3 Bose gas with δ-function interaction

Another model that is receiving considerable attention in the last period, due to its experimental realization in optical lattices, is the Bose gas with δ-function interaction with the Hamiltonian

$$H_{Bose} = \int dx \left[\partial\psi_x^\dagger \partial\psi_x + g\psi^\dagger\psi^\dagger\psi\psi\right]$$

where ψ is a canonical Bose field and $g > 0$ is a coupling constant. It can be shown that the field theoretical problem can be reduced to a quantum mechanical problem with the Hamiltonian

$$\mathcal{H}_N = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j} + 2g \sum_{N \geq k > l \geq 1} \delta(x_k - x_l)$$

which can be solved via Bethe Ansatz. The model is gapless and the central charge c is 1 so the entropy of the gas on a space interval $[0, x]$ also scales as

$$S(x) \to \frac{1}{3}\ln(x), \qquad x \to \infty \qquad\qquad \lozenge$$

References

[1] A. Rényi, *Probability Theory*, North-Holland, Amsterdam, 1970

[2] Bennett C H and DiVincenzo D P 2000 *Nature* **404** 247

[3] H. E. Brandt, *Quantum Information and Computation IV, Proc. SPIE*,Vol. 6244, Bellingham, Washington (2006) pp. 62440G-1-8.

[4] Lloyd S 1993 *Science* **261** 1569; 1994 ibid **263** 695

[5] M. Cramer, J. Eisert, M.B. Plenio, J. Dreissig, Phys.Rev. **A73** 012309 (2006)

[6] C.H. Bennet, H.J. Bernstein, S. Popescu and B. Schumacher,Phys. Rev. **A53** 2046 (1996)

[7] C.Holzhey, F.Larsen and F. Wilczek,Nucl. Phys. **B424** 443 (1994)

[8] G. Vidal, J.I. Latorre, E. Rico and A. Kitaev, Art. No. 227902 (2003) [quant-ph/0211074]

[9] V.E. Korepin, Phys. Rev. Lett. **92** Art. No. 062314 (2004) [cond-mat/0311056]

[10] A. Affleck, T. Kennedy, E.H. Lieb and H. Tasaki, Comm. Math. Phys. **115** 477 (1988)

[11] F.Verstraete, M.A.Martín-Delgado, J.I.Cirac, Phys.Rev.Lett.92, 087201(2004).

[12] F.Verstraete and J.I.Cirac, quant-ph/0311130.

[13] J.J.Garcia-Ripoll, M.A.Martín-Delgado, J.I.Cirac, cond-mat/0404566.

[14] R.B.Laughlin, Phys.Rev.Lett.**50**,1395 (1983).

[15] D.P.Arovas, A.Auerbach, F.D.M.Haldane, Phys. Rev. Lett. **60**, 531 (1988).

[16] H. Fan, V.E. Korepin and Vwani Roychowdhury, Phys. Rev. Lett. **93** Art. No. 227203 (2004) [quant-ph/0406067]

[17] F.D.M.Haldane, Phys.Lett.**93A**,464 (1983); Phys. Rev. Lett. **50**, 1153 (1983).

[18] H. Fan, V.E. Korepin and Vwani Roychowdhury,[quant-ph/0511150]

[19] E. Lieb, T. Schultz and D. Mattis,Ann. Phys. **16** 407 (1961)

[20] E. Barouch and B.M. McCoy, Phys. Rev. **A3** 786 (1971)

[21] M.E. Fisher and R.E. Hartwig, Adv. Chem. Phys. **15**, 333, (1968)

[22] B.Q. Jin, V.E. Korepin,J. Stat. Phys. **116** 79 (2004)

[23] P. Calabrese, J. Cardy, J.Stat.Mech. P002 (2004) [hep-th/0405152]

[24] A.R. Its, B.-Q. Jin, V.E. Korepin, J. Phys. A **38** 2975 (2005) [quant-ph/0409027]

[25] A.R. Its, B.-Q. Jin, V.E. Korepin,[quant-ph/0606178]

[26] F. Franchini, A. R. Its, B.-Q. Jin, V. E. Korepin,J. Phys. A **40** 8467 (2007)[quant-ph/0609098]

[27] F. Franchini, A. R. Its, B.-Q. Jin, V. E. Korepin, [arXiv:0707.2534]

[28] H. Bethe, Zeitschrift fur Physik **71** 205 (1931)

[29] A.Zamolodchikov, V.Fateev, Jadernaya Fizika **32** 581 (1980)

[30] L.A. Takhatajan, Phys.Lett. A **87** 479 (1982)

[31] H.M. Babujian, Nucl.Phys.B **215** 317 (1983)

[32] V.O. Tarasov, L.A. Takhatajan and L.D. Faddeev, Teor. Mat. Phys. **57** 163 (1983)

[33] I. Affleck, FIELD THEORY METHODS AND QUANTUM CRITICAL PHENOMENA, Print-88-0765 (BRITISH COLUMBIA), June 1988. 93pp. Lectures given at Summer School on Fields, Strings and Critical Phenomena. Published in Les Houches Summer School 1988:0563-640 (QCD161:G65:1988)

[34] E.H. Lieb and W. Liniger, Phys. Rev. **130** 1605 (1963)

[35] S. Bravyi,R. Raussendorf, [quant-ph/0610162]

[36] D. G. Angelakis, M. F. Santos, S. Bose, Phys. Rev. **A** 76 031805(R) (2007) [quant-ph/0606159]

[37] M.C.Arnesen, S.Bose, V.Vedral, Phys. Rev. Lett. **87**, 017901 (2001)

[38] J.K. Pachos, M.B. Plenio, Phys. Rev. Lett. **93**, 056402(2004)

[39] S. Anders, M. B. Plenio, W. Dr, F. Verstraete, H.-J. Briegel, quant-ph/0602230

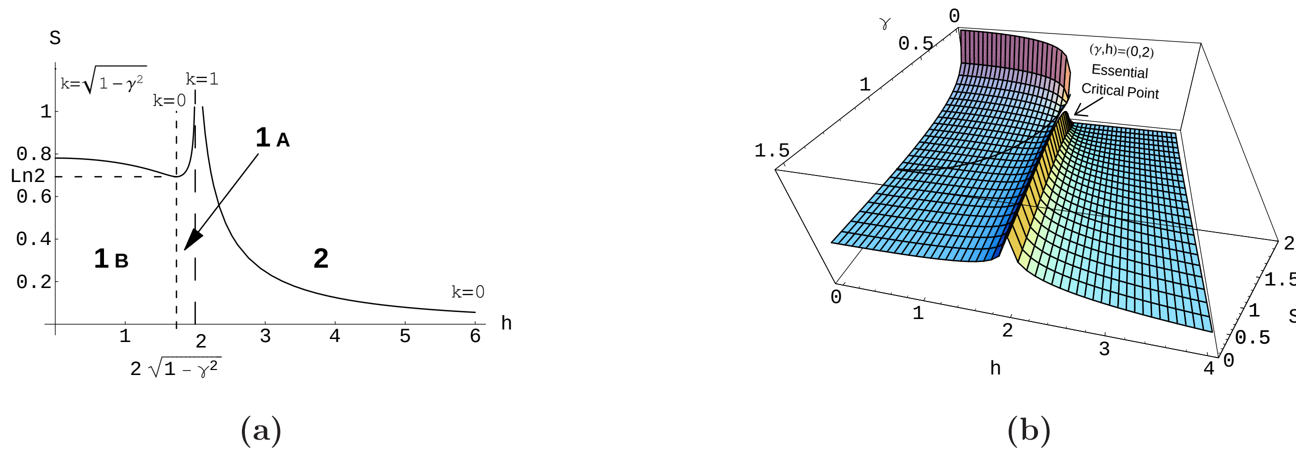


Figure 2: **(a)**The limiting entropy as a function of the magnetic field at constant anisotropy $\gamma = 1/2$. The entropy has a local minimum $S = \ln 2$ at $h = 2\sqrt{1-\gamma^2}$ and the absolute minimum for $h \to \infty$ where it vanishes. S is singular at the phase transition $h = 2$ where it diverges to $+\infty$. The three cases are marked. **(b)** Three-dimensional plot of the limiting entropy as a function of the anisotropy parameter γ and of the external magnetic field h. The local minimum $S = \ln 2$ at the boundary between cases **1**A and **1**B is visible and marked by a continuum line. S diverges to $+\infty$ at the phase transitions $h = 2$ and $\gamma = 0$, $h \leq 2$. The entropy takes every positive value in the vicinity of the multi critical point $(h, \gamma) = (2, 0)$.