Quenching, relaxation, and information transfer in lattice systems

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Joint work with
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Chris M. Dawson, Uli Schollwoeck

Santa Fe, March 2008

- **Motivating question:** Thermalizing open quantum systems

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- **Relaxation dynamics** in closed systems without environments?

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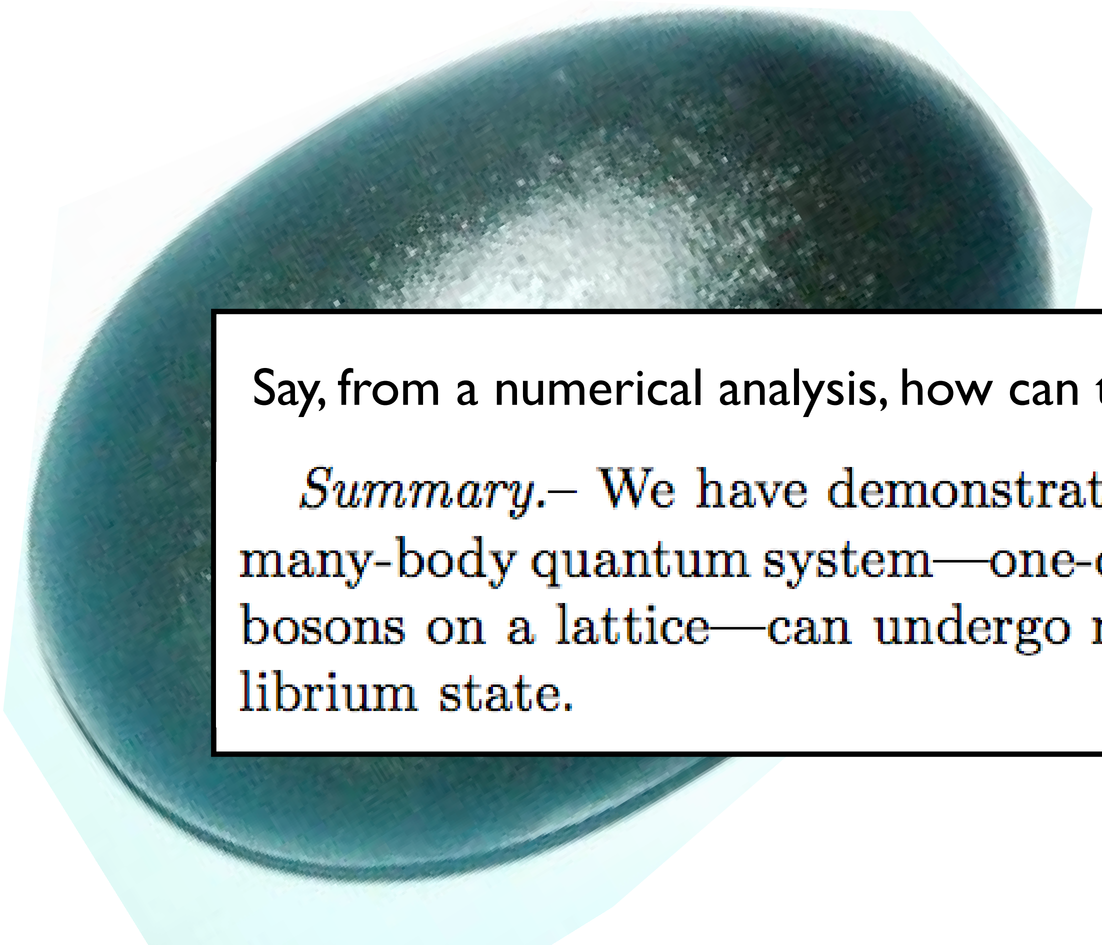
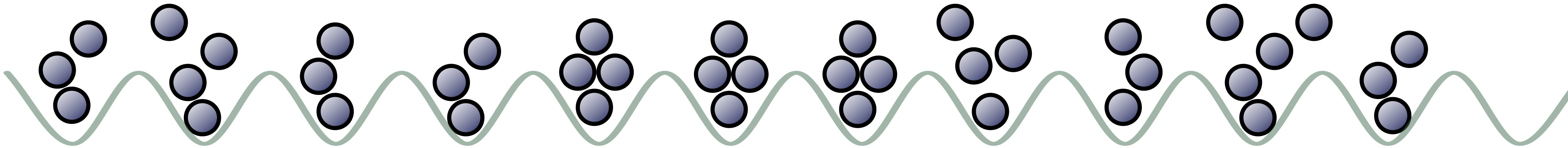
- **Relaxation dynamics** in closed systems without environments?
- In what sense can **closed many-body systems** relax when undergoing time evolution under **local** Hamiltonians?

System



Say, from a numerical analysis, how can this be?

Summary.– We have demonstrated that an integrable many-body quantum system—one-dimensional hard-core bosons on a lattice—can undergo relaxation to an equilibrium state.

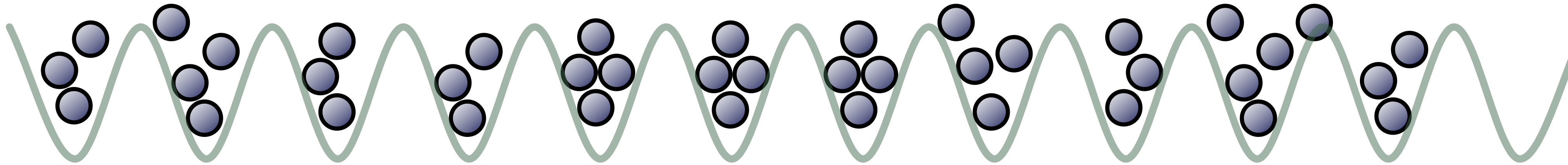


- Setting where "equilibration without an environment" can be studied:

dynamical setting of a sudden quench

- Start with ground state of local Hamiltonian $H = \sum_j h_j$

Calabrese, Cardy, *Phys Rev Lett* **96** (2006)
Eisert, Osborne, *Phys Rev Lett* **96** (2006)
Bravyi, Hastings, Verstraete, *Phys Rev Lett* **97** (2006)
De Chiara, Montangero, Calabrese, Fazio, *J Stat Mech* **0603** (2006)
Cincio, Dziarmaga, Rams, Zurek, *Phys Rev A* **75** (2007)



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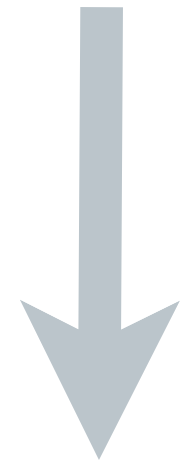

- Sudden change to a new local Hamiltonian $V = \sum_j v_j$

- Study time evolution under V

Calabrese, Cardy, *Phys Rev Lett* **96** (2006)
Eisert, Osborne, *Phys Rev Lett* **96** (2006)
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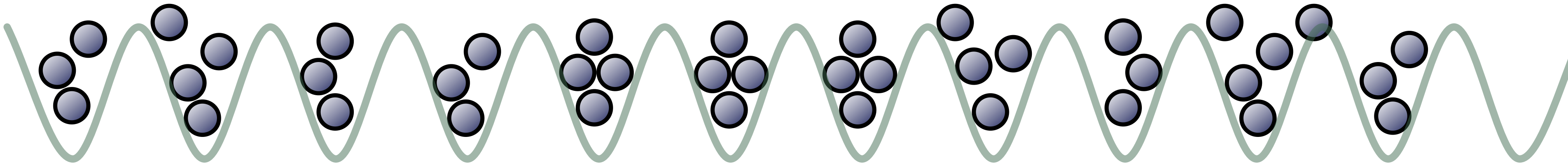


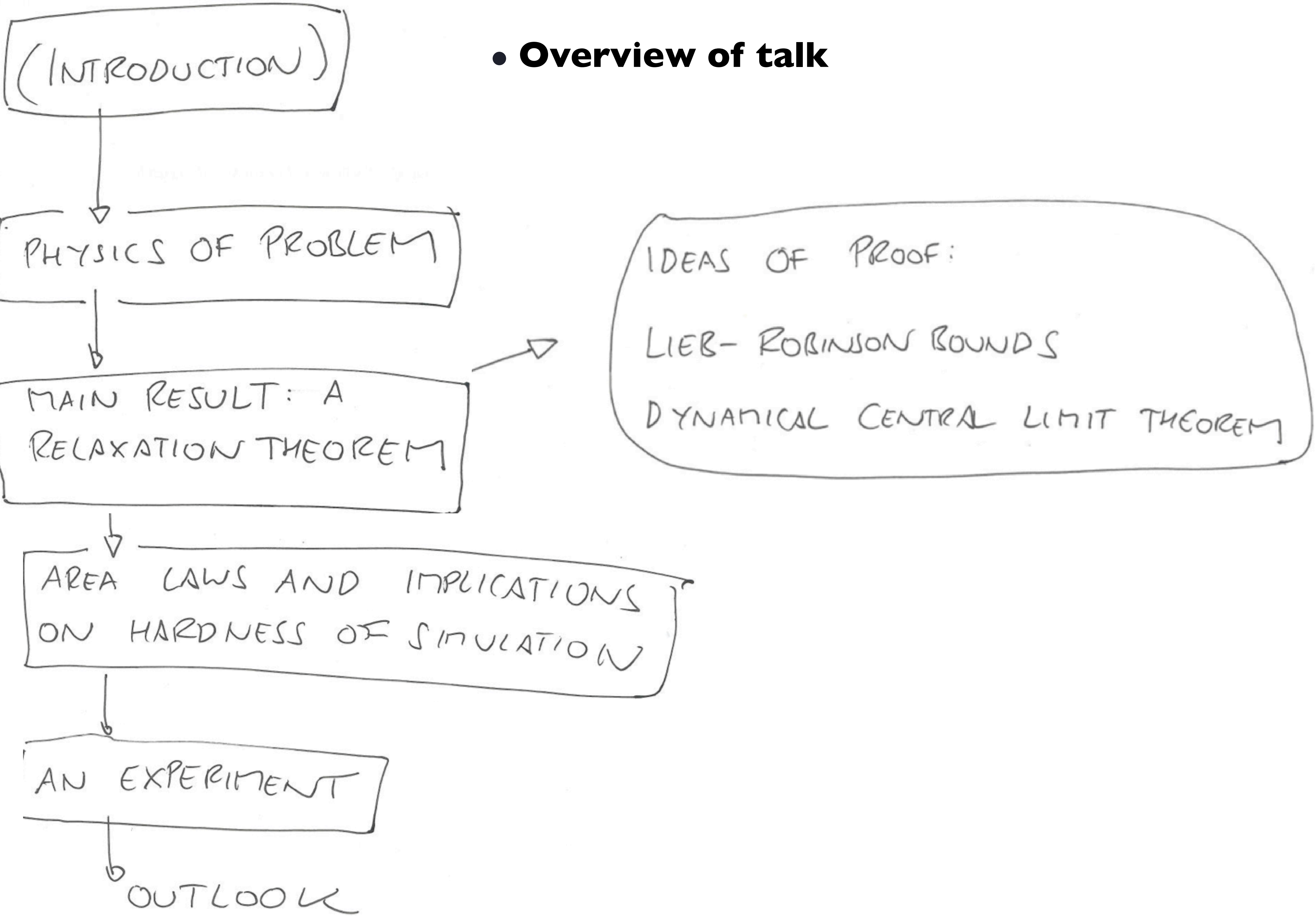
- Renaissance of question as systems become available offering possibility of **probing** such issues:
- **Cold atoms in optical lattices** deliver good control in experiments

Greiner et al, *Nature* **419** (2002)
Tuchmann et al, cond-mat/0504762
Kinoshita et al, *Nature* **440** (2006)

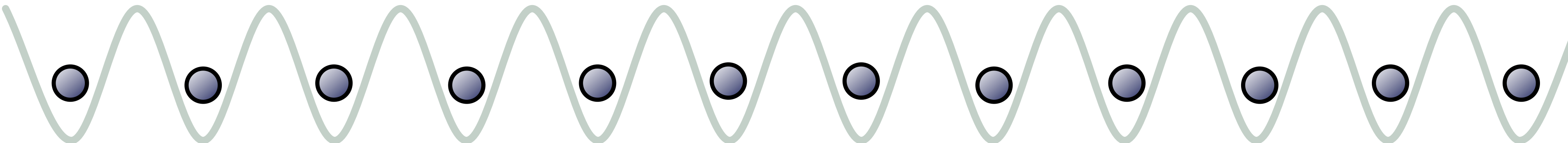
- **This talk:** Setting where one can rigorously study this question

Based on:
Cramer, Dawson, Eisert, Osborne, *Phys Rev Lett* **100** (2008)
Eisert, Osborne, *Phys Rev Lett* **96** (2006)
Cramer, Serafini, Eisert, arxiv:0803.0890
In preparation (2008)

- **Overview of talk**



- Briefly: The physics of the problem

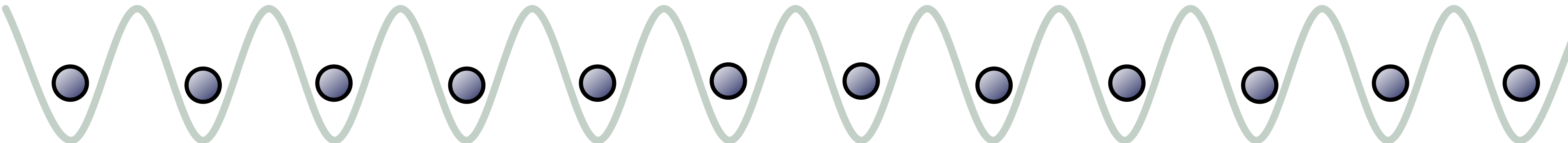


- **Bose-Hubbard model:**

$$H = -J \sum_{\langle j,k \rangle} b_j^\dagger b_k + \frac{U}{2} \sum_{k=1}^{N} b_k^\dagger b_k (b_k^\dagger b_k - 1) - \mu \sum_{k=1}^{N} b_k^\dagger b_k$$

$$V = -J \sum_{\langle j,k \rangle} b_j^\dagger b_k$$

- Standing wave laser light, lattice constant half wavelength, forming **optical lattice**



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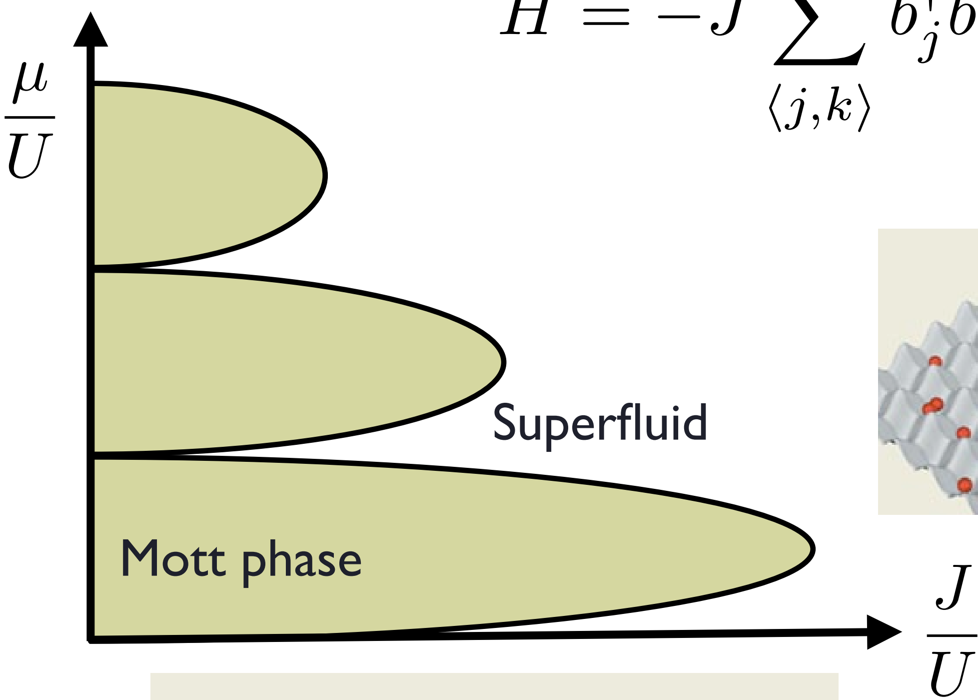
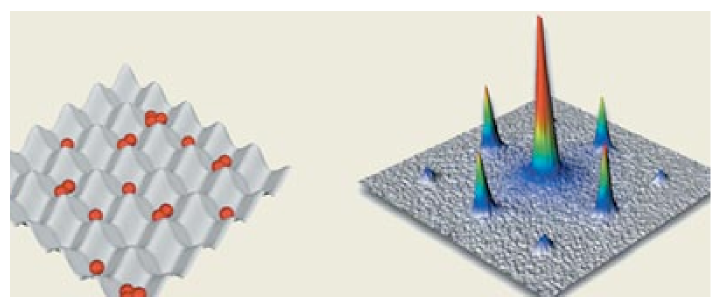
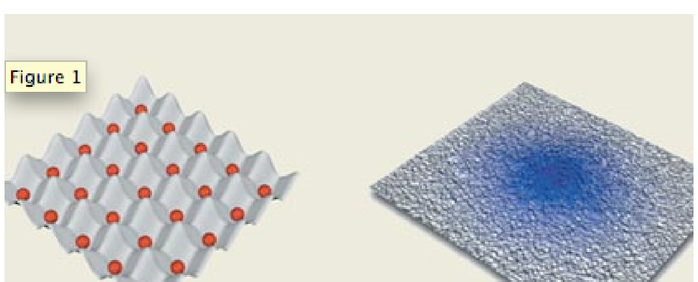
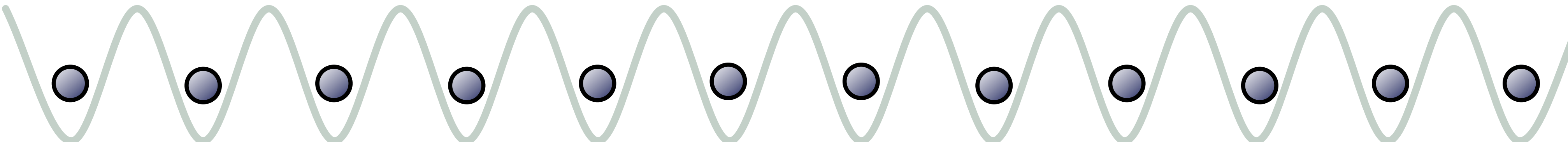






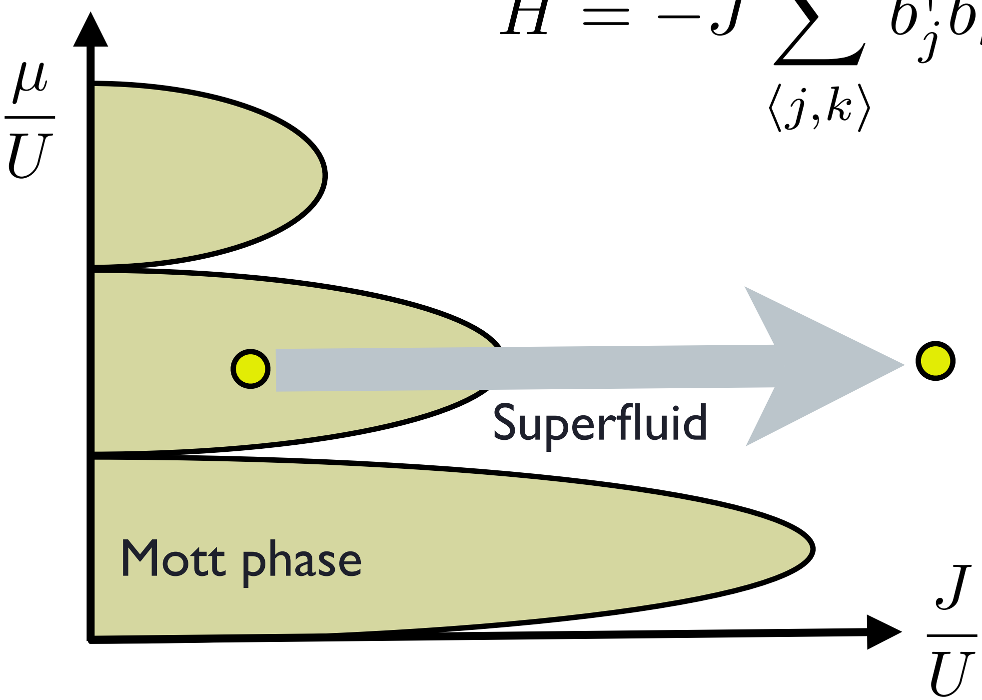
- Phase coherence and absence thereof in Mott and superfluid phase

Greiner et al, *Nature* **419** (2002)



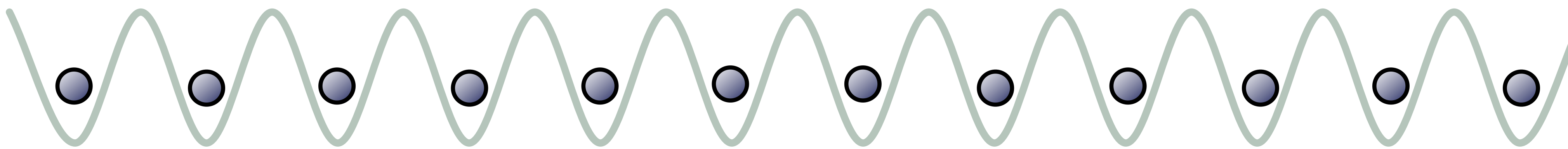
- **Bose-Hubbard model:**

$$H = -J \sum_{\langle j,k \rangle} b_j^\dagger b_k + \frac{U}{2} \sum_{k=1}^{N} b_k^\dagger b_k (b_k^\dagger b_k - 1) - \mu \sum_{k=1}^{N} b_k^\dagger b_k$$



- Here, sudden quench from some **clustering state** from gapped $(\Delta E > 0)$ Mott phase to
- deep **superfluid** V
- Other more general cases later

Cramer, Dawson, Eisert, Osborne, *Phys Rev Lett* **100** (2008)



- **Initial state:**
  - *Product state* of deep Mott phase having m bosons per site $|\psi(0)\rangle = |m\rangle^{\otimes N}$
  - *Clustering state:* Fourth moments of canonical coordinates exist and

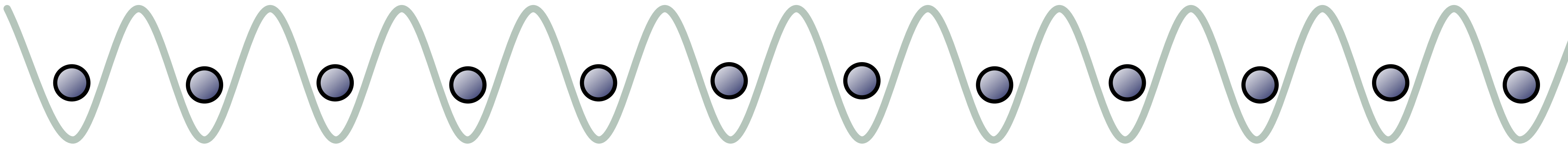
$$\langle \prod_{j\in A\cup B} W_{\xi_j} \rangle_{\rho(0)} - \langle \prod_{j\in A} W_{\xi_j} \rangle_{\rho(0)} \langle \prod_{j\in B} W_{\xi_j} \rangle_{\rho(0)} \leq e^{-\mu \mathrm{dist}(A,B)},$$

$W_{\xi_j} = e^{i(p_j X_j - x_j P_j)}$,Weyl (displacement) operators

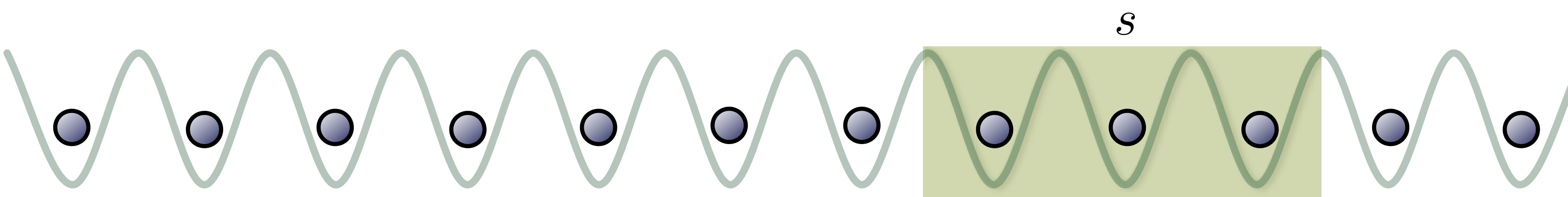
- Then quench and study *time evolution*

$$|\psi(t)\rangle = e^{-itV} |\psi(0)\rangle$$

- Relaxation?



- So, what do we find?
- Is time-dependent non-equilibrium system, so it “wobbles” forever...?



$$|\psi(t)\rangle = e^{-itV}|\psi(0)\rangle \qquad \rho_s(t) = \mathrm{tr}[|\psi(t)\rangle\langle\psi(t)|]$$

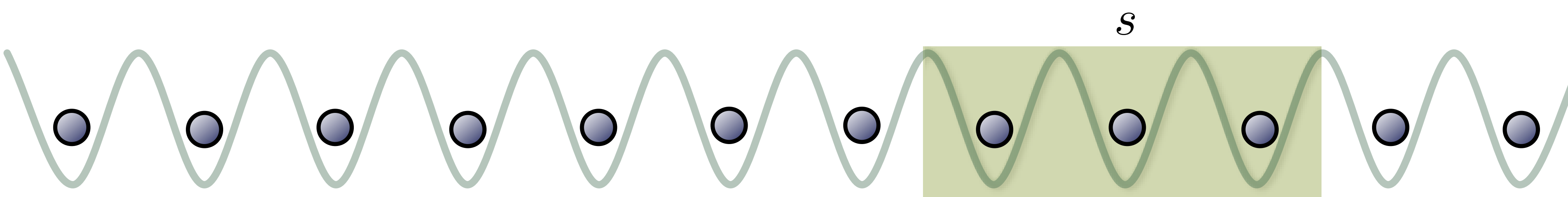
- **The claim:** It does relax exactly for any subblock!

$$\rho_s(t) \rightarrow \rho_G$$

- Remarkably, exact convergence, **no time average**
- Becomes a maximal entropy (Gaussian) state under energy constraint
- Block maximally **entangled** with rest of chain

$$S(\mathrm{tr}_B(|\psi(t)\rangle\langle\psi(t)|)) \rightarrow \max$$

Cramer, Dawson, Eisert, Osborne, *Phys Rev Lett* **100** (2008), cond-mat/0703314



$$|\psi(t)\rangle = e^{-itV}|\psi(0)\rangle \qquad \rho_s(t) = \mathrm{tr}[|\psi(t)\rangle\langle\psi(t)|]$$

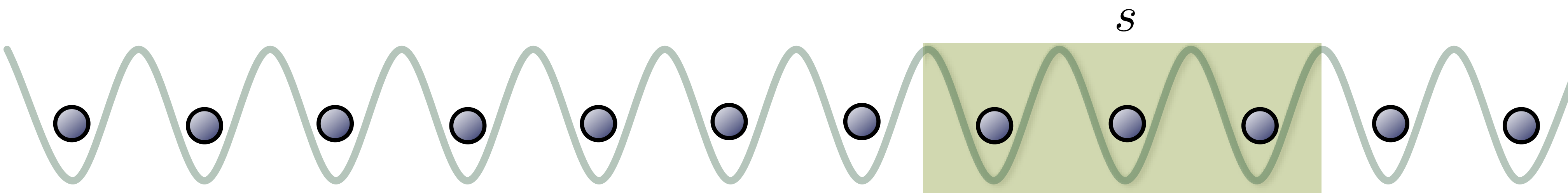
- **Theorem:** Let $\rho(0)$ be a clustering 1D state (e.g., product in deep Mott)

Then, for any $\varepsilon > 0$ and any desired "recurrence time $t_{\mathrm{rec}} > 0$ there ex. a system size N and a relaxation time $t_{\mathrm{rel}} > 0$ such that time evolved state $\rho(t) = e^{-itV}\rho(0)e^{itV}$ satisfies

$$\|\rho_s(t) - \rho_G\|_1 < \varepsilon$$

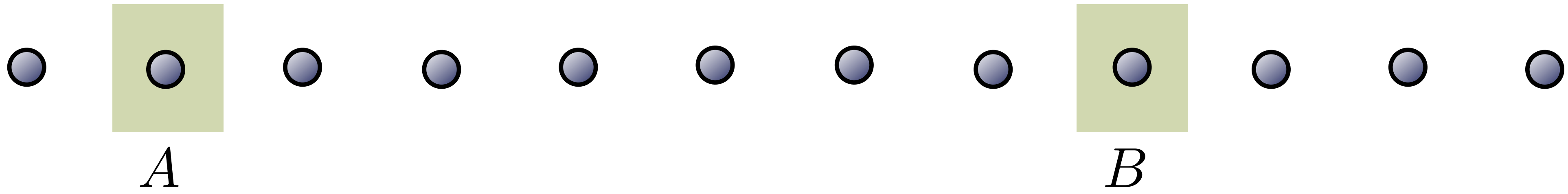
for $t \in [t_{\mathrm{rel}}, t_{\mathrm{rel}} + t_{\mathrm{rec}}]$

Cramer, Dawson, Eisert, Osborne, *Phys Rev Lett* **100** (2008), cond-mat/0703314



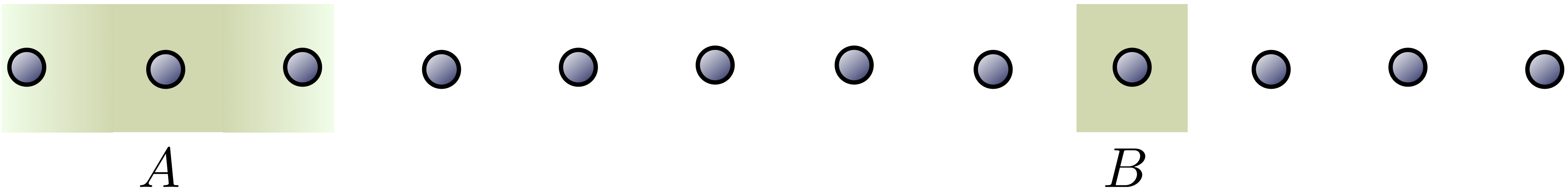
- **So, well, it does relax!**
- How can this be?

- Ideas of proof



- **Observation 1:** There is a **finite speed of information** transfer:

$$\|[A,B]\|_\infty = 0$$



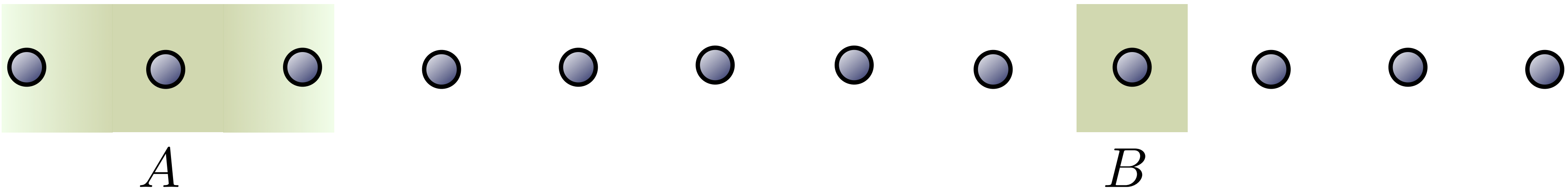
- **Observation 1:** There is a **finite speed of information** transfer:

> - **Lemma (Lieb-Robinson):** For any two (finite-dim) observables, on a finite support, $L = d(A, B)$ apart from each other, we have
>
> $$|||[A(t), B(0)]|||_\infty \leq c\|A\|_\infty \|B\|_\infty \exp(-\mu \mathrm{dist}(A, B) - v|t|)$$

$$A(t) = e^{iHt} A e^{-iHt}$$

v Speed of information transfer, H local Hamiltonian

Lieb, Robinson, *Commun Math Phys* **28** (1972)
Hastings, *Phys Rev Lett* **93** (2004)



- **Observation 1:** There is a **finite speed of information** transfer:

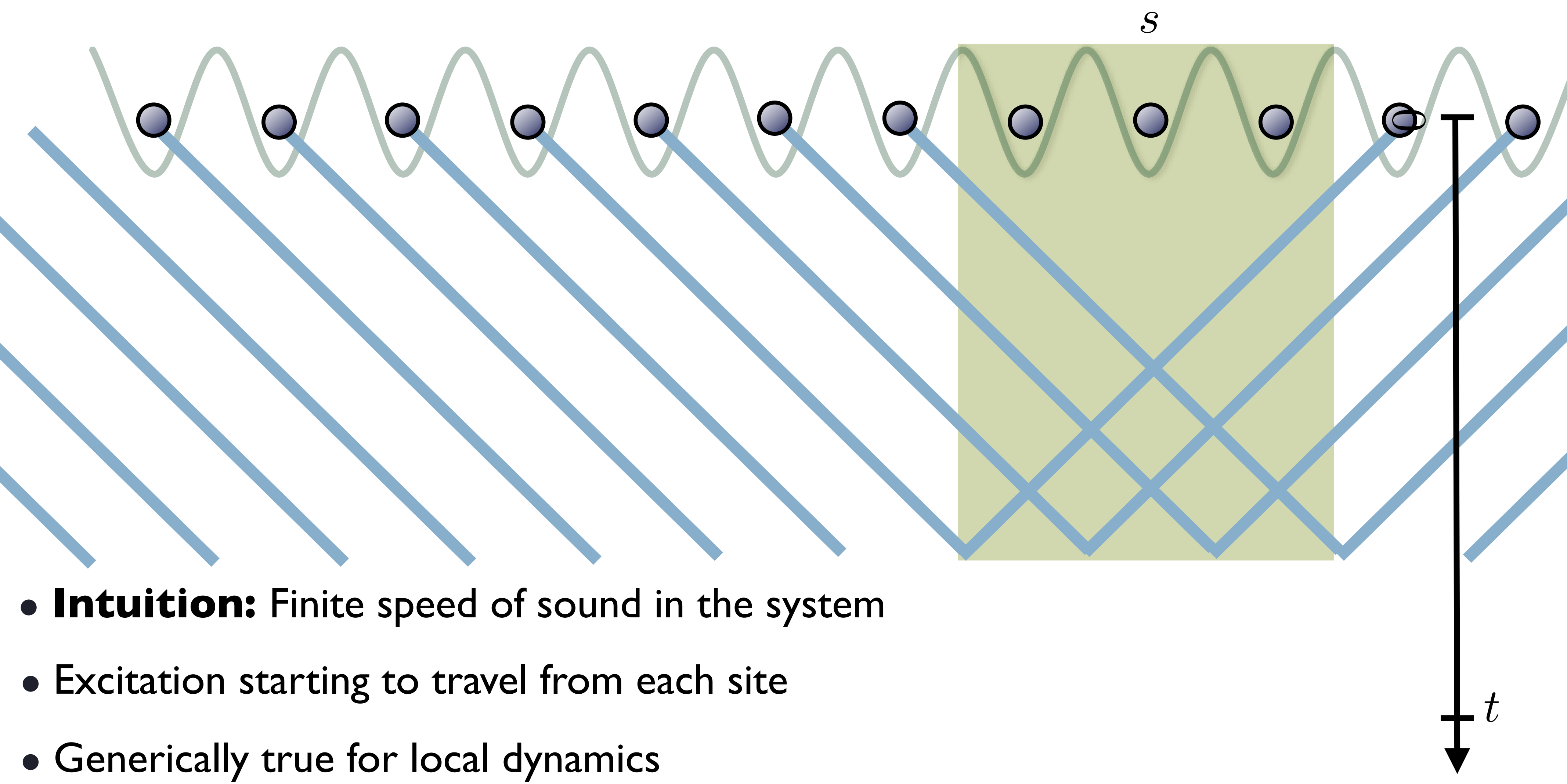
> - **Lemma (harmonic Lieb-Robinson):** Similar statements, e.g., for sites j, k
>
> $$\|[x_j(t), p_k(0)]\}_\infty \leq \frac{\tau^{\mathrm{dist(j,k)/R}} \cosh(\tau)}{(\mathrm{dist}(j,k) - 1)/2))!}$$

$\tau = \max\{\|PX\|_\infty^{1/2}, \|XP\|_\infty^{1/2}\}|t|$, X, P coupling matrices of local Ham

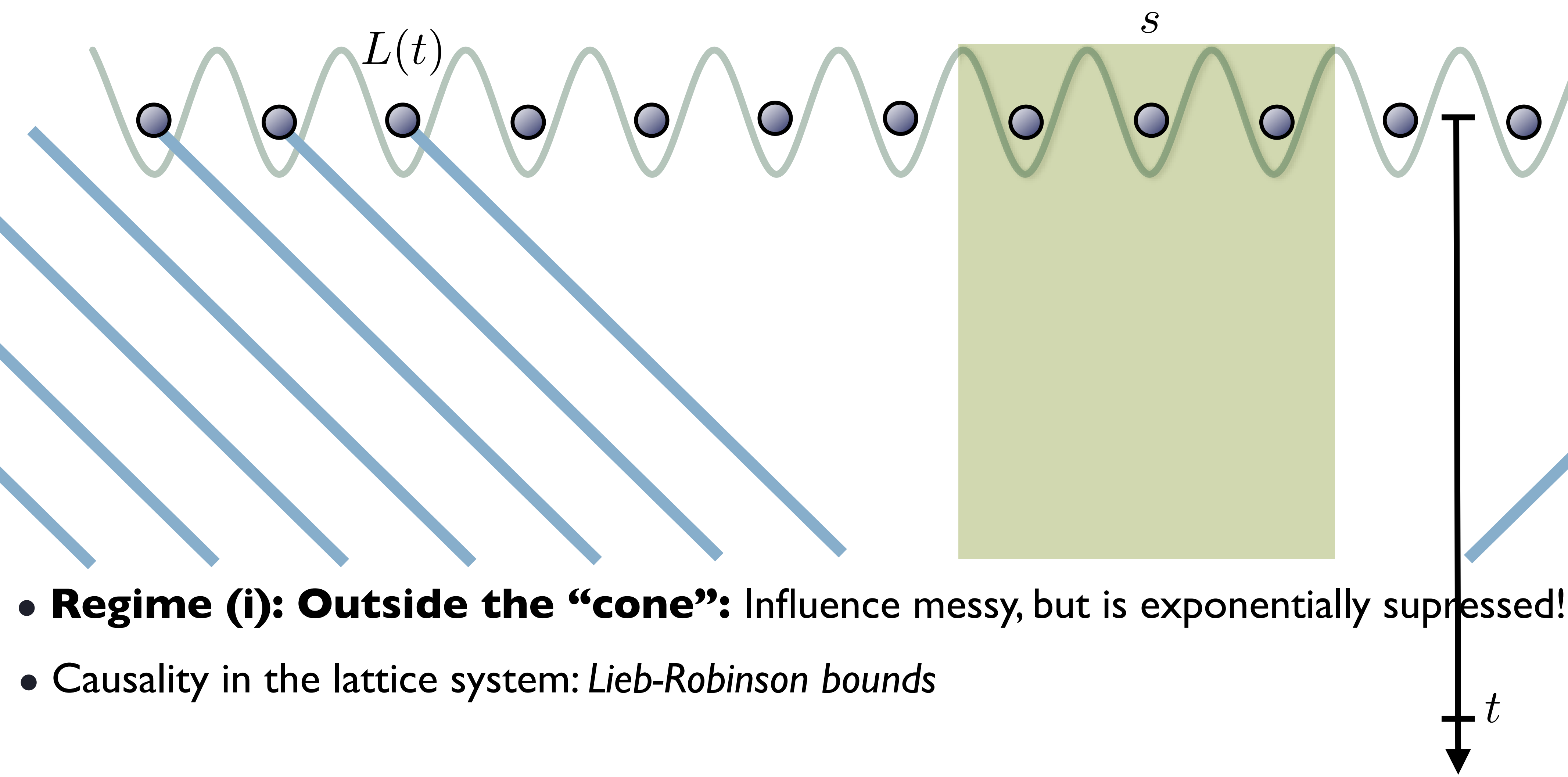
- Gives bounds for $\|[W_\xi(t), W_{\xi'}]\|_\infty$ for **Weyl-operators**

$$W_\xi = e^{i \sum_{j\in A} (p_j X_j - x_j P_j)} \, , \; \xi = (x_1, \dots, x_{|A|}, p_1, \dots, p_{|A|})$$

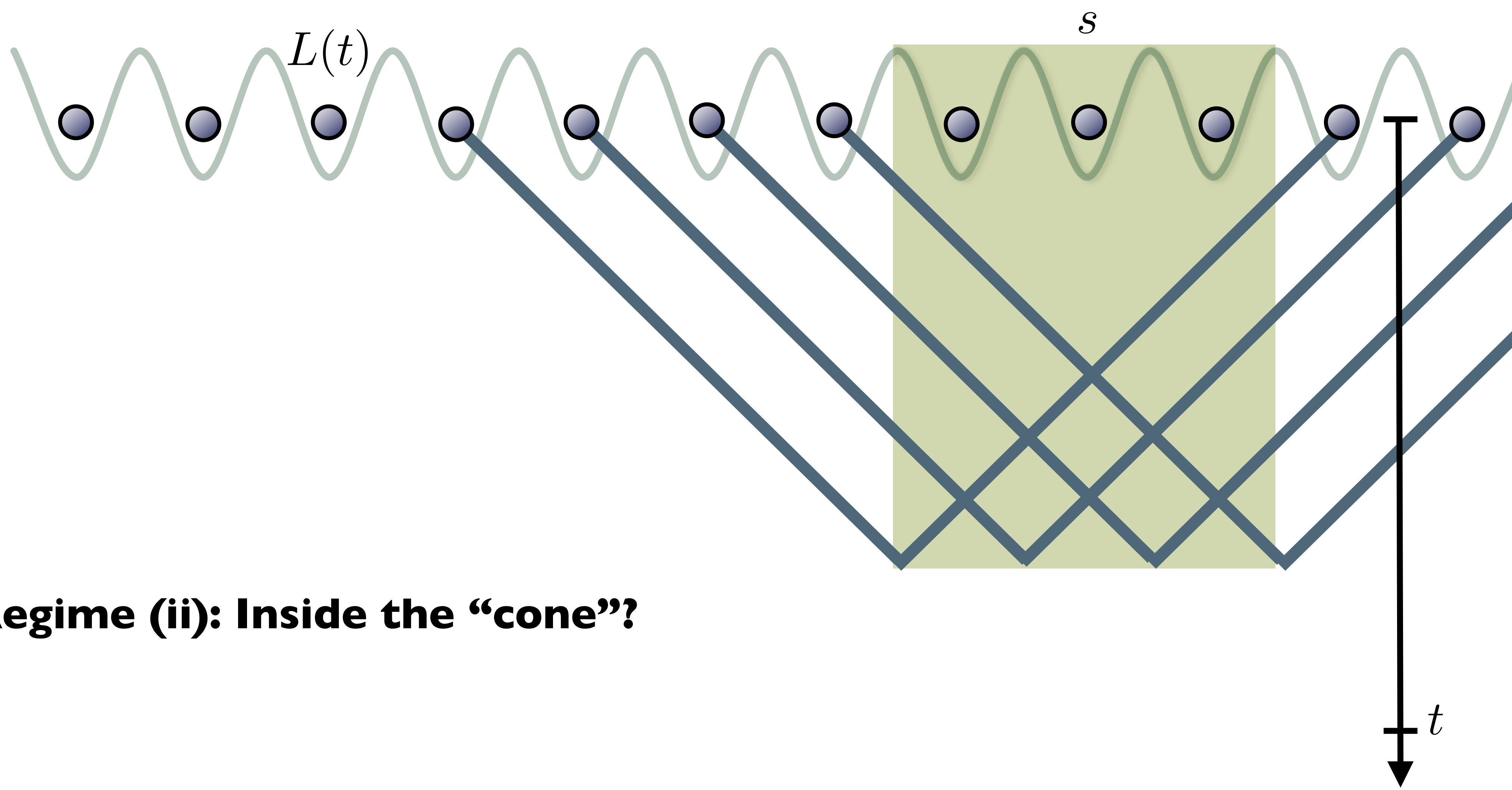
Cramer, Serafini, Eisert, arxiv:0803.0890
Nachtergaele, Raz, Schlein, Sims, arxiv:0712.3820
Buerschaper, Wolf, Cirac, in preparation



- **Intuition:** Finite speed of sound in the system
- Excitation starting to travel from each site
- Generically true for local dynamics



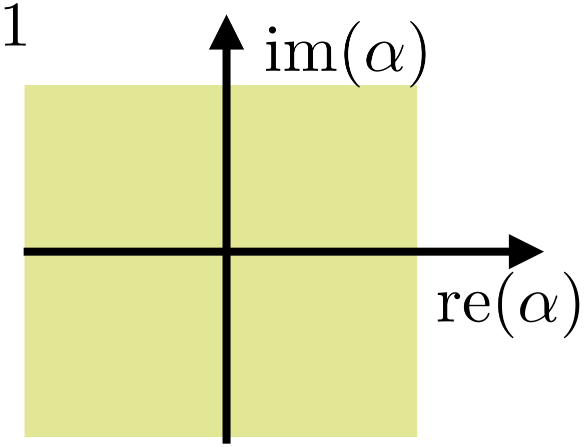
- **Regime (i): Outside the "cone":** Influence messy, but is exponentially supressed!
- Causality in the lattice system: *Lieb-Robinson bounds*



- **Regime (ii): Inside the “cone”?**

- **Phase space picture:**

- For simplicity, let us start from $|m\rangle^{\otimes N}$, and a single site $s=1$



- Characteristic function in phase space $\alpha \in \mathbb{C}$:

$$\chi_j(\alpha;t) = \prod_{k=1}^{N} \langle m| e^{\alpha [V(t)]^*_{j,k} b_k^\dagger - \alpha^* [V(t)]_{i,j} b_k} |m\rangle$$
$$= e^{-|\alpha|^2/2} \prod_{k=1}^{N} L_m(|\beta_{j,k}(t)|^2)$$

where $\beta_{j,k}(t) = \alpha [V(t)]^*_{j,k}$

- $V(t) = e^{-it\mathcal{J}}$, $V_{j,k}(t) = \dfrac{1}{N} \sum_{l=1}^{N} e^{2itJ\cos(2\pi l/N)} e^{2\pi i (j-k) l/N}$

- **Then, collect bounds:**
- For example

$$V_{j,k}(t) \to J_{j-k}(t) \quad \text{(Bessel functions, } |J_l(x)| < x^{-1/3},\ x \geq 0 \text{)}$$

bound Riemann sum error for small $(j-k)/N$

- Collect bounds on $V(t)$ from Lieb-Robinson bounds and properties of Laguerre polynomials
- Gives bounds on, say, $\displaystyle\sum_{k=1}^{N}\sum_{l=2}^{\infty}\frac{(1-L_m(|\beta_{j,k}|^2))^l}{l}$

- **Observation 2:** Take logarithm of characteristic function ($\beta_{j,k}(t) = \alpha[V(t)]^*_{j,k}$)

- **Lemma:**

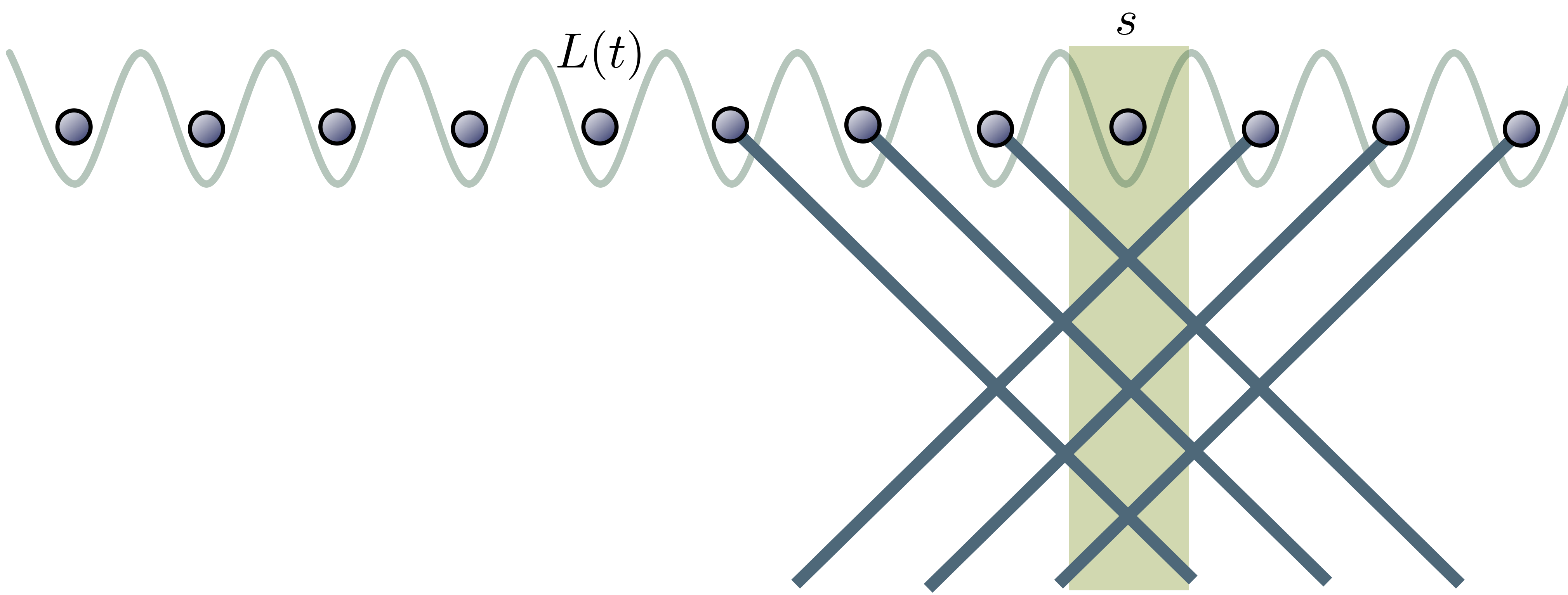
$$\log \chi_j(\alpha;t) = -|\alpha|^2/2 + \log \prod_{k=1}^{N} L_m(|\beta_{j,k}|^2)$$

$$= -|\alpha|^2/2 - m \sum_{k=1}^{N} |\beta_{j,k}|^2 + \sum_{k=1}^{N} l_m(|\beta_{j,k}|^2)$$

$$- \sum_{k=1}^{N} \sum_{l=2}^{\infty} \frac{\left(1 - L_m(|\beta_{j,k}|^2)\right)^l}{l}$$

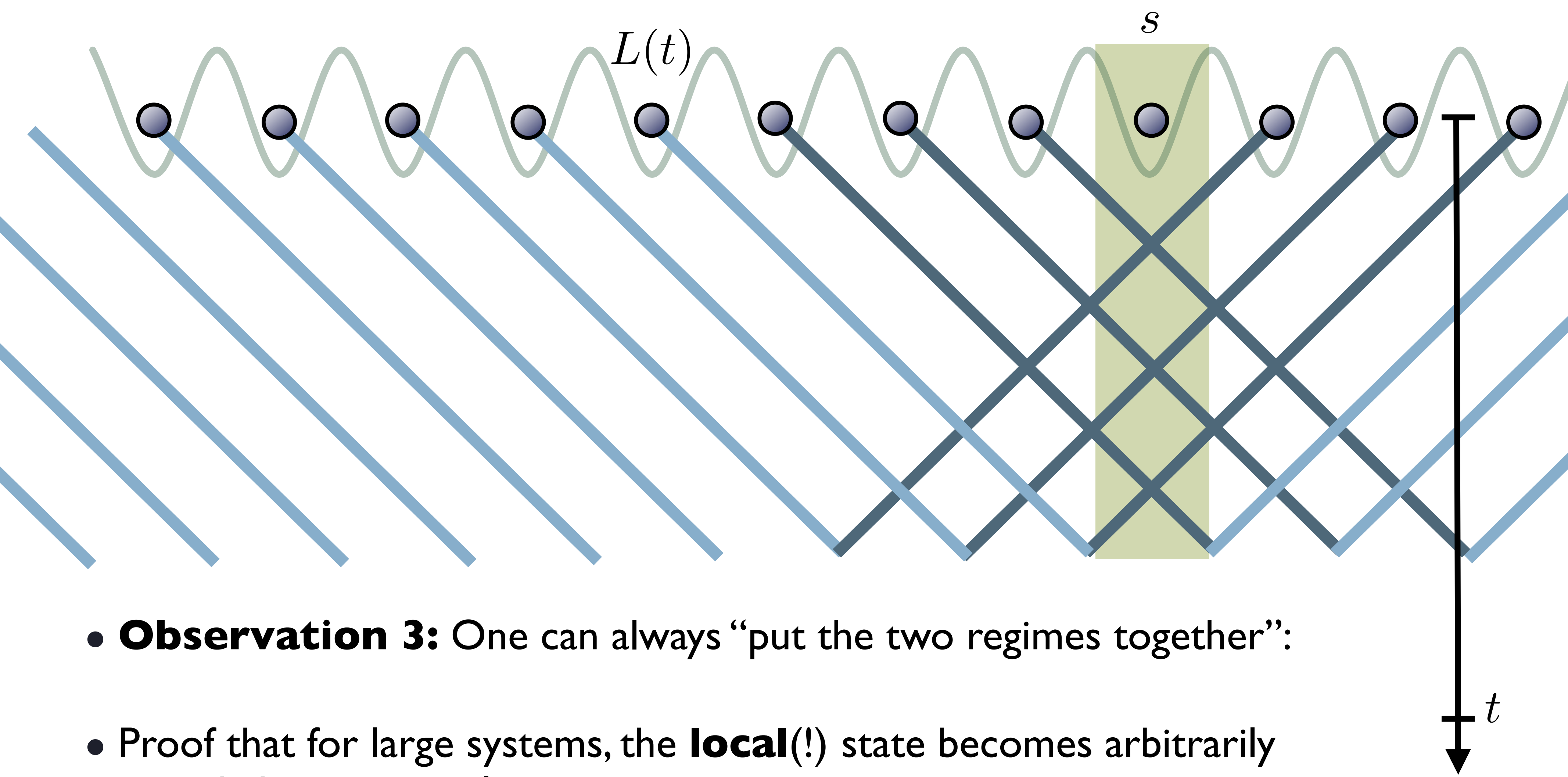
$$= -(m+1)|\alpha|^2/2 + g(\alpha;N,t)$$

- Only the **quadratic leading order term** remains for large times
- Dynamical **central limit theorem!**



$$\chi_j(\alpha;t) = e^{-(m+1/2)|\alpha|^2} + f(\alpha;t)$$

- **Lemma:** Pointwise convergence in phase space for $\chi_j(\alpha;t)$ gives **trace-norm** estimate $\|.\|_1$ for states
- State converges to a **Gaussian** inside the cone, thermal state: **Maximal entropy/ entanglement** for given **energy**

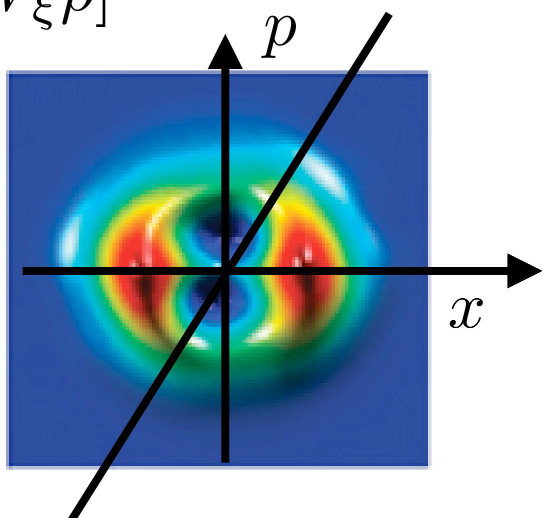


- **Observation 3:** One can always "put the two regimes together":
- Proof that for large systems, the **local**(!) state becomes arbitrarily mixed, the system relaxes

- **More generally**, covering case of clustering initial states:
- Ideas of a **"quantum version of Lindeberg's central limit theorem"**
- Starting point: Take vector $\xi = (x_1, \ldots, x_n, p_1, \ldots, p_n)$ in phase space

  then function $f : \mathbb{R} \to \mathbb{C}$ as $f(x) = \chi(\xi x)$

  is a *classical* characteristic function, when $\chi(\xi) = \mathrm{tr}[W_\xi \rho]$ is *quantum* characteristic function



Projection of Wigner function (Fourier transform of characteristic function) is probability distribution

- **More generally**, covering case of clustering initial states:
- Ideas of a **"quantum version of Lindeberg's central limit theorem":**
  Let $S_j \in \{1,\ldots,N\}$ be mutually disjoint subsets, and let

$\chi_j(x) = \langle\psi| \prod_{k\in S_j} W_{x\alpha_k}|\psi\rangle$ be four times continuously differentiable, with

$$\langle\psi|\Big(\sum_{k\in S_j}(\alpha_j a_j^\dagger - \alpha_j^* a_j)\Big)|\psi\rangle = \langle\psi|\Big(\sum_{k\in S_j}(\alpha_j a_j^\dagger - \alpha_j^* a_j)\Big)^3|\psi\rangle = 0$$

and $A_j := \Big|\langle\psi|\Big(\sum_{k\in S_j}(\alpha_j a_j^\dagger - \alpha_j^* a_j)\Big)^2|\psi\rangle\Big| \leq 1$

Then

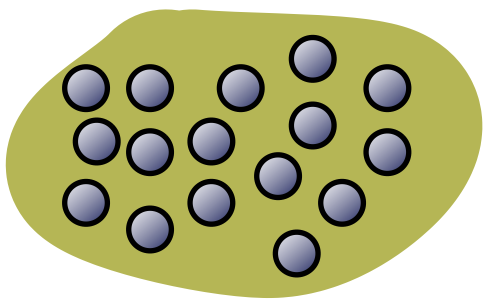
$$\left|\sum_{j=1}^N \log\langle\psi|\prod_{k\in S_j} W_{\alpha_j}|\psi\rangle - \frac{1}{2}\sum_{j=1}^N A_j\right| \leq \frac{1}{4}\sum_{j=1}^N (B_j + A_j \max_k A_k)$$

where $B_j := \Big|\langle\psi|\Big(\sum_{k\in S_j}(\alpha_j a_j^\dagger - \alpha_j^* a_j)\Big)^4|\psi\rangle\Big|$

- Hard work here: Show that quantum lattice system satisfies conditions

In preparation (2008)

- **Intuition for all that: Non-equilibrium dynamics:**

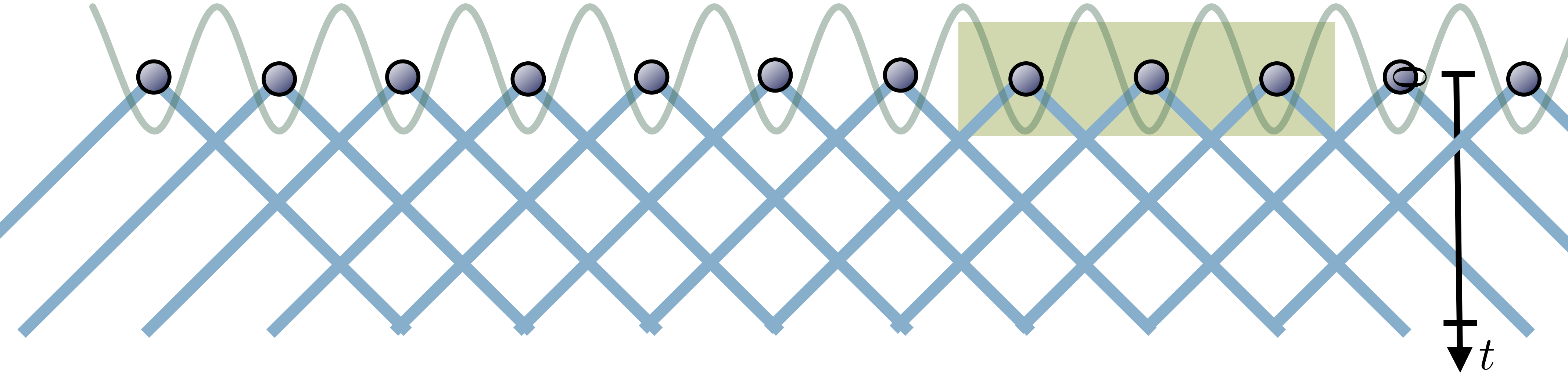


- **Globally**, the *information* of the initial condition is preserved at all times
- **Locally**, the system *looks exactly relaxed*, as if in a thermal state, without time average

- **More general results:**
  - With product states in any dimension
  - >1D any clustering initial state
  - Fermionic models

- Area laws for the entanglement entropy and hardness of simulation

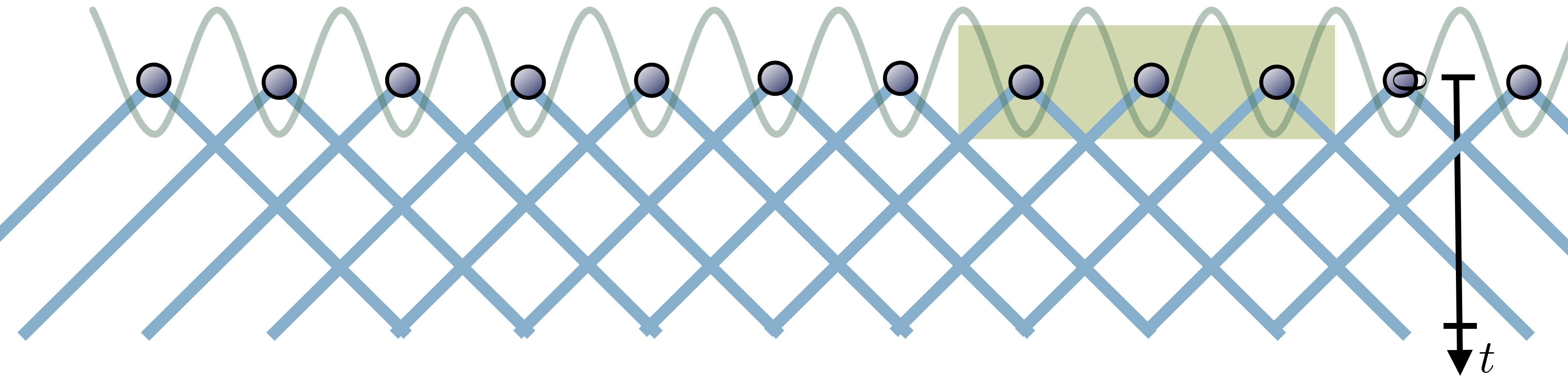
$$S(s) = -\mathrm{tr}[\rho_s(t) \log \rho_s(t)]$$



- **Area laws for entanglement entropy for quenched systems**
  (finite local dimension)

Eisert, Osborne, *Phys Rev Lett* **97** (2006)
Bravyi, Hastings, Verstraete, *Phys Rev Lett* **97** (2006)

$$S(s) = -\mathrm{tr}[\rho_s(t) \log \rho_s(t)]$$



- **Area laws for entanglement entropy for quenched systems:**

Any state $\rho(t) = e^{-itH}\rho(0)e^{itH}$, where H is a local 1D (finite-dim) Hamiltonian and $\rho(0)$ a product satisfies an **area law**:

$$S(s) \leq c_0 t + c_1$$

for some constants c_0, c_1

Proof: E.g., from Lieb-Robinson bounds and Weyl's perturbation theorem

Eisert, Osborne, *Phys Rev Lett* **97** (2006)
Bravyi, Hastings, Verstraete, *Phys Rev Lett* **97** (2006)

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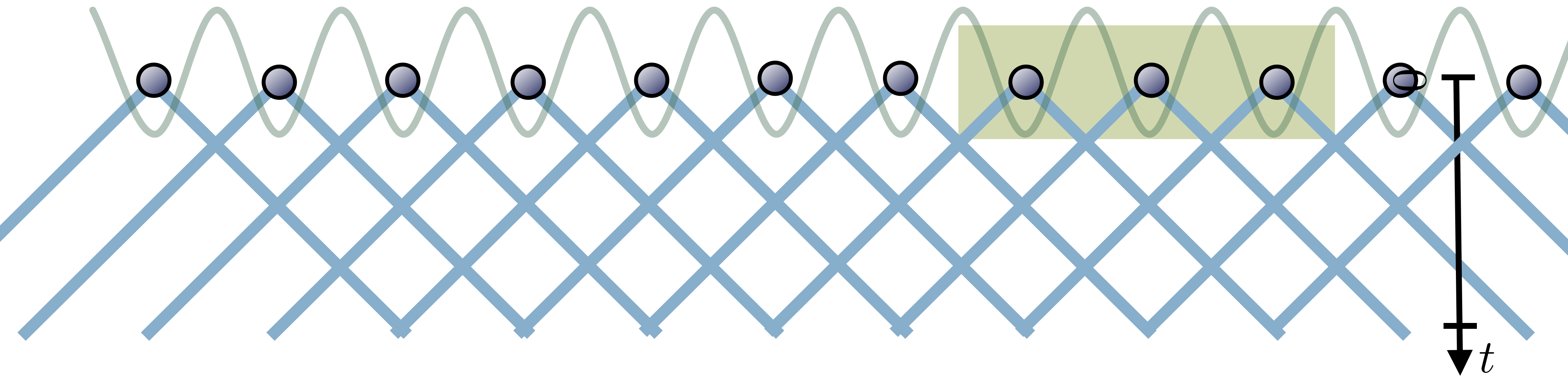


- Unfortunately, previous result shows that **bound is saturated:**

  Take initial product state, 1-norm convergence to Gaussian states implies for entanglement entropy (now for infinite-dim case)

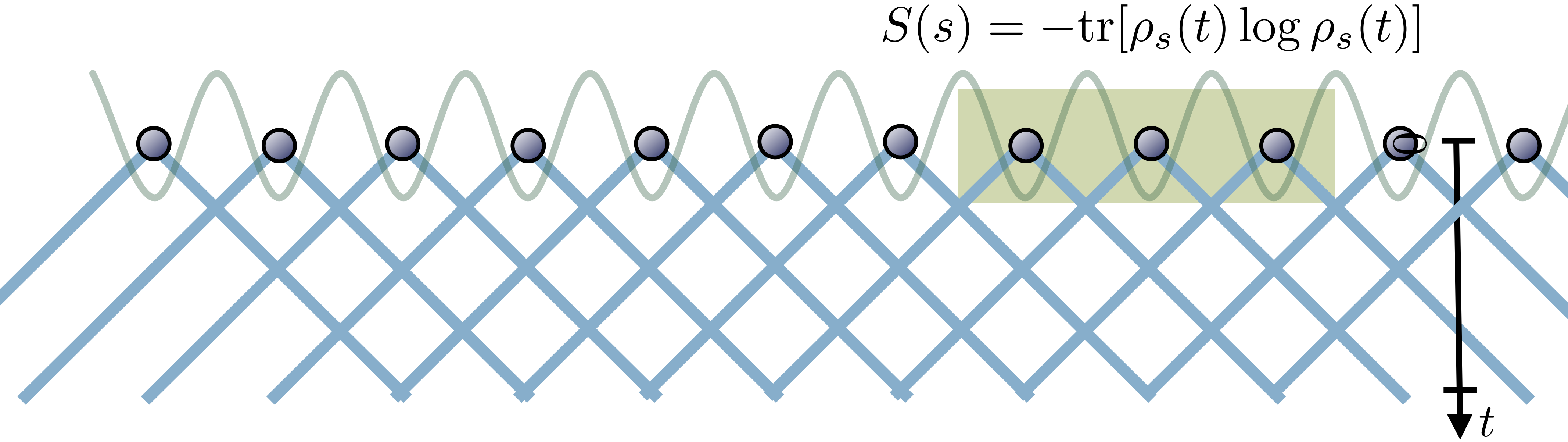
$$S_s(t) \geq c_0 t - f(t)$$

with $f(t) = o(t)$ for $s \geq ct$, $N \geq N_0$

Cramer, Dawson, Eisert, Osborne, *Phys Rev Lett* **100** (2008)

Alternative, more recent proof: Schuch, Wolf, Vollbrecht, Cirac, arxiv:0801.2078

See also Calabrese, Cardy, *J Stat Mech* P10004 (2007)



- No efficient **matrix-product state approximation** of state exists, so t-DMRG cannot work efficiently

$$|\psi\rangle = \sum_{i_1,\ldots,i_N=1}^{d} \mathrm{tr}[A^{(1)}_{i_1} \ldots A^{(N)}_{i_N}]\, |i_1, \ldots, i_N\rangle$$

Cramer, Dawson, Eisert, Osborne, *Phys Rev Lett* **100** (2008)
using Schuch, Wolf, Verstraete, Cirac, *Phys Rev Lett* **100** (2008)
Alternative, more recent proof: Schuch, Wolf, Vollbrecht, Cirac, arxiv:0801.2078
See also Calabrese, Cardy, *J Stat Mech* P10004 (2007)

- Experimental steps

- Very similar situation realizable in **experiment** (+thermal noise, harmonic trap)
- But, hardest thing to probe are **local quantitites**!

- Easier to probe in optical lattices: **Quasi-momentum distribution** from time of flight

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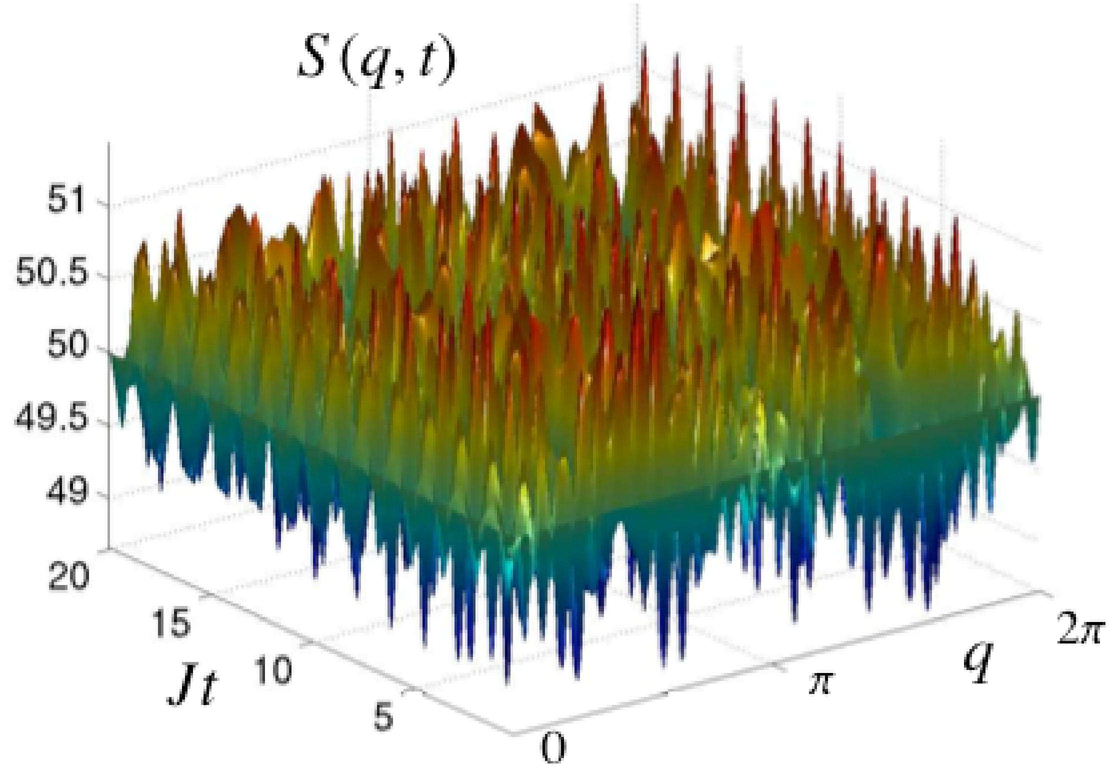
- This is good news, not bad news: Quantity is **"too global"**, shows memory of initial conditions"



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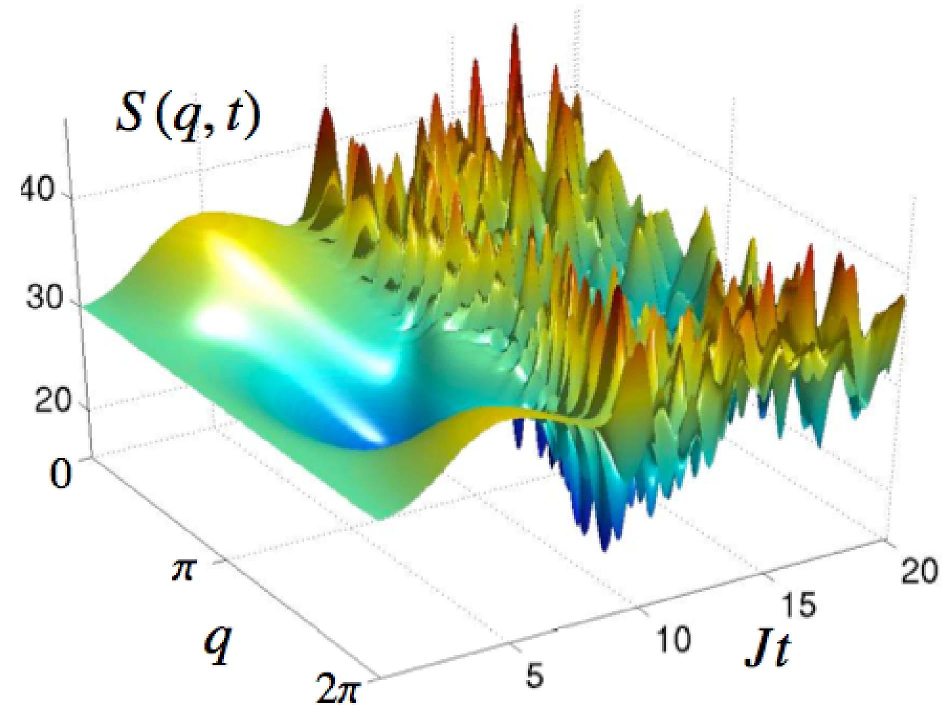
- Or, it probes the boundary conditions/harmonic confining potential (with harmonic trap):



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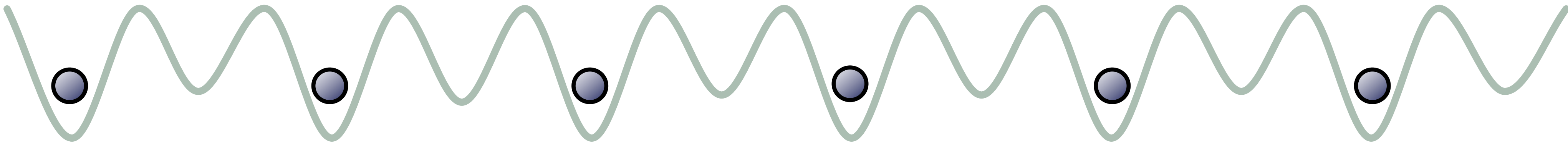
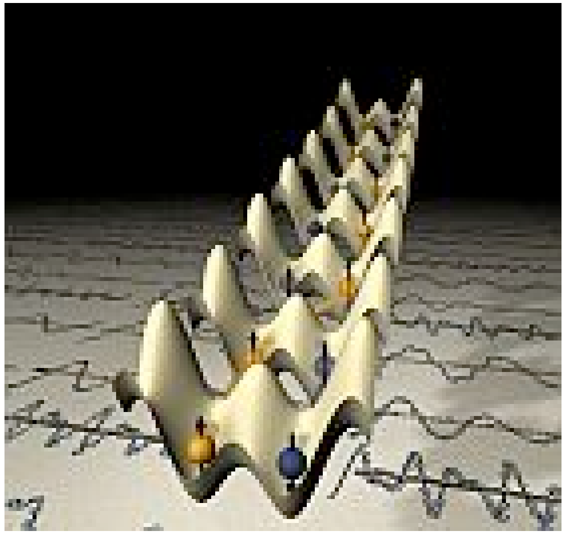
- But: Use **idea of superlattice**! Period 2 properties measurable





Eisert, Cramer, Flesch, Osborne, Schollwock in some order (2008)
Foelling, Trotzky, ... , Bloch, *Nature* **448** (2007)

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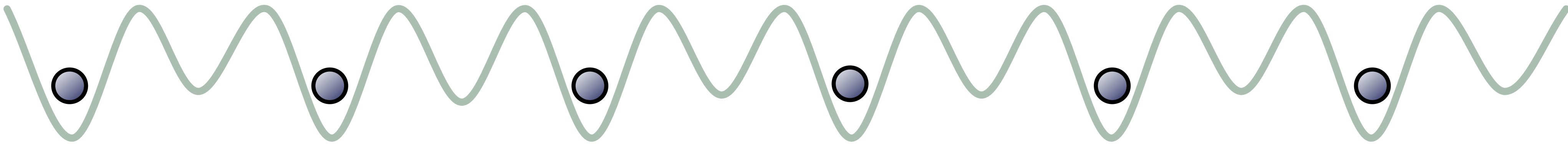
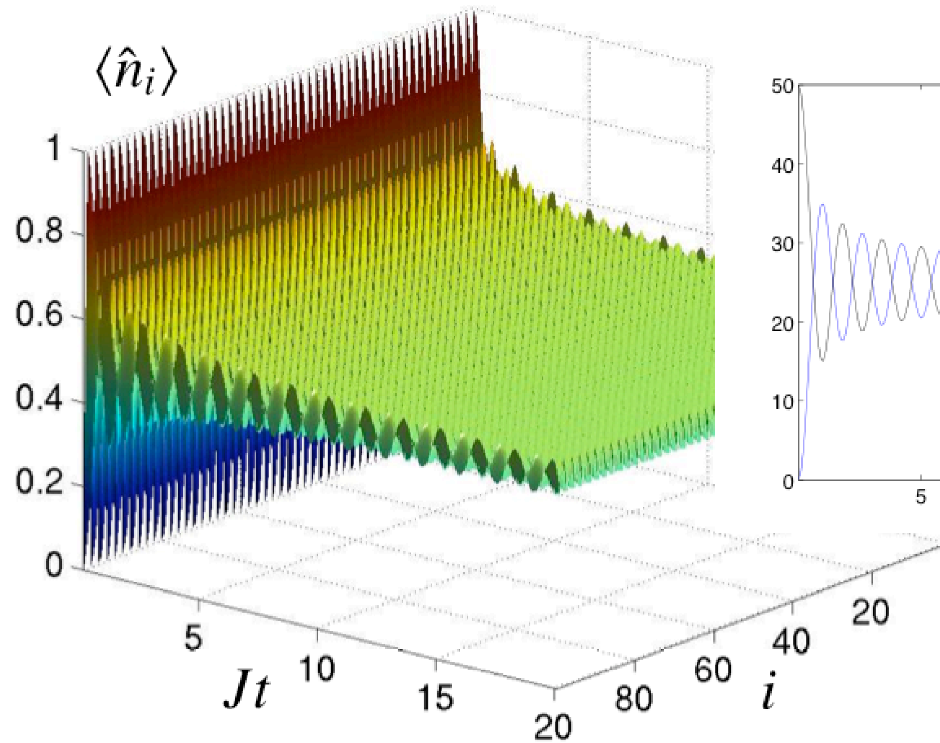
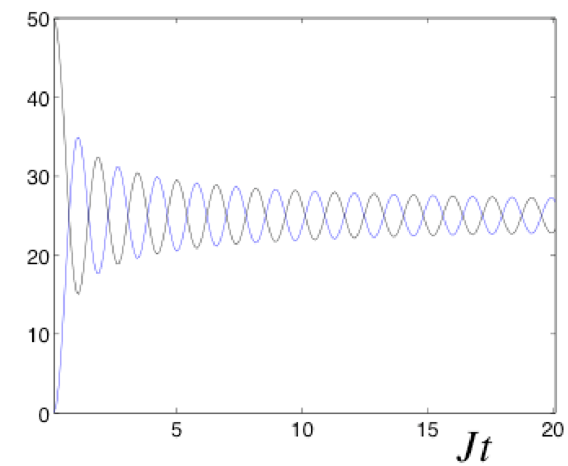


- Even-odd cut is just as if we had **local** dynamics:

  Relaxation as $t^{-1/3}$ for large times





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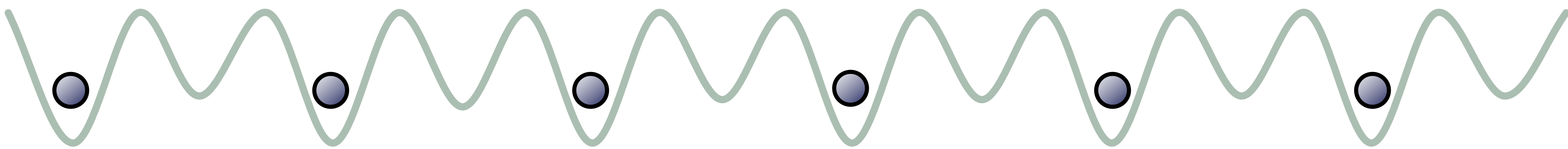
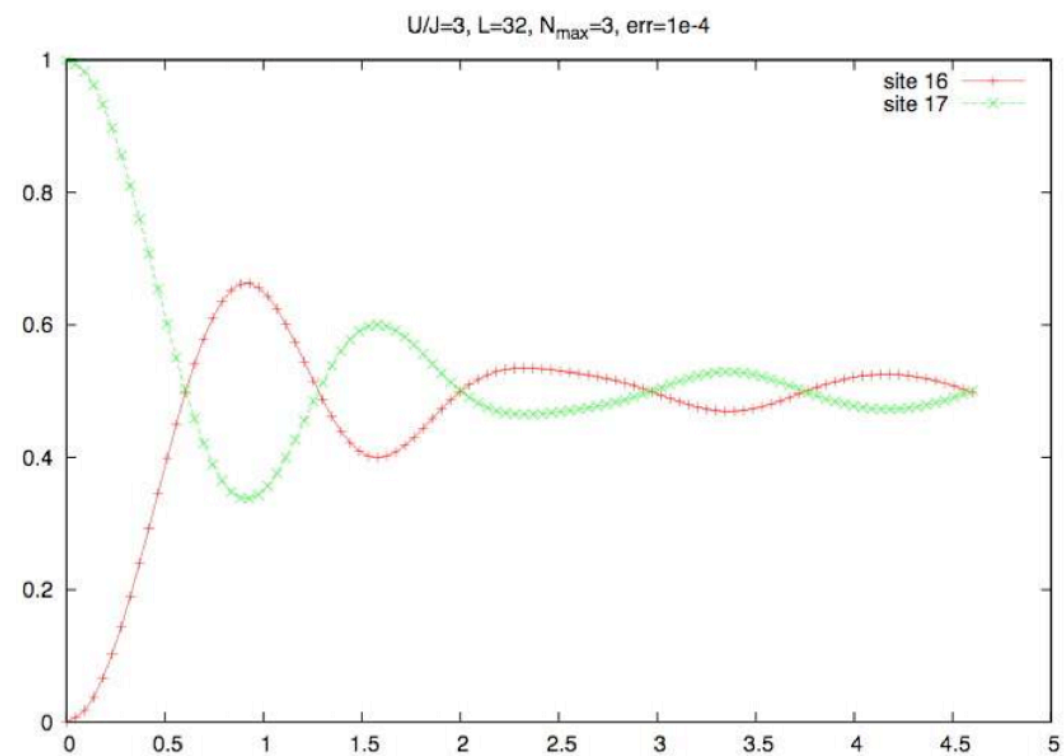


- Extensive numerical work with **t-DMRG:**

  Supports analytical findings in free case for realistic parameters for Rb in optical lattices



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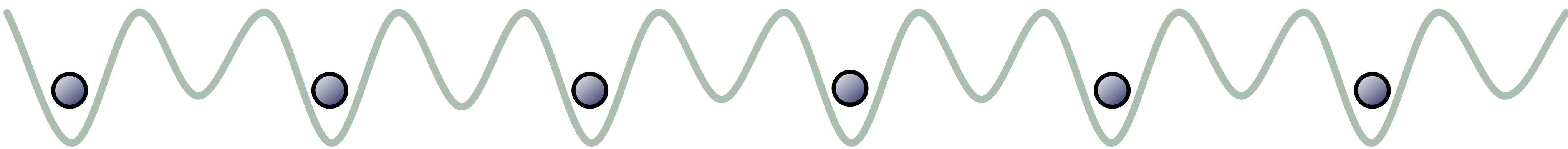
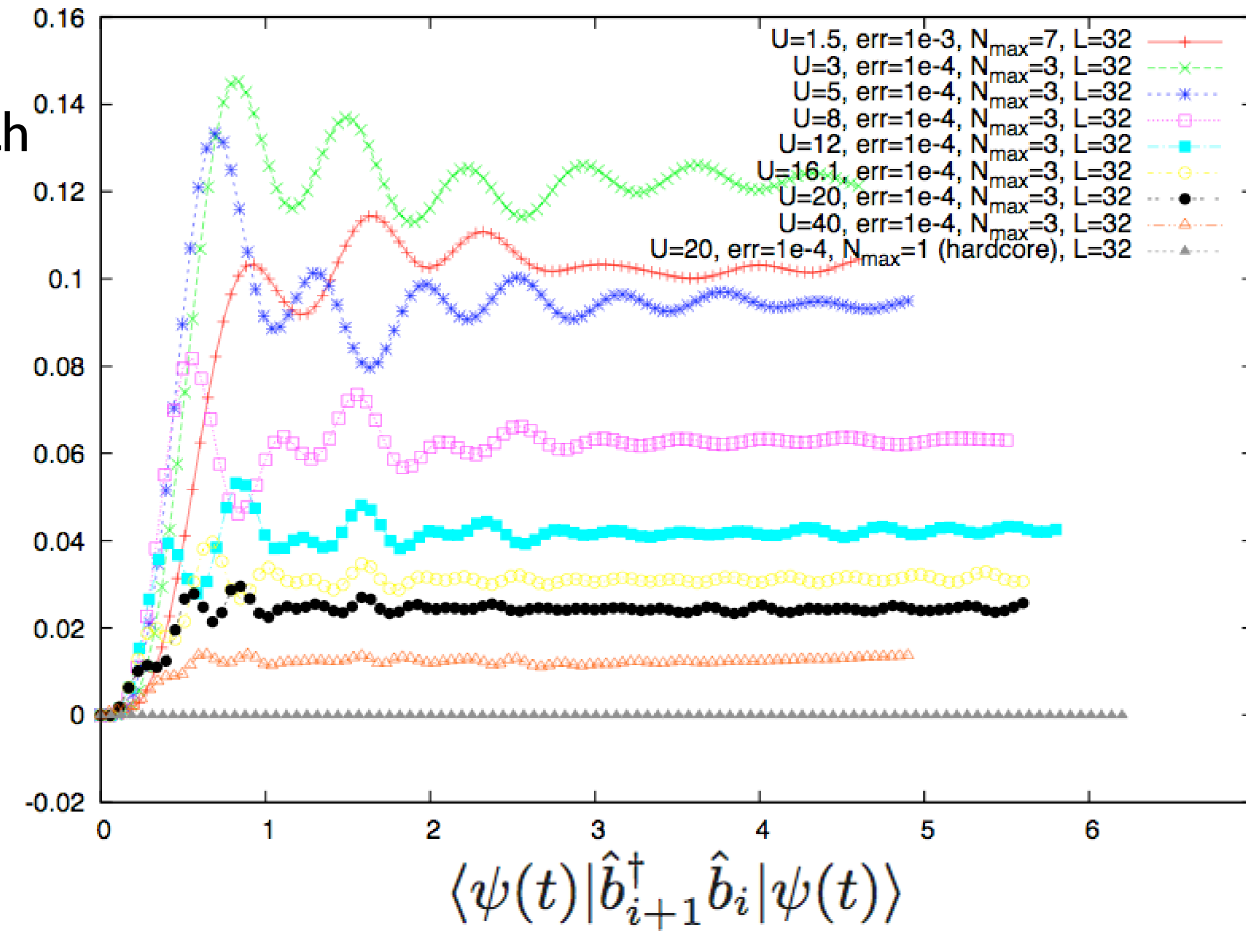
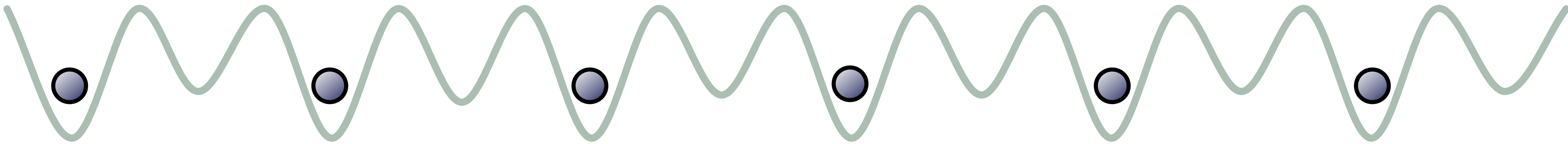


- Extensive numerical work with **t-DMRG:**

  Relaxation to thermal state, see $1/U$ dependence





- So again, experiment could probe non-equilibrium relaxation dynamics:

  - **Globally**, Fourier-transform type quantities do not relax due to *preserved information* of initial condition

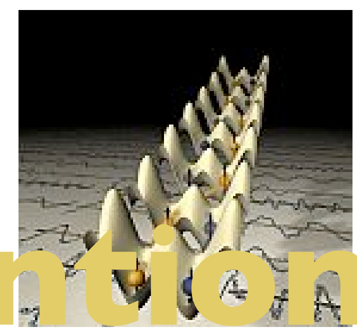
  - **Locally**, or at least with period-2 symmetry, the system *looks relaxed*, can also measure cumulants and higher moments

- **Experiment** using cold Rb atom is in progress this moment in Bloch's group

- Summary and outlook

- Have seen: Quantum lattice systems can **locally, exactly relax** without time average in quenched *non-equilibrium dynamics*

- **Ideas:** - Lieb-Robinson bounds
  - Quantum central limit theorems
- Area laws
- Steps towards experimental realization



Thanks for your attention!

- **Open questions:**

- More on interacting models?
- Relationship to kinematical approaches?
- Random unitaries, unitary 2-designs?
- Relationship to Ackermann numbers? :)