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NUMERICAL MODELLING OF TWO-DIMENSIONAL GAS-DYNAMIC FLOWS ON A VARIABLE-STRUCTURE MESH*

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A method of computing complex two-dimensional gas-dynamic flows on variable-structure meshes is proposed. The medium is represented as a set of point particles and of domains surrounding the particles. These domains are so-called Dirichlet domains. Discretization of the equations is realized on a pattern formed from "Dirichlet neighbours". To construct the difference scheme, the method of support operators is used; with this method, completely conservative schemes can be obtained. Results are given of two-dimensional computations on modelling a Rayleigh-Taylor instability in a closed rectangular vessel.

Introduction.

The need to study complex gas-dynamic flows leads constantly to the development of new methods for performing computing experiments. At present, two main descriptions of the medium are used when modelling flows numerically: the Lagrangian approach in the case of relatively smooth flows /1-3/, and the Eulerian description /4/ for flows with strong

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deformations. At the same time, situations are often encountered in important practical problems in which it is desirable to retain the advantages of both approaches. The use of purely Lagrangian methods leads to strong distortion of the mesh cells, which creep into one another and overlap, thus making it impossible to continue the computation. The main drawback of the Eulerian approach is the difficulty of taking account of contact boundaries. There are various methods of quasi-Lagrangian type, the basic idea of which is to use meshes with Lagrangian nodes, whose connection is not fixed but can vary with time depending on the mutual disposition of the meshes. Such methods are discussed e.g., in /5, 6, 7, Chapter 11, 12, and 8-11/.

The present paper gives a method of computing complex two-dimensional flows on meshes of variable structure. The method has the following distinctive features. The medium is represented as a set of point particles (mesh nodes), which move along with the medium, and domains which surround the particles. These domains are Dirichlet domains or cells (see e.g., /5, 12/). With each particle there is connected a mass which is assumed Lagrangian and fixed in time. All the gas-dynamic quantities relate to particles. Given the configuration of the particles, the Dirichlet cells are constructed afresh at each instant. As a result of the construction of the Dirichlet cells, the "Dirichlet neighbours" are uniquely defined, i.e., the particles which are closest to the given particle in the geometric sense. These particles in fact form the pattern on which the equations of gas dynamics are discretized. The difference scheme is constructed by using the method of support operators /13-15/, which enables completely conservative schemes to be obtained. Matching of the properties of the difference analogues of the operators div and grad is performed both directly, on the basis of integral identities, and by using the variational approach /3/. When introducing the artificial viscosity needed to compute flows with shock waves, and in order to ensure stability of the scheme, the dissipative process is regarded as the consequence of inelastic collisions of particles. This way of introducing viscosity for Lagrangian schemes was proposed in /16/, but it is more natural when Dirichlet cells are used. A similar approach in the one-dimensional case was considered in /11/.

Our algorithm has the following merits. First, the volume of Dirichlet cells depends continuously on the particle coordinates, and remains unchanged, both at the instant when a particle enters the neighbourhood of the given particle, and when another particle leaves. The continuity of the volume determines the continuity of the density for the given particle. Another important property of the Dirichlet volume is that its derivatives with respect to the particle coordinates are likewise continuous. These derivatives appear in the difference equations, so that the acceleration, velocity, internal energy, and pressure are continuous. The pattern of the difference scheme adapts automatically to the solution, so that a reasonable degree of accuracy can be expected. The algorithm can be used to compute complex flows, without emergency situations arising.

The main drawback is that the Dirichlet cells are not Lagrangian; this was pointed out in /7, Chapter 11/. This implies the absence of local approximation for the equation of continuity. It must be said, however, that a similar situation holds for the equations of motion in the widely used Lagrangian schemes of /1, 17/. Also, it is well-known that, in spite of the absence of local approximation, convergence theorems /18/ have been proved for a number of schemes for Laplace's equation. We can expect that a similar situation will hold for the scheme used in our algorithm, though questions concerning accuracy require extra consideration. The next drawback is typical for schemes in which all the quantities are referred to the nodes: in the case of regular rectangular meshes, the scheme is a "through point" scheme, so that stability is reduced and the role of the chosen aritificial viscosity is increased.

1. General description of the algorithm.

The method is designed to solve the equations of gas dynamics in Lagrangian variables:

$$\frac{d\rho}{dt} + \rho \operatorname{div} \mathbf{W} = 0, \tag{1.1}$$

$$\rho \frac{d\mathbf{W}}{dt} - \operatorname{grad} p = 0, \tag{1.2}$$

$$\rho \frac{d\varepsilon}{dt} - p \operatorname{div} \mathbf{W} = 0, \tag{1.3}$$

$$\mathbf{W} = d\mathbf{r}/dt, \qquad F(p, \rho, \varepsilon) = 0. \tag{1.4}$$

where ρ is the density, $\mathbf{W}$ the velocity vector, $\mathbf{r}$ the radius vector, p the pressure, and ε the specific internal energy.

The medium is modelled by a set of point particles and domains connected with them. With each particle there are connected: the mass m_i, the density ρ_i, the components WX_i, WY_i of the velocity vector, the specific internal energy ε_i, the coordinates x_i, y_i, the pressure p_i, and the volume V_i of the corresponding domain. We assume that the particles move with the medium and that their mass remains unchanged in the computational process.

The domains connected with the particles are Dirichlet domains. For a given particle, the Dirichlet domain is defined as the subdomain of the (x, y) plane, all the points of which are closer to the given particle than are the remainder. The Dirichlet cells are convex polygons, which cover the (x, y) plane without gaps or overlaps.

By dividing the domain into Dirichlet cells we can introduce the concept of a neighbour. Given the particle, its neighbours are those whose Dirichlet cells have a common piece of boundary with the cell of the given particle. The neighbours form a pattern, on which Eqs. (1.1)-(1.4) are approximated for the given particle.

The method of support operators /13-15/ is used to construct the difference scheme. As applied to system (1.1)-(1.4), this amounts essentially to replacing the differential operators by their difference analogues DIV and GRAD. The latter are constructed in such a way as to satisfy difference analogues of certain integral identities which are satisfied by the initial differential operators. By thus choosing DIV and GRAD, we can obtain completely conservative differential-difference schemes, where time remains continuous. The methods described in /1/ can be used to construct the completely discrete model when approximating in time.

To be able to compute flows with shock waves, artificial viscosity is introduced into the difference equations. The dissipative process is regarded as a consequence of inelastic collisions of the given particle with its neighbours.

2. Dirichlet cells: properties and method of construction.

1. Suppose we are given in the plane a set of points $\{P_i\}$, $i=1, 2, \ldots, N$. The Dirichlet domain or cell for point P_k is then the domain V_k, any point of which is closer to P_k than are other points of the set $\{P_i\}$.

Note that the Dirichlet cells may be unbounded for certain points. If the computational domain D is bounded, we understand by the Dirichlet cells in it the intersections of the domains V_k with D. Then, those Dirichlet cells which were unbounded on the entire plane, become bounded and part of their boundary consists of pieces of the boundary of D. It will be assumed for simplicity throughout that D is a rectangle with sides parallel to the coordinate axes. In Fig.5 of Sect.5 we show an example of a Dirichlet cell for an actual problem. Given a point P_k, its neighbours are by definition the points P_i for which $\bar{V}_i \cap \bar{V}_k \neq \varnothing$, where $\bar{V}_i$ and $\bar{V}_i$ are the closures of the respective sets. In para.3 of the present section we describe an efficient algorithm for constructing Dirichlet cells and the corresponding lists of neighbours.

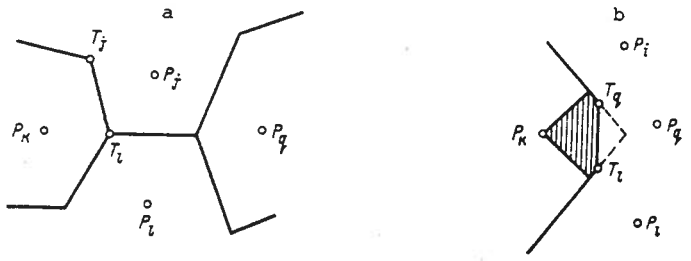

Fig.1

2. Let us consider some geometric properties of Dirichlet cells. First, these cells are convex. Let us see how the coordinates of the vertices of the domain V_k are expressible in terms of the coordinates of the neighbours of the point P_k. We order the neighbours according to angle in the counter clockwise direction. Those neighbours are called consecutive, which stand in such an ordered series. We denote each vertex of V_k by the letter T with the index of the neighbour standing ahead of it when point P_k is circulated counter clockwise. The vertices of V_k are the centres of circles described about the triangles whose vertices are point P_k and two of its consecutive neighbours. In the situation of Fig.1,a, the vertex T_l is the centre of the circle through points P_k, P_l, P_j. The coordinates of the vertex T_l are expressible as follows in terms of the coordinates of P_k, P_l, P_j:

$$x_{T_l} = 0.5\,[A^1(y_{P_l} - y_{P_j}) - A^2(y_{P_l} - y_{P_k})]/D, \tag{2.1a}$$

$$y_{T_l} = 0.5\,[A^1(x_{P_l} - x_{P_j}) - A^1(x_{P_l} - x_{P_j})]/D, \tag{2.1b}$$

where $A^1 = x^2_{P_l} - x^2_{P_k} - y^2_{P_l} - y^2_{P_k}$, $A^2 = x^2_{P_l} - x^2_{P_j} + y^2_{P_l} - y^2_{P_j}$, $D = (x_{P_l} - x_{P_k})(y_{P_l} - y_{P_j}) - (x_{P_l} - x_{P_j})(y_{P_l} - y_{P_k})$.

An important property of the volume (also denoted by V_k of a Dirichlet cell, is its continuous dependence on the position of the point. A second important property of the Dirichlet volume is that the derivatives $\partial V_k/\partial x_{P_q}$, $\partial V_k/\partial y_{P_q}$ are continuous, where q runs over all values from 1 to N.

Let us prove these properties. It must first be remarked that, from the formal standpoint, the volume V_k is a function of several variables, namely, of the coordinates of all points. We know that the differentiability, and hence continuity, of a function of several variables follows from the existence and continuity of the derivatives with respect to the individual arguments, so that it suffices to prove the continuity of the derivatives $\partial V_k/\partial x_{P_q}$, $\partial V_k/\partial y_{P_q}$. The expressions for these derivatives are obtained by direct differentiation, in the light of the fact that only part of the volume V_k depends on the position of the point P_q (see Fig.1,b). First, the expressions for the derivatives were obtained manually, then they were checked by using the REDUCE system of analytic transformations (see e.g., /19/). Let us quote the expression for $\partial V_k/\partial x_{P_q}$ (the derivative with respect to y is computed in the same way):

$$\frac{\partial V_k}{\partial x_{P_\eta}} = (y_{T_\eta} - y_{T_l})\,\frac{x_{P_\eta} - (x_{T_l} + x_{T_\eta})/2}{x_{P_\eta} - x_{P_k}}. \tag{2.2}$$

In the case $x_{P_q} \neq x_{P_k}$ it follows directly from (2.2) that the derivative is continuous, if P_q is a neighbour and remains a neighbour when its coordinates vary with a small neighbourhood. If P_q is not a neighbour of P_k, then $\partial V_k/\partial x_{P_q}=0, \partial V_k/\partial y_{P_q}=0$.

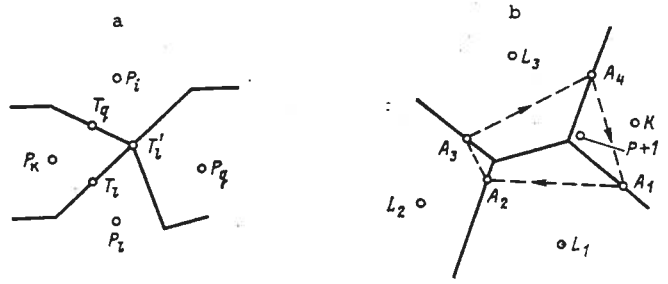

Fig.2

It remains to show that $\partial V_k/\partial x_{P_q}=0$ when the cells V_k and V_q are in contact with a common vertex, or in other words, when the point P_q lies in the circle passing through the points P_k, P_l and P_i. Since, in this case, the points T_l and T_q move to the same point T_l' (Fig.2,a), then $y_{T_l} \to y_{T_q}$ and the required equation follows from (2.2). The case $x_{P_q}=x_P$ is treated similarly, on noting that, since P_kP_q and T_qT_l are mutually perpendicular, we have $(y_{T_q}-y_{T_l})/(x_{P_q}-x_{P_k})=-(x_{T_q}-x_{T_l})/(y_{P_q}-y_{P_l})$, and hence (2.2) can be reduced to the form

$$\frac{\partial V_k}{\partial x_{P_q}} = -(x_{T_q}-x_{T_l})\frac{x_{P_q}-(x_{T_l}+x_{T_q})/2}{y_{P_q}-y_{P_k}}.$$

Now consider how the derivative $\partial V_k/\partial x_{P_k}$ behaves. Since the volume of the Dirichlet cell remains unchanged when all points are given a parallel shift, we can write

$$V_k(x_{P_k}-t, x_{P_q}+t, x_{P_l}+t, \ldots, y_{P_k}, y_{P_q}, y_{P_l}, \ldots) = \text{const},$$

so that

$$\frac{\partial V_k}{\partial t} = \sum_{i \in \Pi(k)} \frac{\partial V_k}{\partial x_{P_i}} = 0, \tag{2.3}$$

where the pattern $\Pi(k)$ consists of the point k itself and all its neighbours. A similar relation holds for $\partial V_k/\partial y_{P_k}$. From (2.3) we have

$$\frac{\partial V_k}{\partial x_{P_k}} = -\sum_{i \in \Pi'(k)} \frac{\partial V_k}{\partial x_{P_i}}, \tag{2.4}$$

where the pattern $\Pi'(k)$ consists of neighbours of the point k. By (2.4), the derivative $\partial V_k/\partial x_{P_k}$ is continuous.

3. The algorithm for constructing the Dirichlet cells in fact reduces to finding the neighbours for each point. When the neighbours are found, the coordinates of the vertices of the Dirichlet cells are given by (2.1). The main ideas underlying our method are taken from /12/.

Assume that the set of points considered is numbered from 1 to N in some way. We will first consider separately the first P points on the assumption that neighbourhoods are established between them while disregarding the other points. Assume that the numbers of the neighbours, for each point, are stored in a special list and are ordered according to angle, counter-clockwise. The point with number $P+1$ is then brought into consideration, and the set of first $P+1$ points is considered. We then have to find the neighbours for point number $P+1$ from among the first points and correct the list of neighbours for the first P points. We shall describe this procedure in the case when the Dirichlet cells for the $P+1$ points do not contain, as pieces of boundary, pieces of the sides of the bordering rectangle.

That one of the first P points which is closest to point $P+1$ is first found; call it K (see Fig.2,b). We then drop a perpendicular through the mid-point of the segment joining points K and $P+1$, and seek the intersection of this perpendicular with a side of the Dirichlet cell for point K, as shown in Fig.2,b, i.e., on moving along the perpendicular to the point of intersection, which we denote by A_1, point $P+1$ remains on the right. The side intersected by the perpendicular uniquely defines one of the neighbours of the point, call it L_1. Point L_1 is put first in the list of neighbours for the point $P+1$. We then drop a perpendicular through the mid-point of the segment joining points $P+1$ and L_1. This perpendicular cuts two sides of the Dirichlet cell for point L_1. One of the points of intersection is the same as A_1, since A_1 is the centre of the circle passing through points L_1, K, and $P+1$.

The second point (call it A_2) defines the number of the next neighbour of point $P+1$.

call it L_2. In order to order counter-clockwise, this number is placed first in the list of neighbours of point $P-1$, while number L_1 is placed second. This process is continued until the Dirichlet cell of point $P-1$ is closed. The list of neighbours of the new point $P-1$ is thus constructed.

Now consider how the lists of neighbours of other points must be corrected. First, we change the list of neighbours of only those points which were in the neighbourhood of the point $P-1$. Say we have to correct th list of point L. We first see which of the two sides are cut by the perpendicular dropped through the mid-point of segment $L, P+1$. Let N_1, N_2, be the numbers of the respective neighbours. Then, from the old list of neighbours of point L, we remove all the neighbours whose numbers lie in the list between N_1 and N_2, and in their place we introduce the neighbour with number $P+1$. If there are no other numbers between N_1 and N_2, the list is increased as a result of introducing number $P+1$ between N_1 and N_2. In the situation shown in Fig.2,b, from the list of neighbours of point L_1 we strike out L_3, and in its place put $P+1$, etc.

Notice that the must laborious stage in the algorithm is that connected with choosing the nearest point for the newly introduced point. We propose the following procedure for realizing this stage. We choose one point (call it K_1) among the first points, and compute the distance from $P+1$ to K_1 and all its neighbours. We compare these distances and at the next approximation to the nearest point we take point K_2, the distance to which is least. If $K_1=K_2$, the required point is found. Otherwise, the procedure is repeated, starting with point K_2, etc. It is easily shown that this process leads to the nearest point after a finite number of steps. The number of operations per point is then $O(N^{1/2})$, whereas $O(N)$ operations are needed in the direct method. Obvicusly, the efficiency of the algorithm for finding the nearest point depends on the choice of the initial point K_1.

In the general algorithm, described in Sect.1, the procedure for constructing the Dirichlet cells is used at the new $(n+1)$-th time layer. The lists of neighbours are then known at the previous n-th time layer. It is natural to use this information to choose the point K_1; in fact, as K_1 we choose a neighbour of point $P+1$ at the n-th time layer. The points are then numbered and run through in such a way that these neighbours are among the first processed points. Examples of the construction of Dirichlet cells and some data about the efficiency of the algorithm are given in Sect.5.

3. Construction of the difference scheme.

1. As applied to Eqs.(1.1)-(1.4), the method of support operators /13-15/ is as follows. Let a completely conservative difference scheme be required; then we first have to see what properties of the differential operators imply the satisfaction of the laws of conservation in the differential case.

We know (see e.g., /20/) that, for Eqs.(1.1)-(1.4), the laws of conservation follow from the fact that operators div and grad have the divergence form

$$\int_V \operatorname{div} \mathbf{A}\, dv = \oint_S (\mathbf{A}, \mathbf{n})\, ds, \qquad \int_V (\mathbf{c}, \operatorname{grad} \varphi)\, dv = \oint_S \varphi\, (\mathbf{c}, \mathbf{n})\, ds,$$

and from the fact that the following identity holds:

$$\int_V \varphi \operatorname{div} \mathbf{A}\, dv - \int_V (\mathbf{A}, \operatorname{grad} \varphi)\, dv = \oint_S \varphi\, (\mathbf{A}, \mathbf{n})\, ds, \tag{3.1}$$

where S is the surface bounding the volume V, $\mathbf{n}$ is the outward normal to S, φ and $\mathbf{A}$ are scalar and vector functions, and $\mathbf{c}$ is a constant vector function.

In the method of support operators, the difference analogues DIV and GRAD of the operators div and grad, are constructed in such a way that the difference analogue of (3.1) is satisfied, i.e., one operator, say DIV, is constructed directly, while GRAD is found from the conditions for satisfying the difference analogue of (3.1). It is natural to choose the divergence form of DIV. The divergence of GRAD, needed for complete conservativeness, is ensured if DIV is chosen so that it vanishes on a constant mesh vector function /20/.

2. Let us use the approach described to construct the difference scheme for Eqs.(1.1)-(1.4). Notice that DIV cannot be specified arbitrarily here. For, to approximate the equation of continuity we use in the algorithm the relation

$$\rho_i = m_i / V_i. \tag{3.2}$$

On reducing (3.2) to a form similar to (1.1), we obtain the explicit form of DIV. We differentiate (3.2) with respect to time, while noting that $dm_i/dt=0$:

$$\frac{d\rho_i}{dt} - \rho_i \left[\frac{1}{V_i} \frac{dV_i}{dt} \right] = 0. \tag{3.3}$$

Comparing (3.3) and (1.1), we see that, in the difference case, corresponding to div we have the expression in brackets in (3.3), i.e.,

$$(\mathrm{DIV}\, \mathbf{W})_i = \frac{1}{V_i} \frac{dV_i}{dt}. \tag{3.4}$$

Since V_i is not explicitly dependent on time, while its vertices are determined by the

position of the neighbours, we can write

$$(\mathrm{DIV}\,\mathbf{W})_i = \frac{1}{V_i}\sum_{k\in\Pi(i)}\left(\frac{\partial V_i}{\partial x_k}WX_k + \frac{\partial V_i}{\partial y_k}WY_k\right). \tag{3.5}$$

Consider the properties of DIV. First, this operator has divergence form, which is a direct consequence of (3.4). Second, for a constant vector function, expression (3.5) vanishes. This follows from (2.3) and (3.4). As we have remarked, from this there follow the equations

$$\sum_{k\in\Pi(i)}\frac{\partial V_i}{\partial x_k}=0, \qquad \sum_{k\in\Pi(i)}\frac{\partial V_i}{\partial y_k}=0. \tag{3.6}$$

Thus DIV satisfies the requirements stated in Para.1 of the present section.

3. In accordance with the ideas of the method of support operators, to construct GRAD we use the difference analogue of identity (3.1). Assume for simplicity that the line integral in (3.1) vanishes. We then use the following difference analogue of (3.1):

$$\sum_i \varphi_i(\mathrm{DIV}\,\mathbf{A})_i V_i + \sum_i (AX_i GX_i + AY_i GY_i)V_i = 0, \tag{3.7}$$

where GX and GY correspond to the x- and y-components of vector GRADφ and are subject to definition. Regarding (3.7) as an identity in AX_i and AY_i, we obtain the following expressions for the operator components:

$$GX_i = -\frac{1}{V_i}\sum_{k\in\Pi(i)}\frac{\partial V_k}{\partial x_i}\varphi_k, \qquad GY_i = -\frac{1}{V_i}\sum_{k\in\Pi(i)}\frac{\partial V_k}{\partial y_i}\varphi_k.$$

If we define the scalar products in the spaces of mesh scalar and vector functions by the equations

$$(|\varphi, \psi|) = \sum_i \varphi_i\psi_i V_i, \qquad (|\mathbf{A}, \mathbf{B}|) = \sum_i (AX_i BX_i + AY_i BY_i)V_i, \tag{3.8}$$

then we can write (3.7) as the operator equation

$$\mathrm{GRAD} = -\mathrm{DIV}^*, \tag{3.9}$$

in the same way as in the differential case.

4. Using the above operators DIV and GRAD, the differential-difference equations for (1.2) and (1.3) may be written as

$$\rho d\mathbf{W}/dt = -\mathrm{GRAD}\,p, \qquad \rho d\varepsilon/dt = -p\,\mathrm{DIV}\,\mathbf{W}. \tag{3.10}$$

The difference analogue of (1.1) can be written, in accordance with (3.2)-(3.5), as

$$d\rho/dt + \rho\,\mathrm{DIV}\,\mathbf{W} = 0.$$

The properties of DIV and the construction of GRAD ensure that the scheme is completely conservative /20/.

5. The use of the difference analogue of (3.1) implies the so-called direct method of matching the properties of DIV and GRAD. Let us consider what scheme is implied by the variational method of matching /9, 20, 21/.

In accordance with the Hamilton-Ostrogradskii variational principle, motion of the medium occurs in such a way as to obtain the extremum of the functional of action

$$S = \int_{t_1}^{t_2} \mathcal{L}(t)\,dt, \tag{3.11}$$

where $\mathcal{L}$ is the Lagrangian, defined as the difference between the potential and kinetic energies. Variation of (3.11) must be made in the light of the law of conservation of mass, the adiabaticity condition, and the kinematic relations.

In the case of a discrete medium, the Lagrangian $\mathcal{L}_h$ has the form

$$\mathcal{L}_h = \sum_i m_i\left(\frac{(\mathbf{W}, \mathbf{W})_i}{2} - \varepsilon_i\right). \tag{3.12}$$

The above relations may be written as

$$m_i = \mathrm{const}, \tag{3.13a}$$

$$m_i d\varepsilon_i = -p_i dV_i, \tag{3.13b}$$

$$dx_i/dt = WX_i, \qquad dy_i/dt = WY_i. \tag{3.13c}$$

To construct the difference equations, the variation of the functional (3.11) must be equated to zero. The variations of all quantities must then be expressed in terms of variations of the coordinates x and y:

$$\delta S=\int_{t_0}^{t_1}\Big[\sum_i\big[(\mathbf{W},\delta\mathbf{W}_i)-\delta\varepsilon_i\big]m_i\Big]dt=\int_{t_0}^{t_1}\Big[\sum_i(m\mathbf{W},\delta\mathbf{W}_i)_i\Big]dt-\int_{t_0}^{t_1}\Big(\sum_i m_i\delta\varepsilon_i\Big)dt.$$

In view of (3.13b,c), we obtain

$$\delta S=\int_{t_0}^{t_1}\Big[\sum_i\Big(\Big|m\mathbf{W},\frac{d\delta\mathbf{r}}{dt}\Big|\Big)_i\Big]dt-\int_{t_0}^{t_1}\Big(\sum_i p_i dV_i\Big)dt.$$

Next, we transform the first integral by using integration by parts and condition (3.13a), and to transform the second, we use (3.4) and (3.13c) in turn. The result is

$$\delta S=-\int_{t_0}^{t_1}\Big[\sum_i\Big(\Big|\rho\frac{d\mathbf{W}}{dt},\delta\mathbf{r}\Big|\Big)_i V_i+\sum_i p_i(\mathrm{DIV}\,\delta\mathbf{r})_i V_i\Big]dt.$$

Using definition (3.8) of the scalar product, we obtain

$$\delta S=-\int_{t_0}^{t_1}\Big[\Big(\Big|\rho\frac{d\mathbf{W}}{dt},\delta\mathbf{r}\Big|\Big)+(|p,\ \mathrm{DIV}\,\delta\mathbf{r}|)\Big]dt. \tag{3.14}$$

Using the concept of adjoint operator, we can write

$$(|p,\mathrm{DIV}\,\delta\mathbf{r}|)=(|\mathrm{DIV}^*p,\delta\mathbf{r}|).$$

From this and (3.14), since $\delta\mathbf{r}$ is arbitrary, it follows that the condition $\delta S=0$ is equivalent to the equation

$$\rho d\mathbf{W}/dt=\mathrm{DIV}^*p, \tag{3.15}$$

which approximates the equation of motion (1.2). Recalling Eq.(3.9), it can be seen that Eq.(3.15) is the same as the first of Eqs.(3.10). Moreover, it is obvious that we have the same difference analogues of the equations of continuity and the equation for the internal energy; for the variational method of matching, these equations are required as connections.

4. Artificial viscosity.

1. When introducing artificial viscosity into our discrete model, we start from the idea of /16/, whereby the artificial dissipative process is regarded as a consequence of inelastic collisions of particles modelling the discrete medium.

Let us explain this idea in the one-dimensional case. We will assume that a non-uniform Lagrangian mesh, whose nodes have coordinates $\{x_i\}$ (Fig.3), is introduced into the computational domain.

As in the two-dimensional case, the quantities m_i, ρ_i, p_i, u_i, ε_i, and V_i are connected with each node. In the one-dimensional case the Dirichlet cell boundaries are located at the mid-point between nodes; they are marked by asterisks in Fig.3. When introducing the artificial dissipative process, we assume that the mass m_i is distributed over all its Dirichlet cell. We next consider the process of inelastic collision of particle numbered i with its neighbours, i.e., particles $i-1$ and $i+1$. We assume that collision occurs at points corresponding to the boundaries of the Dirichlet domain. Only part of the mass of the corresponding cells participates in the collision. When particles i and $i-1$ collide, part of the mass of particle i, call it $\delta m_{i,i+1}$, is introduced into the collision process, and part $\delta m_{i+1,i}$ of particle $(i+1)$. The first subscript refers to t he particle number, whose part is cosidered, while the second indicates the collision with which the particle the mass participates. As a result of inelastic collision of masses $\delta m_{i,i-1}$, $\delta m_{i+1,i}$, a particle with mass $\delta m_{i+1,i}+\delta m_{i,i-1}$ the velocity $\bar u$ is formed. By the law of conservation of momentum,

$$(\delta m_{i,i+1}+\delta m_{i+1,i})\bar u=\delta m_{i,i-1}\bar u_i+\delta m_{i+1,i}\bar u_{i-1}, \tag{4.1}$$

where $\bar u_i$ and $\bar u_{i+1}$ are the velocities of masses $\delta m_{i,i+1}$ and $\delta m_{i+1,i}$ respectively. If we assume that $\bar u_i=u_i$, $\bar u_{i+1}=u_{i-1}$, we obtain from (4.1):

$$\bar u=\frac{\delta m_{i-1,i}u_{i-1}-\delta m_{i,i+1}u_i}{\delta m_{i,i-1}-\delta m_{i,i+1}}.$$

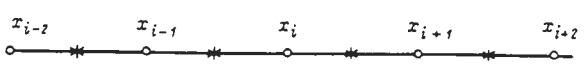

Fig.3

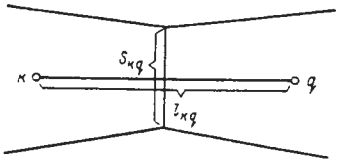

Fig.4

We next assume that thepart of mass $\delta m_{i,i+1}$ having velocity $\bar{u}$, collides inelastically with mass $m_i-\delta m_{i,i+1}$ having velocity u_i, while the part $\delta m_{i+1,i}$ similarly collides with mass $m_{i+1}-\delta m_{i+1,i}$. A similar process occurs with particles i and $i-1$, in which parts of the masses, $\delta m_{i,i-1}$ and $\delta m_{i-1,i}$, participate in the collision.

As a result, a particle with mass m_i is again formed, with new velocity $\hat{u}_i$, given by the law of conservation of momentum:

$$\hat{u}_i=\frac{1}{m_i}\Big[(m_i-\delta m_{i,i+1}-\delta m_{i,i-1})u_i+\frac{\delta m_{i,i-1}}{\delta m_{i,i-1}+\delta m_{i+1,i}}(\delta m_{i,i+1}u_i+\delta m_{i-1,i}u_{i-1})+\frac{\delta m_{i,i-1}}{\delta m_{i,i-1}+\delta m_{i-1,i}}(\delta m_{i-1,i}u_{i-1}+\delta m_{i,i-1}u_i)\Big].$$

It was shown in /16/ that, given a special choice of the quantities δm, we can obtain with this approach, for Lagrangian schemes, several familiar types of artificial viscosity: linear, quadratric, and composite /22/. In /16/. For the case of one- and two-dimensional Lagrangian schemes, the properties of this dissipative process are treated in detail, and notably, the question of the variation of the kinetic energy and of the corresponding change in the equation for the internal energy.

2. Consider the two-dimensional case. We assume that the particle only participates in inelastic collision with its neighbours. The collision with each particle occurs along their common piece of Dirichlet cell boundary, while the actual collision with each neighbour is similar to the one-dimensional case, i.e., as a result of the collision of particle k with its neighbour q (Fig.4), the projection of the velocity vector at the node on the direction of the segment joining k and q, varies by the amount

$$\frac{1}{m_k}\Big\{(m_k-\delta m_{kq})(u_k)_q+\frac{\delta m_{kq}}{\delta m_{kq}+\delta m_{qk}}\times[\delta m_{kq}(u_k)+\delta m_{qk}(u_q)_q]\Big\},$$

where $(u_k)_q$ etc. are the velocity projections at the corresponding point. Similarly for collisions with other neighbours. The resultant expression for the new velocity at node k is

$$\hat{\bar{w}}_k=\sum_{i\in\Pi(C)}\Big\{\Big(1-\frac{\delta m_{ki}}{m_k}\Big)(u_k)_i-\frac{\delta m_{ki}}{m_k(\delta m_{ki}+\delta m_{ik})}[\delta m_{ki}(u_k)_i+\delta m_{ik}(u_k)_k]\Big\}\mathbf{e}_{ki},$$

where $\mathbf{e}_{ki}$ is the unit vector along the segment ki.

The second important question is the choice of the δm_{kq}. Since we assume that the collisions occur only with neighbours, it follows from arguments of continuity that δm_{kq} must be zero at the instant of departure of the point from the neighbourhood. Moreover, it is natural to require that the dissipative process should be stronger as the points come closer. This demand is satisfied by the choice

$$\delta m_{kq}=\rho_k S_{kq}\Delta t c_k.$$

Here, S_{kq} is the length of the common boundary of cells V_k and V_q, Δt is the time step, and c_k is a characteristic velocity. In the computations whose results are given in Sect.5 c_k was taken to be the velocity of sound at node k, which corresponds in the one-dimensional case to linear viscosity. Notice that the process of introducing artificial viscosity gives expressions similar to those for so-called "contour viscosity", which was used in /7, Chapter 8/ when describing the Lagrangian method.

5. Example of the computation.

Consider the results of computing the development of Rayleigh-Taylor instability in a closed rectangular vessel in a gravitational field of force $g=1$. The problem is posed as follows. In the closed rectangle with rigid walls there are two fluids: a heavy fluid with initial density $\rho_1=10$ at the top, and a lighter fluid with density $\rho_2=1$ at the bottom. The equations of state for the fluids were taken as $p_1=50(\rho-10)$ and $p_2=50(\rho-1)$. The boundary between the fluids at the initial instant is shown by the heavy line in Fig.5. Due to the instability of this configuration, in the course of time the heavy fluid must flow to the bottom and displace the light fluid upwards.

In Fig.5 we show the initial configuration of Dirichlet cells; the asterisks denote the heavy fluid particles, and the points the light particles. In Fig.6 we consider the instant for which purely Lagrangian methods usually give satisfactory results. The first critical instant is connected with the impact of the heavy fluid on the bottom of the vessel (Fig.7). The next characteristic instant is linked with the impact of the fluid moving along the bottom with the left-hand wall (Fig.8). It can be seen in Fig.9 how at a later instant the wave reflected from the left-hand wall tips over, and the particles of heavy fluid drop from the upper boundary, where they were compressed by the bubble of surfacing light fluid. In Fig.10 we show the configuration when all the heavy fluid has gone to the bottom.

6. Discussion.

1. At the next stage, a theoretical and numerical study of the accuracy of our scheme is

required. The stability of the scheme and the choice of step Δt are important topics. A study of the stability for a similar scheme in /7/, Chapter 11/ suggests that step Δt will not tend to zero as the points approach one another, but will depend on certain characteristics of the Dirichlet cell, e.g., on its diameter.

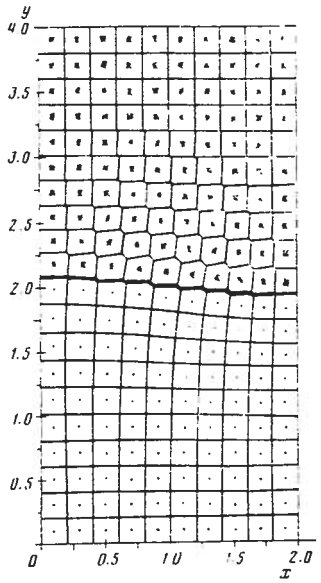

Fig.5

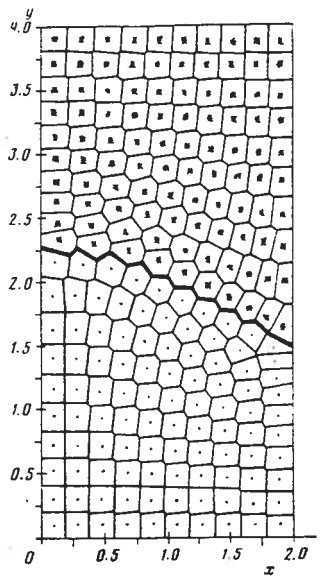

Fig.6

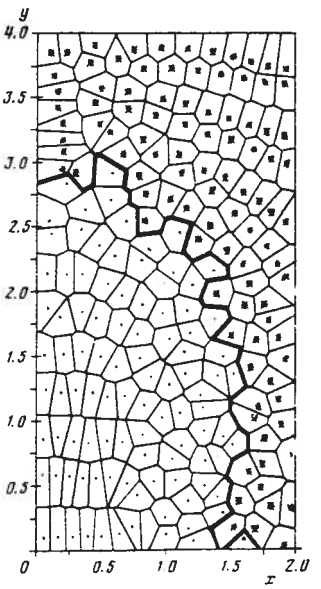

Fig.7

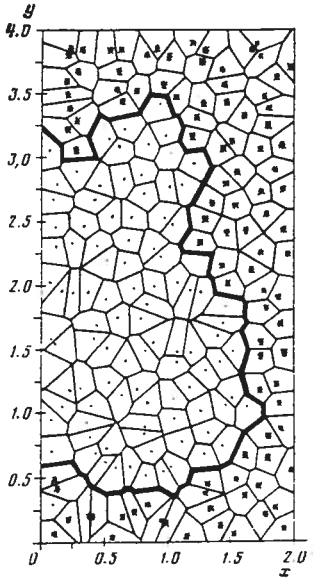

Fig.8

2. A second group of questions needing consideration concerns the realization of the implicit scheme. As in /23/, it is proposed to realize the method of parallel chords with iterations over the pressure.

3. From the stand-point of solving practical problems, the question of realizing boundary conditions of the free boundary type is important. This can be done e.g., by immersing the computational domain in the medium where the pressure depends only weakly on the density and has a given space and time distribution.

4. When computing actual problems, certain physical effects such as heat conduction, magnetic field diffusion, etc., must be taken into account. In this connection it becomes necessary to construct schemes which utilize Dirichlet cells for the processes. The method of support operators can prove an effective means here for constructing schemes.

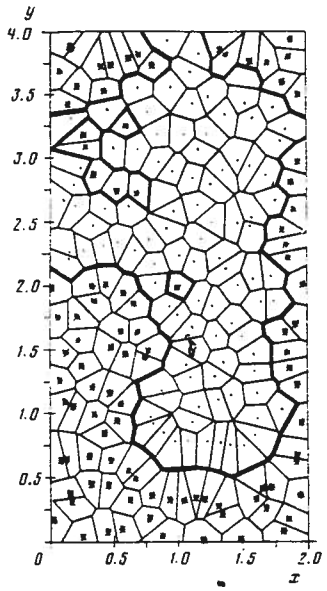

Fig.9

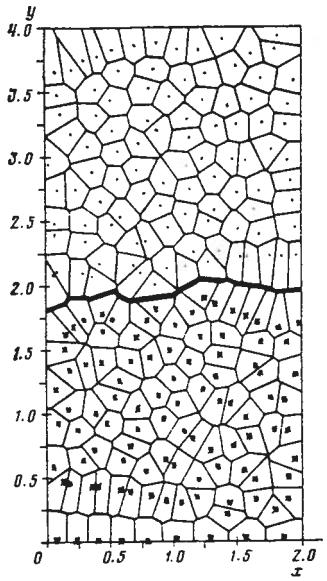

Fig.10

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SHORT COMMUNICATIONS

NEW ALGORITHMS FOR CALCULATING DISCRETE FOURIER TRANSFORMS*

A.M. GRIGORYAN

Effective methods are proposed for calculating a multidimensional discrete Fourier transform based on a new representation of it.

This paper describes a new general approach to the consideration of an arbitrary multidimensional discrete Fourier transform (MDFT), the basic idea of which lies in the possibility of a single-valued representation of each component transform as a one-dimensional vector that corresponds to it. Such an approach allows independent calculations of the MDFT to be carried out on each group on readouts that do not intersect with another group of readouts, into which the whole domain of definition of the spectrum is divided in a defined way, that allows effective algorithms to be constructed for calculating the MDFT by means of a minimum number of one-dimensional DFTs.

The case of a two-dimensional DFT is described in detail and the new algorithms that correspond to it are compared with the least algorithms worked out up to the present time for calculating a two-dimensional DTF, based on the Kooley-Tukey method /1, 2/, on polynomial transforms /3/, and also on the operation of two-dimensional "butterflies" /4/. The corresponding algorithm is considered as a special case and for a one-dimensional Fourier transform.

1. Vector representation of the spectrum by a MDFT.

Let us consider an arbitrary block $(f_{k_1,\dots,k_n})$ for an n-dimensional discrete signal, whose dimensions, for simplicity, will be considered equal i.e. $1 \leqslant k_i \leqslant N$, $i=1, 2, \dots, n$, for some integer N. Each spectral component in the readout $(p_1, \dots, p_n)$ where $p_i \in Z_N$ $i=1, 2, \dots, N$, $i=1, 2, \dots, n$, which apart from a normalized multiplier, is equal to

$$F_{p_1,\dots,p_n} = \sum_{k_1=1}^{N} \dots \sum_{k_n=1}^{N} f_{k_1,\dots,k_n} W^{k_1 p_1 + \dots + k_n p_n}, \tag{1}$$

where $W=W_N=\exp(2\pi i/N)$, can be represented as the N-dimensional vector

$$F_{p_1,\dots,p_n} = (f_{p_1,\dots,p_n,1}, \dots, f_{p_1,\dots,p_n,N}), \tag{2}$$

for which

$$F_{p_1,\dots,p_n,t} = \sum_{i=1}^{N} f_{p_1,\dots,p_n,i} W^{it}.$$

For this, as follows from (1), each component of the vector (2) must be calculated by summation of the values of the initial signal at readouts of the corresponding sets:

$$T_{p_1,\dots,p_n,t} = \left\{(k_1, \dots, k_n): 1 \leqslant k_i \leqslant N, i = 1, 2, \dots, n, \sum_{i=1}^{n} k_i p_i = t \bmod N\right\}, \tag{3}$$

i.e.

*Zh.vychisl.Mat.mat.Fiz., 26, 9, 1407-1412, 1986