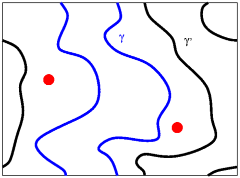
Insight on quantum topological order from finite temperature considerations



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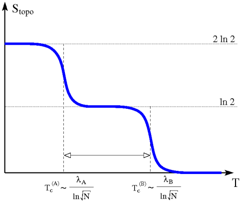

Conference on Classical and Quantum Information Theory,
Santa Fe NM, March 24, 2008

Outline

General context

A toy model: the toric code in two dimensions

Finite temperature behaviour

- thermal fragility in spite of gap "protection"
- classical origin of quantum topological information?

From qubits to pbits in three dimensions

Conclusions

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Topological phases of matter

Das Sarma, Freedman, Nayak, Simon, Stern 2007

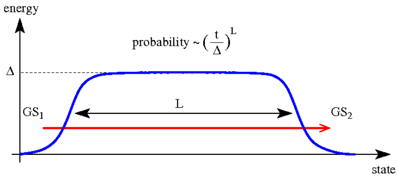
"A system is in a topological phase if, at low temperatures, energies, and wavelengths*, all observable properties (e.g. correlation functions) are invariant under smooth deformations (diffeomorphisms) of the spacetime manifold in which the system lives."

(i.e., *all observable properties are independent of the choice of spacetime coordinates*).

[*] "By 'at low temperatures, energies, and wavelengths', we mean that diffeomorphism invariance is only violated by terms which vanish as $\sim \max\{e^{-\Delta/T}, e^{-|x|/\xi}\}$ for some non-zero energy gap Δ and finite correlation length ξ."

How robust is really the topological protection?

Non locality $\Rightarrow$ *strong* protection from zero-temperature perturbations (non-local tunneling events are suppressed exponentially in system size)



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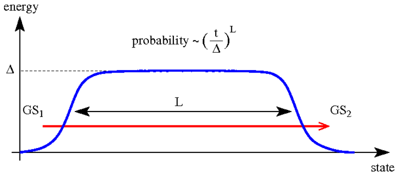
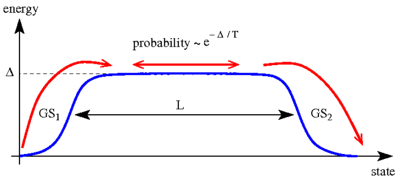




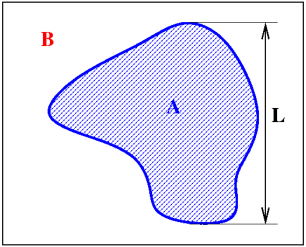
What about thermal fluctuations?

- *above* the energy barrier: no topological protection
- there is no local order parameter: 'ordinary' protection arguments (e.g., magnets) do *not* apply

The topological entropy (I)

Given a system described by the ground state density matrix $\rho = |\Psi_0\rangle\langle\Psi_0|$

$$
\begin{aligned}
\rho_{\mathcal{A}} &= \mathrm{Tr}_{\mathcal{B}}\, \rho \\
S_{\mathcal{A}} &= -\mathrm{Tr}\left[\rho_{\mathcal{A}} \log \rho_{\mathcal{A}}\right] \\
&= \alpha L - \gamma + \ldots
\end{aligned}
$$

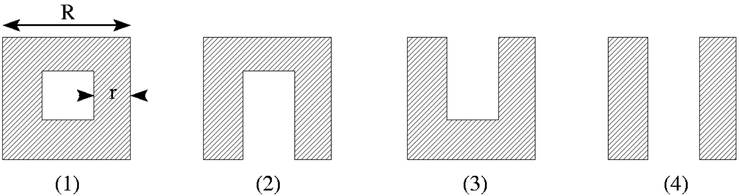


where $\gamma > 0$ is a universal constant of global origin, dubbed the topological entropy (Kitaev, Preskill 2006, Levin, Wen 2006)

The topological entropy (II)

Levin, Wen 2006

The topological contribution γ can be obtained directly by choosing the following four bipartitions



and taking an appropriate linear combination

$$S_{\text{topo}} \equiv \lim_{r,R\to\infty} \left(-S_{1\mathcal{A}} + S_{2\mathcal{A}} + S_{3\mathcal{A}} - S_{4\mathcal{A}}\right) = 2\gamma$$

where each term is given by the Von Neumann (entanglement) entropy of the corresponding bipartition

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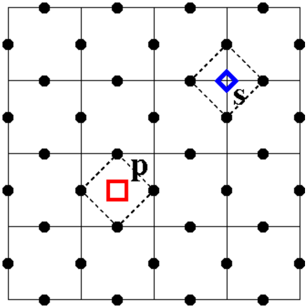
The toric code (I)

Kitaev 1997

$$
\begin{aligned}
H &= -\lambda_A \sum_s A_s - \lambda_B \sum_p B_p \\
B_p &= \prod_{j \in p} \hat{\sigma}_j^z \qquad A_s = \prod_{j \in s} \hat{\sigma}_j^x
\end{aligned}
$$

where λ_A, λ_B are real, positive parameters



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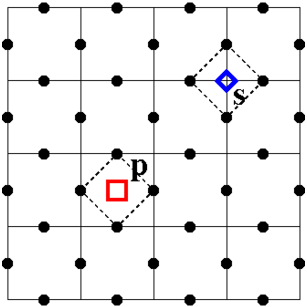


$$[B_p, B_{p'}] = 0 \qquad [A_s, A_{s'}] = 0 \qquad [A_s, B_p] = 0$$

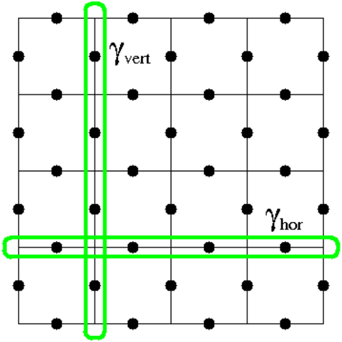
One can diagonalise the Hamiltonian at the same time as the $N-1$ independent B_p operators, and $N-1$ independent A_s operators (N being the number of sites).

The toric code (II)

The ground state is 4-fold degenerate $2^{2N-2(N-1)} = 4$,

$\Rightarrow$ four topological sectors identified by the eigenvalues of the non-local operators

$$\Gamma_{\mathrm{hor}} = \prod_{i \in \gamma_{\mathrm{hor}}} \hat{\sigma}_i^z \quad \Gamma_{\mathrm{vert}} = \prod_{i \in \gamma_{\mathrm{vert}}} \hat{\sigma}_i^z$$



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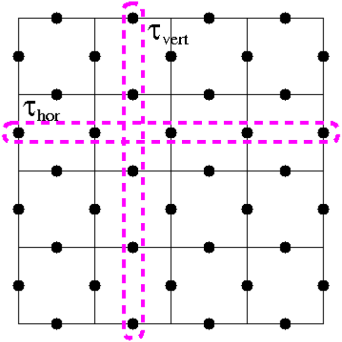

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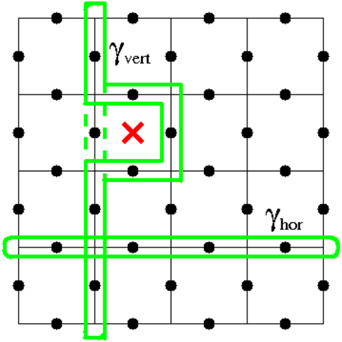

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Outline

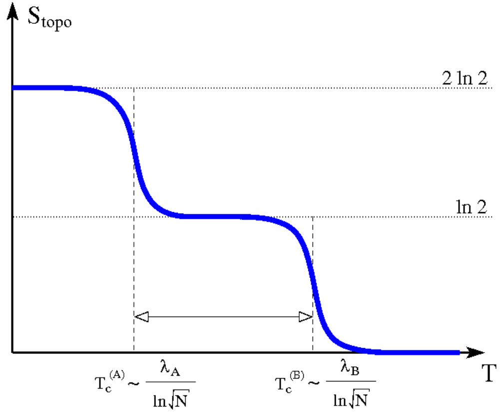
Finite T behaviour of the topological entropy (I)

$\rho(0) = |\Psi_0\rangle\langle\Psi_0| \longrightarrow \rho(T) = e^{-\beta H}/Z$, yet one can obtain an *exact expression* for both $S_{\mathcal{A}}$ and $S_{\rm topo}$ CC + Chamon 2007

- at finite system size N and temperature T, $S_{\rm topo}$ is a function of $N \ln[\tanh(\lambda_{A/B}/T)] \sim NT$, and the $N \to \infty$ and $T \to 0$ limits do not commute:
  - if the thermodynamic limit is taken first, the topological entropy vanishes identically
  - if the zero temperature limit is taken first, we recover the known result $S_{\rm topo}(T = 0) = 2 \ln 2$

Finite T behaviour of the topological entropy (II)

$$H = -\lambda_A \sum_s \prod_{j \in s} \hat{\sigma}_j^x - \lambda_B \sum_p \prod_{j \in p} \hat{\sigma}_j^z$$



(For finite N and assuming for simplicity $\lambda_A < \lambda_B$)

The vanishing occurs when the average number of defects in the system approaches one: $N\, e^{-\lambda_{A/B}/T} \sim 1$

Intuitive picture

order parameter to distinguish topological sectors:

at *zero* temperature $T = 0$:

$$\Gamma_0 = \left\langle \prod_{i \in \gamma} \hat{\sigma}_i^z \right\rangle_0 = \pm 1$$

independent of γ

$$\longrightarrow$$

at *finite* temperature T:

$$\Gamma(T) = \frac{1}{N_\gamma} \sum_{\{\gamma\}} \left\langle \prod_{i \in \gamma} \hat{\sigma}_i^z \right\rangle_T$$

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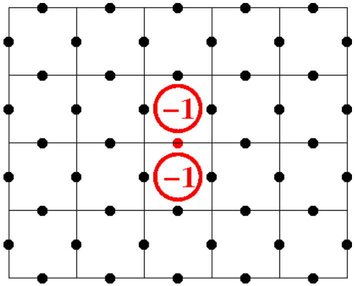
<u>low energy defects</u>:

plaquettes with $B_p = -1$

- they appear in pairs
- they are deconfined



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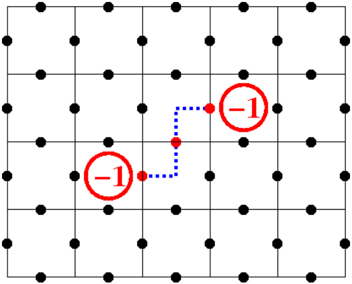

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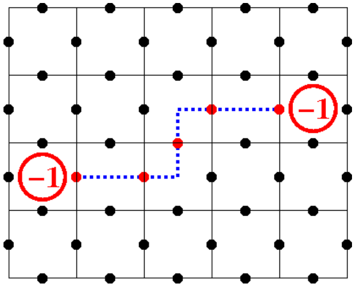

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independent of γ

$\longrightarrow$

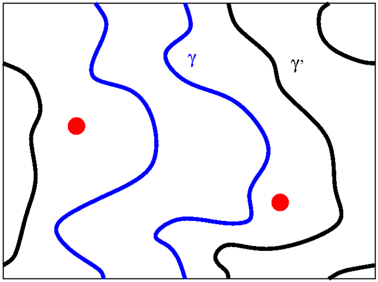
at *finite* temperature T:

$$\Gamma(T) = \frac{1}{N_\gamma} \sum_{\{\gamma\}} \left\langle \prod_{i\in\gamma} \hat{\sigma}_i^z \right\rangle_T$$

Two (deconfined) defects immediately spoil the order parameter: $\Gamma(T) \simeq 0$

$\Downarrow$

protection only if
NO defects at all!



Compare with a ferromagnetically ordered phase

at *zero* temperature $T = 0$:

$$M_0 = \langle \hat{\sigma}_i^z \rangle_0$$

independent of i

$\longrightarrow$

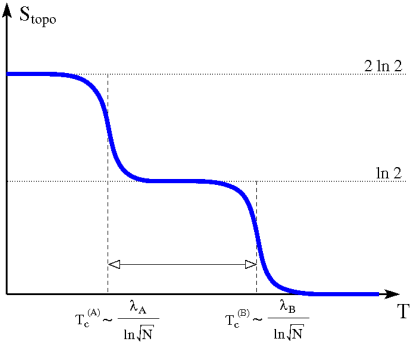
at *finite* temperature T:

$$M(T) = \frac{1}{N} \sum_i \langle \hat{\sigma}_i^z \rangle_T$$

Thermal fluctuations act on individual spins, and one needs a finite density of them to affect the order parameter, which is therefore exponentially protected: $\delta M \sim e^{-\Delta/T}$

Further considerations

- in the limit of $\lambda_B \to \infty$, $\lambda_A \to 0$ ($\lambda_B \to 0$, $\lambda_A \to \infty$), half of the topological entropy is preserved at finite T, and the system becomes a classical hard-constrained model $\Rightarrow$ classical topological order CC + Chamon 2006



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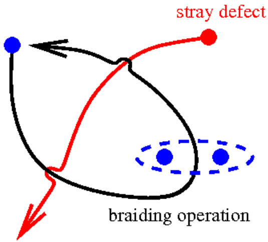
- the two contributions in the Hamiltonian (λ_A and $\lambda_{\mathcal{B}}$) are in fact additive:

$$S_{\mathcal{A}}(T) = S_{\mathcal{A}}^{(P)}(T/\lambda_B) + S_{\mathcal{A}}^{(S)}(T/\lambda_A)$$
$$S_{\rm topo} = S_{\rm topo}^{(P)}(T/\lambda_B) + S_{\rm topo}^{(S)}(T/\lambda_A)$$

The non-vanishing quantum topological entropy arises from the plaquette and star terms in the Hamiltonian as two equal and *independent* (i.e., classical) contributions

Conclusions from the 2D toric code

- topological order is fragile as compared to Landau-Ginzburg ordered systems with a gap
- in experiments, larger systems give higher topological quantum protection, but require $T \sim 1/\ln N$ to be safe from thermal fluctuations



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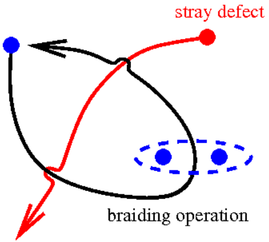

- the topological nature of this model has a classical origin

- quantum mechanics arises from the ability to create coherent superpositions

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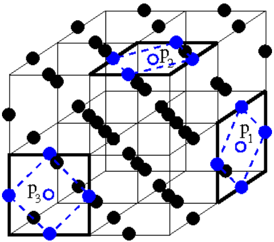
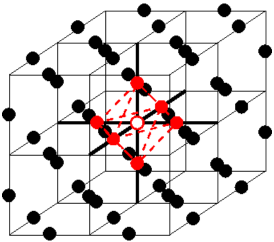
What about three dimensions? CC + Chamon (in preparation)

- on a simple cubic lattice, the plaquette operators remain planar (4-spin) terms, but the star operators become three-dimensional (6-spin) terms

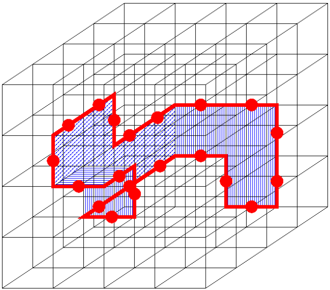


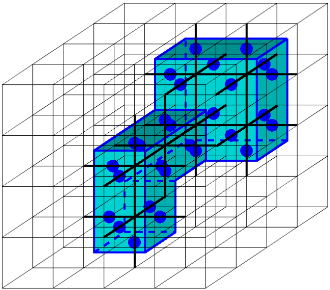


$$H = -\lambda_A \sum_s \prod_{i \in s} \hat{\sigma}_i^x - \lambda_B \sum_p \prod_{i \in p} \hat{\sigma}_i^z$$

Membranes vs Loops (I)

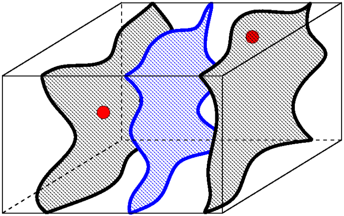
- on a simple cubic lattice, the plaquette operators remain planar (4-spin) terms, but the star operators become three-dimensional (6-spin) terms
- the dual loop structure is replaced by closed membranes on the dual lattice



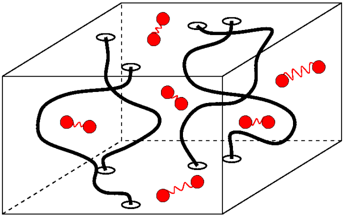


Membranes vs Loops (II)

- on a simple cubic lattice, the plaquette operators remain planar (4-spin) terms, but the star operators become three-dimensional (6-spin) terms
- the dual loop structure is replaced by closed membranes on the dual lattice
- non-local operators distinguishing the different sectors are winding loops and winding membranes

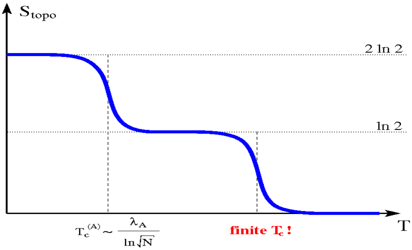


fragile



ROBUST!

Finite temperature behavior



The two contributions (λ_A and $\lambda_{\mathcal{B}}$) are again additive:

$$S_{\mathcal{A}}(T) = S_{\mathcal{A}}^{(P)}(T/\lambda_B) + S_{\mathcal{A}}^{(S)}(T/\lambda_A)$$

$$\Downarrow$$

Classical origin of topological information is confirmed (no need for hard constraints in 3D)

From qubits to pbits

The quantum topological information becomes probabilistic (classical) topological information at finite temperature:

- the loop sectors are still well defined
- loss of membrane contribution $\Rightarrow$ loss of coherence within each sector

$$|\Psi(T=0)\rangle = \psi_1 \sum_{\alpha\in 1} |\alpha\rangle + \psi_2 \sum_{\alpha'\in 2} |\alpha'\rangle + \ldots$$

$$T \neq 0 \quad : \qquad \begin{cases} \mathcal{P}_1(T) = |\psi_1|^2 \\ \mathcal{P}_2(T) = |\psi_2|^2 \\ \ldots \end{cases}$$

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- the quantum nature originates from the ability to create coherent superpositions

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Can we use our intuition on classical systems to engineer / investigate new classes of quantum topologically ordered systems?

What about quantum Hall states and other topologically ordered systems in the continuum?