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Numerical Simulation of Two-Dimensional Flows by the Free-Lagrangian Method

MIKHAIL YU. SHASHKOV
ANDREY V. SOLOVJOV

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NUMERICAL SIMULATION OF TWO-DIMENSIONAL FLOWS BY THE FREE-LAGRANGIAN METHOD

Mikhail Yu. Shashkov

Andrey V.Solovjov

Abstract

The free-Lagrangian method is usedfor computing two-dimensional gas-dynamic flows with complex structure. The free-Lagrangian methods have some characteristic features: First, a set of point particles is used which moves together with the medium and the surrounding domain. Second, the connections between the particles are not fixed but can vary with time depending on the mutual disposition of the nodes.

In this article we give a survey on a long period of work expennded on the development, realization and verification of a new, purely Lagrangian, conservative free-Lagrangian method. The term "purely Lagrangian " means that the mass of particles is not changed, no mass, momentum and energy exchange appears between particles, no interpolation or smoothing procedures are used; the cell connected with a particle contains only one material, that is there are no " mixed " cells. In our method the network structure is determined by the Dirichlet tessellations of the computational domain. It is shown by a computational experiment that the volume of a Dirichlet cell varies more smoothly than that of comparable approaches when the structure of the grid changes. The continuity of the volume determines, for Lagrangian methods, the continuity of density of the given particle. Another important property of the Dirichlet volume is that its derivatives with respect to particle coordinates are likewise continuous. These derivatives appear in the difference equations, so that acceleration, velocity, internal energy, and pressure are continuous.

The conservative difference scheme is obtained by using the method of support-operators. The approximation property of the constructed difference scheme is investigated. Some approaches to the treatment of free boundaries are described. Results of computation are presented for some two-dimensional problems.

Present address: National Center of Mathematical Modeling,
USSR Academy of Sciences, Miusskaya Sq. 4, Moscow, 125047 USSR.
e-mail: demos!ibrae.ins.su!pbl.@fuug.fi

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Introduction.

1. The interest in complex gas-dynamic flows frequently induces to develop new methods for performing computational experiments. At present, two main descriptions of the medium are used for numerically modeling of flows : the Lagrangian approach in situations of relatively smooth flows [1-3], and the Eulerian description for flows with strong deformations [4]. At the same time, situations are often encountered in which important practical problems make it desirable to exploit the advantages of both approaches.

Most of the important practical problems deal with several materials of essentially different physical properties. Multi-material problems, which inherently contain interfaces, are best suited to the Lagrangian method of computation. This is because Lagrangian methods resolve interfaces crisply, in contrast to Eulerian methods, which diffuse or smear them out.

Unfortunately Lagrangian methods based on fixed-connectivity quadrilateral meshes do not have adequate flexibility to follow convoluted interface shapes and frequently fail due to excessive mesh distortion. Mesh cells creep into one another and overlap, thus making it impossible to continue the computation.

2. There are many free-Lagrangian methods [5-12]. The basic ideas used in (to the author's knowledge) all free-Lagrangian methods was described in [5]. Let us characterize these ideas. First, a set of point particles and of surrounding domains is used for describing the medium. The particles move together with the medium. The shape and size of these domains are determined by the arrangement of the particles. Secondly, the connections between particles are not fixed but can vary with time depending on the mutual disposition of the points. There are many different approaches to determine the structure of connections. After determination we are given the set of "closest" particles for each given particle. This set of particles is a pattern or stencil for difference scheme construction.

The pioneer work on free-Lagrange constructions was done by Pasta and Ulam [5] in USA and D'yachenko [6] in USSR. The term "free-Lagrange" was used first by Crowley [8] and is used widely now all over the world.

3. In this article we will describe the purely Lagrangian

free-Lagrangian method. The term "purely Lagrangian" means that the mass of particles is not changed, there is no mass, momentum and energy exchange between particles, no interpolation or smoothing procedures are used; the cell, connected with a particle, contains only one material, there are no " mixed " cells.

In our method the grid structure is determined by the Dirichlet tessellations of the computational domain. This geometrical decomposition is well known for convex domains [13,14], one possible definition for unconvex domains will be given in Sect.1 of this paper. In Sect.1 we also briefly describe how to construct Dirichlet tessellations. When the structure of grid is fixed we have many possibilities for constructing a correspondence domain (domain surrounding the particle). We prefer Dirichlet or Voronoi cells. We will compare them with other types of correspondence domains in Sect.1. It is shown by a computational experiment that the volume of a Dirichlet cell is varied with fewer singularities when the grid structure changes. This is a very important property because the continuity of volume, for Lagrangian methods, causes the continuity of density, for the given particles. Another important property of the Dirichlet volume is that its derivatives with respect to particle coordinates are likewise continuous. These derivatives occur in the respective difference equations, so that acceleration, velocity, internal energy, and pressure are continuous.

Conservative difference schemes will be obtained by means of support operators [15] in Sect.2. The first order differential operators div and grad occur in the equations of gas dynamic. Following the support operators method we shall specify some integral identity from which the conservation laws follow. Then we will construct the difference operators DIV and GRAD which are analogues of the differential operators div and grad. The difference scheme constructed in this way will be fully conservative.

The approximation property of the constructed difference scheme will be investigated in Sect.3. The local approximation of the equation of motion is shown to be of first order. The same order of approximation, in the integral sense, is valid for the continuity equation.

When some the artificial viscosity is requested to compute flows with shock waves, and in order to ensure the stability of the scheme, the dissipative process shall be regarded as the consequence of inelastic collisions of the particles - Sect.4. This way of introducing viscosity for Lagrangian schemes was proposed in [16], but it seems to be more natural to use Dirichlet cells. A similar approach to the one-dimensional case was considered in [17].

Most considerably, free-Lagrangian methods have the advantage to cope with a free boundary. Some approaches to free boundaries will be described in Sect.5.

There are important problems in which the evolution of a small perturbation is investigated. One problem of this type is that one connected with the stability of compression of spherical laser targets [26]. The evolution of a small distortion of spherical form of some target is investigated by a computational experiment. The according difference scheme has to preserve the symmetry of the original physical problem, and must not introduce additional distortions. An appropriate scheme is presented in Sect.6.

There are many other possibilities to use Dirichlet cells for constructing difference schemes. The Godunov scheme will be described in Sect.7.

Some computational results will be shown in Sect.8. As a first sample computational experiment, we consider a plane shock wave which moves with constant velocity and runs against the obstacle given by a semi-infinite right angle. As a second computational experiment, let us consider the problem of the interaction of a shock with some bubble.

Dirichlet Cells, Definition, Construction, and Properties

1.Regard some finite set of given particles $\{P_k\}$ on the Cartesian plane. A Dirichlet cell D_i is that part of the plane containing all points x which are closer to P_i than to any other particle from the set $\{P_k\}$:

$$D_i = \{ x: d(x,P_i) < d(x,P_k) \text{ for all } k \neq i \}.$$

Dirichlet cells cover the whole plane (leaving out the boundaries) without gaps and overlapping. On some bounded region R, the Dirichlet cell D_i for a given point P_i is the intersection of the Dirichlet cell on the plane, with the region R:

(1) $$D_i = \{ x: d(x,P_i) < d(x,P_k) \text{ for all } k \neq i, x \in R \}.$$

Such cells are the polygonal domains. An example is given in Fig.1 where R is a rectangle.

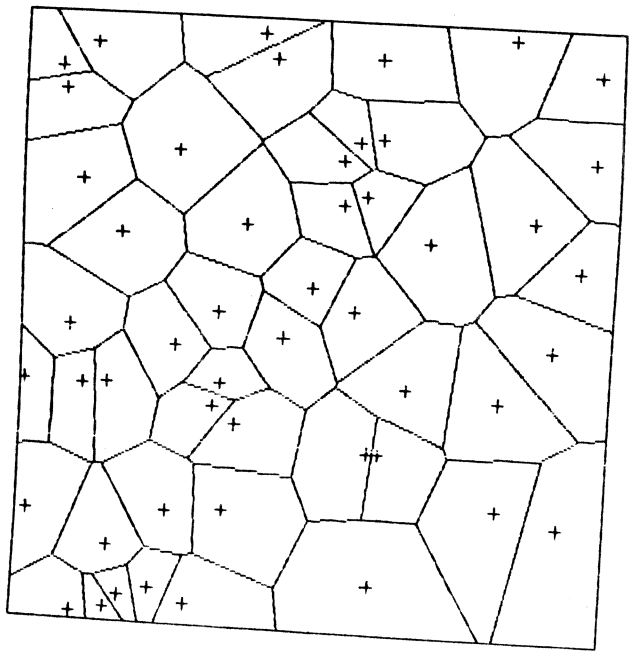

Fig.1

Two particles whose cells have some common part of boundary, are named neighbour particles. The corresponding cells are called neighbour cells - Fig.2.

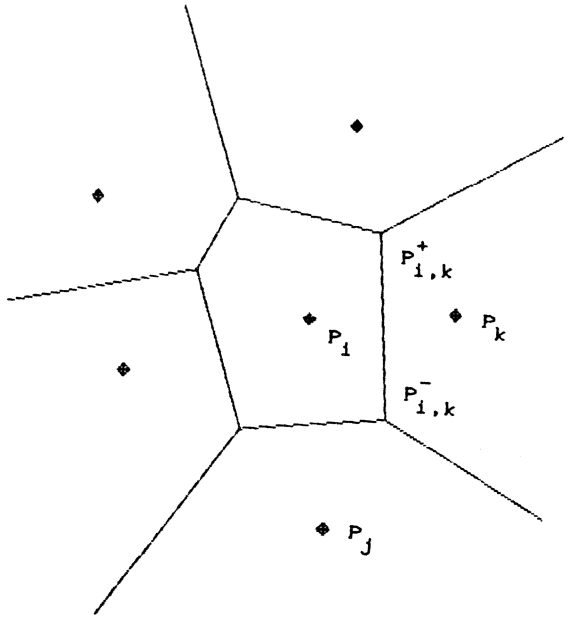

Fig.2

2. The algorithm for constructing the Dirichlet cells in fact reduces to find the neighbours for each point. When the neighbours are found, the coordinates of the vertices of the Dirichlet cells are given by following relations:

(2)
$$x_{ijk} = \frac{(y_k-y_i)(x_j^2-x_i^2+y_j^2-y_i^2)-(y_j-y_i)(x_k^2-x_i^2+y_k^2-y_i^2)}{2[(y_k-y_i)(x_j-x_i)-(y_j-y_i)(x_k-x_i)]}$$

$$y_{ijk} = \frac{(x_i-x_k)(x_j^2-x_i^2+y_j^2-y_i^2)-(x_i-x_j)(x_k^2-x_i^2+y_k^2-y_i^2)}{2[(y_k-y_i)(x_j-x_i)-(y_j-y_i)(x_k-x_i)]}$$

because every vertex of a Dirichlet cell is equidistant one from three neighbouring particles.

The main idea underlying the method we use, is taken from

[14]. Assume that the set of points considered is numbered from 1 to N in some way. We will consider separately the first P points on the assumption that the neighbours are established disregarding the other points. Assume that the indices of neighbours, for each point, are stored in a special list and are ordered counter-clockwise. The point with number P+1 is then brought into consideration and the set of the first P+1 points is considered. We then have to find the neighbours for point number P+1 from among the first points and correct the lists of neighbours for the first P points. For the reason of simplicity, we shall describe this procedure only for the special case of some point number P+1 whose cell does not contain, as pieces of boundary, pieces of the sides of the bordering rectangle.

That one of the first P points which is closest to point P+1 is found first, call it K (see Fig.3).

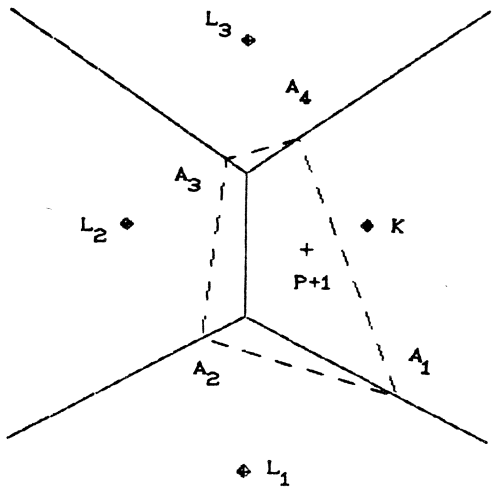

Fig. 3

We then draw a perpendicular through the center of the line from K to P+1 and determine the intersection of this perpendicular with a side of the Dirichlet cell for point K, as shown in Fig. 3, i.e., on moving along the perpendicular to the point of intersection, which we denote by A_1, point P+1 remains to the right. The side

met by the perpendicular uniquely defines one of the neighbours of point P+1, call it L_1 . Point L_1 is put first into the list of neighbours for the point P+1. We then draw a perpendicular through the center of the line from P+1 to L_1 . This perpendicular cuts two sides of the Dirichlet cell for point L_1. One of the points of intersection is A_1, since A_1 is the center of the circle passing through the points L_1, K and P+1.

The second point (call it A_2) defines the number of the next neighbour of point P+1, call it L_2. In order to establish a counter-clockwise ordering, this number is placed in front of the previous neighbours in the list of point P+1 , i.e. L_1. This process is continued until the Dirichlet cell of point P+1 is closed. In this way the list of neighbours of a new point P+1 is constructed.

Now consider how the lists of neighbours of the points must be corrected. First, we change the lists of those points which were in the neighbourhood of point K. Let L be such a point. We first look at which of the two sides are cut by the perpendicular drawn through the center of line L,P+1. Let N_1, N_2 be the numbers of the respective neighbours. Then, from the old list of neighbours of point L, we remove all those neighbours whose numbers lie in the list between $N_1$and N_2 , and in place of them we introduce the neighbour with number P+1. If there are no numbers between N_1 and N_2, the list is inlarged as result of introducing P+1 between N_1 and N_2. In the situation shown in Fig. 3, we delete K from the list of neighbours of L_2, and replace it by P+1.

Notice that the most laborious step in the algorithm is choosing the closest point for a newly introduced point. We propose the following procedure for realizing this step. We choose one point (call it K_1) from the first points, and compute the distance from P+1 to K_1 and to all of its neighbours. We compare these distances and select the closest point, say K_2,as the next approximation. If $K_1=K_2$ the required point is found. Otherwise, the procedure is repeated, starting with point K_2. It can easily be shown that this process leads to the nearest point after a finite number of steps. The number of operations is $O(N^{1/2})$, whereas O(N) operations are needed in the direct method. Obviously, the efficiency of the algorithm for finding the closest point depends on the choice of the initial point K_1.

In the general algorithm, the procedure for constructing the Dirichlet cells is used at the new (n+1)-th time layer. The lists of neighbours are then known at the previous n-th time layer. It is quite obvious to use this information to choose the point K_1, in fact, we choose K_1 as a neighbour of point P+1 at the n-th time layer. The points are then numbered and run through in such a way that these neighbours are among the points which are to be processed first.

3. Let us discuss some fundamental properties of the area V_i of a Dirichlet cell. An important property is the continuity with respect to the position of the points. A second property of the Dirichlet area is that the derivatives $\partial V_i/\partial x_k$, $\partial V_i/\partial y_k$ are continuous, where k runs from 1 to N. Let us prove both properties. It must first be remarked that, from the formal standpoint, the volume is a function of several variables, namely, of the coordinates of all points. We know that the differentiability, and hence continuity of a function of several variables follows from the existence and continuity of the derivatives with respect to the arguments, therefore it is sufficient to prove the continuity of the derivatives $\partial V_i/\partial x_k$, $\partial V_i/\partial y_k$. The expressions for the Dirichlet area and for the derivatives are

$$V_i = \sum_{S_i} \frac{x^+_{i,k} + x^-_{i,k}}{2} \left(y^+_{i,k} - y^-_{i,k} \right) \tag{3}$$

$$\frac{\partial V_i}{\partial x_k} = \frac{y^+_{i,k} - y^-_{i,k}}{x_k - x_i} \left(x_k - \frac{x^+_{i,k} + x^-_{i,k}}{2} \right), \qquad k \neq i \tag{4}$$

$$\frac{\partial V_i}{\partial y_k} = \frac{y^+_{i,k} - y^-_{i,k}}{x_k - x_i} \left(y_k - \frac{y^+_{i,k} + y^-_{i,k}}{2} \right) \tag{5}$$

Here the set S_i contains all neighbours of particle P_i, furthermore $x^+_{i,k}$, $y^+_{i,k}$ and $x^-_{i,k}$, $y^-_{i,k}$ are the coordinates of the ends of the side corresponding to the pair i,k. These derivatives are obtained [10] by direct differentiation, making use of the fact that only some part of the area V_i depends on the position of the point K (see Fig.4).

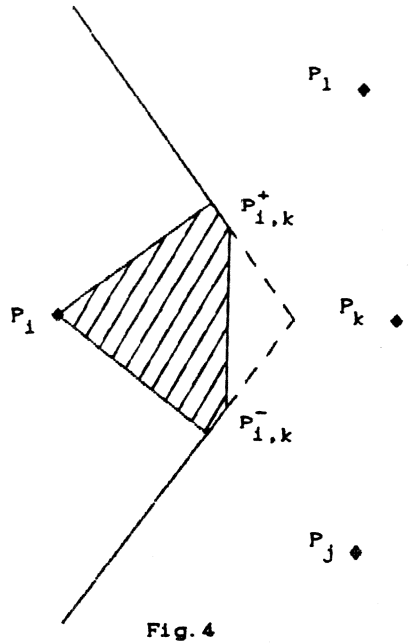

Fig.4

In the case $x_k \neq x_i$ it follows directly from (4),(5) that the derivatives are continuous, if point K is a neighbour and remains a neighbour when its coordinates vary within a small surrounding. When K is not a neighbour of I the derivatives are equal to zero. It remains to show that $\partial V_i/\partial x_k$, $\partial V_i/\partial y_k = 0$ when the common side of cells I and K degrades to one point F , or in other words, when the point K lies on the circle passing through the points I, J, K (see Fig. 5). Since, in this case, the points $P^+_{i,k}$ and $P^-_{i,k}$ fall upon to the same point F, the asserted equation follows from (4),(5). The case $x_k = x_i$ is treated similarly: Since the line connecting points I and K, and side $P^+_{i,k}$, $P^-_{i,k}$ are mutually perpendicular, we have $(y^+_{i,k}-y^-_{i,k})/(x_k-x_i) = -(x^+_{i,k}-x^-_{i,k})/(y_k-y_i)$ and hence, for example (4) can be reduced to the form

$$\frac{\partial V_i}{\partial x_k} = - \frac{x^+_{i,k} - x^-_{i,k}}{y_k - y_i} \left(x_k - \frac{x^+_{i,k} + x^-_{i,k}}{2} \right). \tag{6}$$

Now consider the derivative $\partial V_i/\partial x_i$. Since the volume of any Dirichlet cell remains unchanged when all points are moved by a parallel shift, we can write

$V_i(x_i+t, x_j+t, x_k+t, \ldots, y_i, y_j, y_k, \ldots) = \text{const}$, for all t,

so that

$$\partial V_i/\partial t = \sum_{\hat{S}_i} \partial V_i/\partial x_k = 0,$$

where the set $\hat{S}_i$ consists of the point I itself and of all of its neighbours. From this relations we have

$$\partial V_i/\partial x_i = -\sum_{S_i} \partial V_i/\partial x_k$$

hence this derivative is continuous.

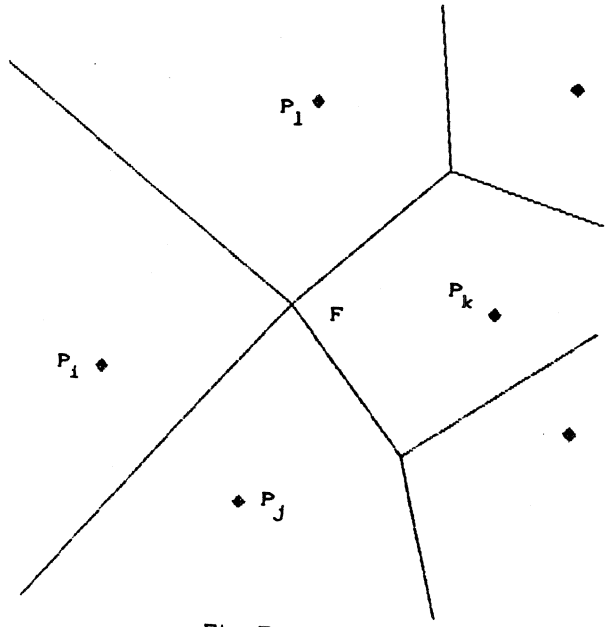

Fig.5

4. As mentioned in the introduction, several different correspondence domains can be constructed when a set of points and some neighbourhood relation is fixed . Now let us consider two further types in order to show that Dirichlet cells have the best continuity properties.

The division of the plane by Dirichlet cells allows to define Dirichlet neighbours. The neighbours of a given point are those points whose Dirichlet cells have some side in common with the cell of the point under consideration. If we connect all pairs of neighbours we obtain a triangulation, that is the Delaunay triangulation.

We start from the Dirichlet cell. Dirichlet cells are convex polygonal domains. They cover all the plane without overlapping and gaps. Each Dirichlet cell always contains its corresponding point. Every polygon vertex is the center of the circle circumscribing one of the triangles of the Delaunay triangulation.

A second kind of cell (the median type) is used in [9]: The set of polygon vertices is defined to include both the triangle centers (points of median intersections) and the centers of the lines connecting the neighbour points. This polygon may be unconvex, but it always contains its corresponding point.

Both types of correspondence domains have geometrical interpretations. However, a third approach is possible. Instead of the boundary, the definition can also be based on the area, by means of some formula. One possible formula is given in [8]. To explain this formula, we consider all triangles from the Delaunay triangulation containing some given point. The area of each triangle is distributed over the three points in proportion to the angles.

In order to investigate the properties of correspondence domains we propose the following numerical experiment. A set of random points (marked by symbol x and ■ in Fig.6) is regarded as fixed in the unit square .

One additional point is moved with constant speed through the initial grid along the broken line upwards. The distance between the line and point 31 is 0.1 percent of the average distance between the points. The movement starts at time 0.275 and ends at time 0.675. The minimal distance between the moving point and point 31 is reached at time 0.5. At each moment, the meshes for all points (including the moved point) are built up again. We look after point 31 marked by a black rectangle.

In Fig. 7, the behavior of the area of cell 31 is shown. The broken line represents the Dirichlet cell and the other the cell of median type. It is obvious that the area of the Dirichlet cell

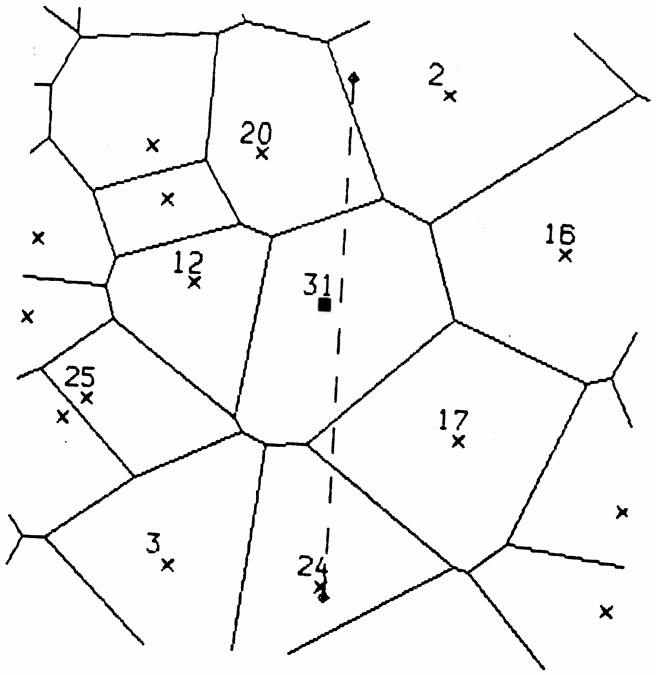

Fig.6

has only one discontinuity near 0.5 as we will see. Formally, this cell area is a function of the coordinates of all points from the set $\{P_i\}$. If $P_i=P_k$ for any $k\neq i$, then the area of cell P_i is undefined since the direction of the line P_iP_k is undefined. We are near this situation at the moment 0.5. The side turning is large while the point shift is small.

The area function of the median type has many discontinuities caused by structural changes of neighbourhood. The amplitude of discontinuity weakly depends on the distance between the point and the line of motion.

Analogous results are obtained for the geometrical object of the third type. We have the same moments of discontinuities, but other amplitudes [30].

Below we consider the configuration of Dirichlet cells and of cells of the median type at the moment t=0.32. This is the moment in which the mesh structure changes.

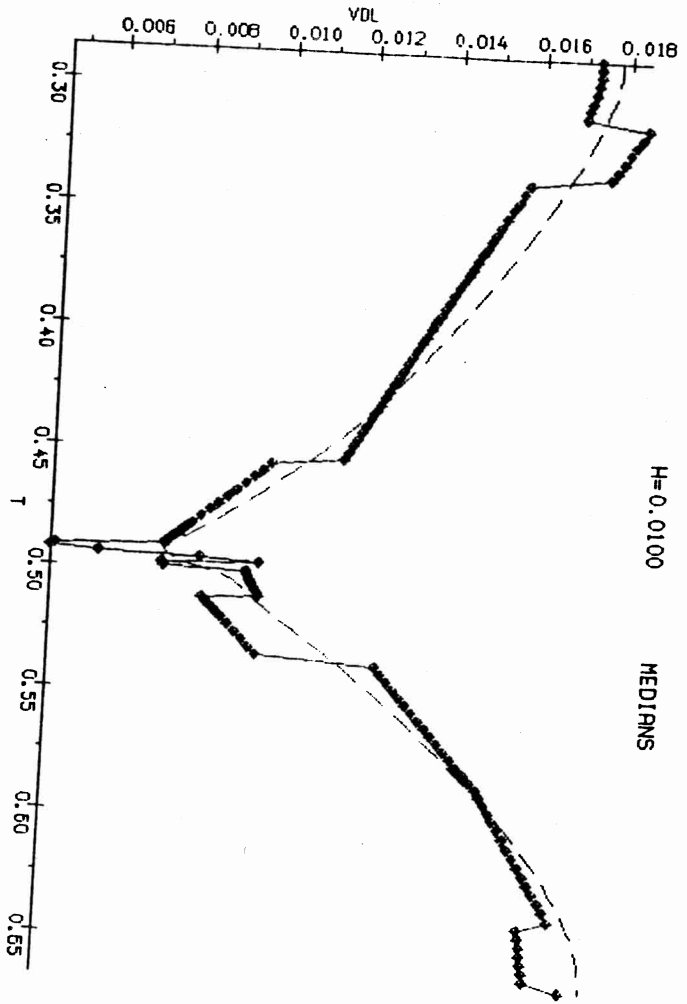

Fig.7

We show two configurations, for t=0.3239 (before the neighbours change) and t=0.3274 (after changing). The moving point has the number 32. Broken lines indicate the moment t=0.3239 and solid line the moment t=0.3274 (Fig.8,9).

The configurations of Dirichlet cells for these moments are shown in Fig. 8. It is obvious that cell structure and cell area are not changed. This fact corresponds to the continuity of the area function for the Dirichlet cell at t=0.32, confer Fig. 7.

The configurations of the cells of the median type are shown in the Fig. 9. At the first moment the points 31 and 25 are neighbours and at the second moment they are not. The area of cell 31 changes because the number of neighbours of the point 31 changes.

One could suppose that the cells of the median type have the better properties when the point passes in a short distance. This is related to the fact that the coordinate formulas for cell vertices are obtained by interpolation of the coordinates of some points. But the interpolated coordinates only behave continuous as long as the neighbour structure is not changed.

It follows from our experiment (Fig. 6-9), and from the geometrical point of view that the Dirichlet cell possesses the best properties of continuity.

5. Let us try to generalize the concept of Dirichlet tessellation for non-convex domains. We consider some polygonal domain E and some finite set $\{P_i,\ i=1..N\}$ of particles (fixed points) consisting of the boundary vertices and of inner points.

A point $x \in E$ belongs to the correspondence domain O_i of particle P_i if:

a) the line x,P_i lies within the domain E,

b) the relation $d(x,P_i)<d(x,P_j)$ holds for all P_j, $j \neq i$ with x,P_i inside the domain E.

It can be proved [31] that these correspondence domains are polygonal domains that satisfy the above mentioned properties on Dirichlet cells and that they convert into Dirichlet cells when the domain E becomes convex.

The above definition is not constructive. But we can also give constructive definition in terms of polygonal intersections and unions and prove its equivalence. The algorithm for constructing correspondence domains is based on this definition [31]. Two

examples of our tessellation for unconvex domains are shown in Fig.10,11.

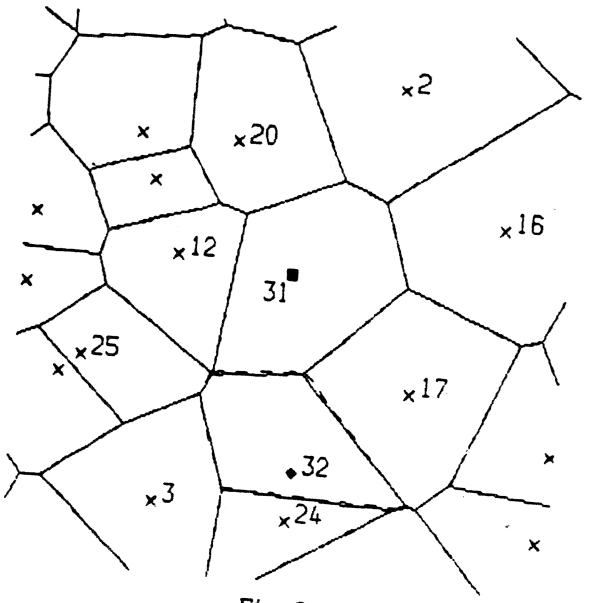

Fig.8

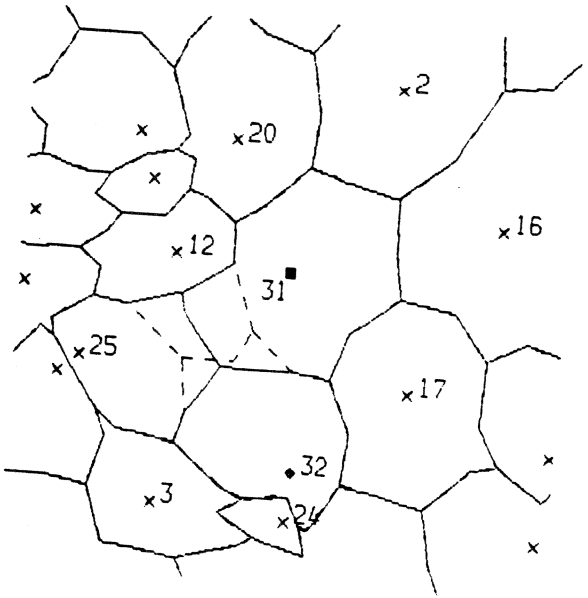

Fig.9

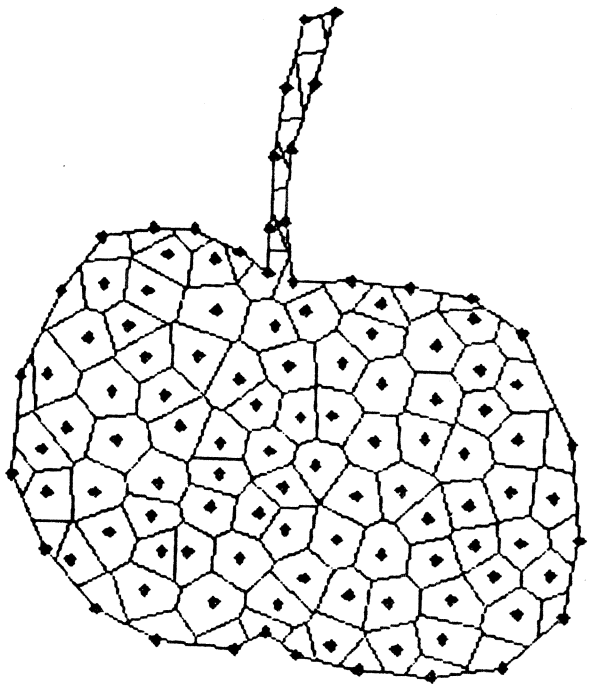

Fig.10

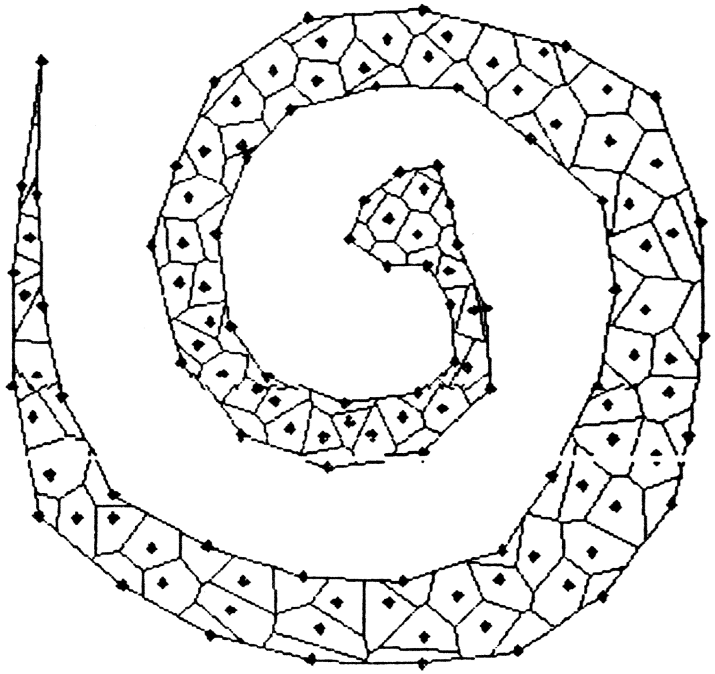

Fig.11

The difference scheme

The aim of our method is to solve the system of gas-dynamic equations in Lagrangian form:

$$\frac{d\rho}{dt} + \rho \operatorname{div} W = 0, \tag{7}$$

$$\rho \frac{dW}{dt} + \operatorname{grad} p = 0, \tag{8}$$

$$\rho \frac{d\varepsilon}{dt} + p \operatorname{div} W = 0, \tag{9}$$

$$W = \frac{dr}{dt}, \tag{10}$$

$$F(p,\rho,\varepsilon)=0, \tag{11}$$

In the equations the first order differential operators div and grad are used. We want to construct the difference scheme for this system by means of the support-operators [15].

Following this method we have to present some integral identities which comprise the conservation laws. As we know, the conservation laws for equations (7) - (11) follow from the equations

$$\int_V \operatorname{div} \mathbf{A}\, dv = \oint_S (\mathbf{A}, \mathbf{n})\, ds, \tag{12}$$

$$\int_V (\mathbf{C}, \operatorname{grad} p)\, dv = \oint_S p\,(\mathbf{A}, \mathbf{n})\, ds, \tag{13}$$

$$\int_V p \operatorname{div} \mathbf{A}\, dv + \int_V (\mathbf{A}, \operatorname{grad} p)\, dv = \oint_S p\,(\mathbf{A}, \mathbf{n})\, ds \tag{14}$$

where S is the surface of the volume V, **n** is the outer normal of S, p and **A** are arbitrary scalar and vector functions, **C** is a constant vector function. The equations (12) and (13) mean that the conservation laws of mass and momentum are valid. Equation (14) means that the full energy depends only on the surface force.

Next we construct the difference operators DIV and GRAD, the difference analogues of the differential operators div and grad. The difference operators are constructed such that the difference analogue of equation (14) will be valid. One of the operators (DIV, for example) is defined ad hoc and the other (GRAD) is obtained from the analogue of equation (14). It is natural that the difference operator DIV should be divergent. Besides, if the operator DIV is zero for constant vector functions on the grid,

the difference operator GRAD will be divergent, too, and the difference scheme will be fully conservative.

Now we construct the difference scheme for the system of gas-dynamic equations (7) - (9) using the method of support operators. First we construct the operator DIV. Let us approximate the equation of continuity in the following form:

$$\rho_i = m_i / V_i.$$

The derivative of ρ_i with respect to time is the following:

$$\frac{d\rho_i}{dt} = \frac{1}{V_i}\frac{dm_i}{dt} - \frac{m_i}{V_i^2}\frac{dV_i}{dt} = -\frac{\rho_i}{V_i}\frac{dV_i}{dt} \tag{15}$$

Here we took into account that $dm_i/dt=0$. Comparing the equations (7) and (15) we conclude that the difference operator DIV must have the form

$$(\operatorname{DIV} W)_i = \frac{1}{V_i}\frac{dV_i}{dt} \tag{16}$$

Since the volume V_i is a function of the coordinates of particle P_i and of its neighbours, we can write

$$(\operatorname{DIV} W)_i = \frac{1}{V_i}\sum_{k\in S_i}\left(\frac{\partial V_i}{\partial x_k}\frac{dx_k}{dt} + \frac{\partial V_i}{\partial y_k}\frac{dy_k}{dt}\right) = \tag{17}$$

$$= \frac{1}{V_i}\sum_{k\in S_i}\left(\frac{\partial V_i}{\partial x_k}Wx_k + \frac{\partial V_i}{\partial y_k}Wy_k\right).$$

Here the set S_i consists of particle P_i and of all of its neighbours.

Let us investigate the properties of the difference operator DIV. So, as the volume V_i doesn't change if the velocities of particle P_i and of all of its neighbours are identical, then the operator DIV is zero if **W** is a constant vector function. Furthermore, the difference operator DIV has a divergent form. This follows from equation (16).

In order to derive the difference operator GRAD we use the difference analogue of the integral identity (14). Note that the contour integral in the right part of equation (14) vanishes. Hence the difference analogue is

$$\sum_i p_i(\operatorname{DIV} \mathbf{A})_i V_i + \sum_i \left(Ax_i(\operatorname{GRAD}_x p)_i + Ay_i(\operatorname{GRAD}_y p)_i\right) V_i = 0 \tag{18}$$

Regard this expression as an identity in Ax_i and Ay_i. Then we have

(19) $(\mathrm{GRAD}_x\ p)_i = -\frac{1}{V_i}\sum_{k\in S_i}\frac{\partial V_k}{\partial x_i}p_k$, $(\mathrm{GRAD}_y\ p)_i = -\frac{1}{V_i}\sum_{k\in S_i}\frac{\partial V_k}{\partial y_i}p_k$.

If we define the inner products as scalar and vector functions by the equations

$$(p,q) = \sum_i p_i q_i V_i, \qquad (A,B) = \sum_i (Ax_i Bx_i + Ay_i By_i)V_i$$

then we can write (18) as operator relation GRAD = -DIV* in the same way as for the differential operators.

The difference scheme corresponding to the system of equations (7) - (9) is

$$\frac{d\rho_i}{dt} = -\rho_i\ (\mathrm{DIV}\ W)_i\ ,$$

$$\rho_i\frac{dW_i}{dt} = -\ (\mathrm{GRAD}\ p)_i\ ,$$

$$\rho_i\frac{d\varepsilon_i}{dt} = -\ p_i\ (\mathrm{DIV}\ W)_i\ ,$$

$$W_i = \frac{dr_i}{dt},$$

$$F_i(p_i,\rho_i,\varepsilon_i) = 0.$$

The time is continuous, in these equations . It remains to descretize the time. We can, for example, choose an implicit discretization. Notice, that the lists of neighbours change for all particles at each step of time. The properties of DIV and the construction of GRAD ensure that the scheme is conservative.

It is very important that the coefficients of the difference scheme are functions of the volume of Dirichlet cells and the derivatives $\partial V_i/\partial x_k$, $\partial V_i/\partial x_k$ .It is proved in Sect.1 that these quantities are smooth with respect to grid structure.The continuity of volume implies the continuity of density. The continuity of the derivatives with respect to the particle coordinates implies the continuity of acceleration, velocity, internal energy, and pressure.

Approximation.

The main drawback is that Dirichlet cells are not Lagrangian, this was pointed out in [9]. Therefore local approximation is missing for the equation of continuity [19]. It must be said, however, that a similar situation holds for the equations of motion in difference schemes for equation in Lagrangian form [18] which are widely used. Also, it is well-known, that in spite of missing local approximation, convergence theorems [20] are valid for a number of schemes for Laplace's equation.

We introduce some notation. Let r_h be the projection from the space of continuous scalar functions to scalar functions defined on the particles of the grid:

$$(r_h u)_{ij} = u(x,y)\big|_{x=x_i,y=y_i}.$$

Let $(R_h W)_i = ((r_h WX)_i\ ,(r_h WY)_i)$ be the projection from the space of continuous vector functions to vector functions being defined on particles. Then the grid functions of the truncation errors of div and grad are:

$$\psi = r_h(\mathrm{div}\ W) - \mathrm{DIV}(R_h W),\ \varphi = R_h(\mathrm{grad}\ u\ ) - \mathrm{GRAD}(r_h u)$$

Let us assume that the Dirichlet grid satisfies the following conditions. There is a constant h, such that $d(D_i) \leqslant C_1 h$, $h_{ik} \geqslant C_2 h$, where $d(D_i)$ is the minimal diameter of the circles containing cell D_i, with center in point I, and h_{ik} is the distance between particle I and K. Making use of this assumption and of relations (3)-(5) we can prove that V_i is $O(h^2)$, and $\partial V_i/\partial x_k$, $\partial V_i/\partial y_k$ are $O(h)$. We also assume that all functions, p, W etc., are smooth.

Let us consider the difference operator GRAD. First we notice that the following equation is fulfilled:

(20) $$\frac{\partial}{\partial x_i}\sum_{S_i} V_k = \sum_{S_i}\frac{\partial}{\partial x_i}V_k = 0.$$

The same relation holds for the derivative with respect to y. These relations follow from geometrical properties, that is the sum of volumes V_i and all neighbours of point I does not depend on the position of point I. By using Taylor expansion we obtain

$$u_k = u_i + \frac{\partial u}{\partial x}\Big|_i(x_k - x_i) + \frac{\partial u}{\partial y}\Big|_i(y_k - y_i) + \xi_{ik},\ |\xi_{ik}| \leqslant Ch^2.$$

Substituting this expansion into GRAD and using (3)-(5), (20) we obtain

$$(\mathrm{GRAD}_x(r_h u))_i = -\frac{1}{V_i}\left(-\frac{\partial u}{\partial x}\Big|_i \cdot V_i + O(h^3)\right) = r_h\left(\frac{\partial u}{\partial x}\Big|\right)_i + O(h)$$

and the same results for GRAD_y. Thus we proved the first order of local approximation for grad, and hence for the equation of motion.

Let us consider the difference operator DIV. There is no local approximation for the general case, this can be shown by counter-example. In spite of this fact it can be shown [21] that the equation of continuity is approximated in the integral sense with first order. Let us discuss some details.

First we consider the Eulerian form of the continuity equation

$$\int_V \frac{\partial \rho}{\partial t}\, dV + \oint_{\partial V} \rho\, (W,n)\, ds = 0, \tag{21}$$

where V is an arbitrary Eulerian volume. The difference analogue of this equation is constructed as follows. We consider only particles in the volume V, multiply the equation of continuity by the volume of the appropriate Dirichlet cell and sum up the obtained expressions. The result will be

$$\Sigma \frac{d\rho_i^h}{dt} V_i + \Sigma\, \rho_i\, (\mathrm{DIV}\, W^h)_i\, V_i = 0, \tag{22}$$

where the superscript h means that we deal with a solution of the difference equation. Substituting the projections of ρ and W the solutions of the differential equations, into (22) we obtain:

$$\Sigma \frac{d}{dt}(r_h\rho)_i\, V_i^0 + \Sigma\, (r_h\rho)_i\, (\mathrm{DIV}\, R_h W)_i\, V_i^0 = \phi, \tag{23}$$

where the volume V_i^0 is the volume of the appropriate Dirichlet cell and where the Dirichlet structure is constructed by means of those particle coordinates x,y that are exact solutions of the differential equations. Making use of the approximation property of operator grad and of the relation

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + (W, \mathrm{grad}\, \rho)$$

we obtain

$$\phi = \int_V \frac{\partial \rho}{\partial t}\, dV + \oint_{\partial V} \rho\, (W,n)\, ds + O(h), \tag{24}$$

and using (21) we finally obtain that ϕ is of linear order in h.

Similar considerations lead to a first order approximation of the integral equation of continuity in Lagrangian form

$$\frac{d}{dt}\int_V \rho\, dV = 0.$$

Artificial viscosity.

The artificial dissipative process is regarded as a consequence of inelastic collisions of particles modeling the medium. Let us explain this idea in the one-dimensional case. We will assume a non-uniform Lagrangian grid, whose nodes have coordinates $\{x_i\}$. As in the two-dimensional case, the quantities m, ρ, p, u, ε, and V are related to each node. In the one-dimensional case, the boundaries of Dirichlet cells are located at the mid-points between nodes. When introducing the artificial dissipative process, we assume that the mass m_i is distributed over the whole Dirichlet cell. We next consider the process of inelastic collision of particle numbered I with its neighbors, i.e., with particles I-1 and I+1. We assume that collision happens at the boundary of the Dirichlet cells. Only parts of the mass of the corresponding cells participate in the collision. When particles I and I+1 collide, part of the mass of particle I, call it $\delta m_{i,i+1}$, is involved in the collision process, and also part $\delta m_{i+1,i}$ of particle I+1. The first subscript refers to the particle, whose part is considered, while the second indicates the counterpart. As a result of inelastic collision of masses $\delta m_{i,i+1}$ and $\delta m_{i+1,i}$, a particle with mass $\delta m_{i,i+1} + \delta m_{i+1,i}$ and the velocity $\bar{u}$ is formed. By the law of the conservation of momentum

$$(25) \qquad (\delta m_{i,i+1} + \delta m_{i+1,i})\,\bar{u} = \delta m_{i,i+1}\,\bar{u}_i + \delta m_{i+1,i}\,\bar{u}_{i+1}$$

where $\bar{u}_i$ and $\bar{u}_{i+1}$ are the velocities of masses $\delta m_{i,i+1}$ and $\delta m_{i+1,i}$ respectively. If we assume $\bar{u}_i = u_i$ and $\bar{u}_{i+1} = u_{i+1}$, we obtain

$$(26) \qquad \bar{u} = (\delta m_{i,i+1}\,u_i + \delta m_{i+1,i}\,u_{i+1})/(\delta m_{i,i+1} + \delta m_{i+1,i}).$$

We next assume that the part of mass $\delta m_{i,i+1}$ having velocity $\bar{u}$, collides inelastically with the mass $m_i - \delta m_{i,i+1}$ having velocity u_i, while the part $\delta m_{i+1,i}$, similarly collides with the mass $m_{i+1} - \delta m_{i+1,i}$ A similar process proceeds with particles I and I-1, where parts of the masses, $\delta m_{i,i-1}$ and $\delta m_{i-1,i}$, participate in the collision. As a result, a particle with mass m_i again is formed, with new velocity $\hat{u}$ given by the law of the conservation of momentum:

$$\hat{u}_i = \{ (m_i - \delta m_{i,i+1} - \delta m_{i,i-1})\,u_i +$$

$$+ \delta m_{i,i+1}\,(\delta m_{i,i+1}\,u_i + \delta m_{i+1,i}\,u_{i+1})/(\delta m_{i,i+1} + \delta m_{i+1,i}) +$$

$$+ \delta m_{i,i-1}\,(\delta m_{i,i-1}\,u_i + \delta m_{i-1,i}\,u_{i-1})/(\delta m_{i,i-1} + \delta m_{i-1,i}).$$

This equation may be written in the next form:

$$(27) \qquad m_i(\hat{u}_i - u_i) = F_{i,i+1} + F_{i,i-1}$$

where

$$F_{i,i+1} = \frac{\delta m_{i,i+1}\,\delta m_{i+1,i}}{\delta m_{i,i+1} + \delta m_{i+1,i}}\,(u_{i+1} - u_i)$$

$$F_{i,i-1} = \frac{\delta m_{i,i-1}\,\delta m_{i-1,i}}{\delta m_{i,i-1} + \delta m_{i-1,i}}\,(u_{i-1} - u_i)$$

We notice that $F_{i,i-1} = -F_{i-1,1}$. The law of the conservation of momentum follows from this relation. The equation (27) describes how velocity is varied due to viscosity force.

Let us consider the dissipation of the kinetic energy. From the law of the conservation of full energy we obtain that the variation of full energy is equal to the work of viscosity forces:

$$\Delta E_K + \Delta E_I = \Delta A$$

where the variation of kinetic energy is

$$\Delta E_K = \frac{\hat{u}_i + u_i}{2}\,F_{i,i+1} + \frac{\hat{u}_i + u_i}{2}\,F_{i,i-1}$$

and the work of viscosity forces ΔA is equal

$$\Delta A = F_{i,i+1}\,\tilde{u}_{i,i+1} + F_{i,i-1}\,\tilde{u}_{i,i-1}$$

where the $\tilde{u}_{i,i+1}$, $\tilde{u}_{i,i-1}$ are some velocities which describe the moves of ends of Dirichlet cells. If we define these quantities by

$$\tilde{u}_{i,i+1} = \frac{\delta m_{i+1,i}(\hat{u}_{i+1} + u_{i+1})/2 + \delta m_{i,i+1}(\hat{u}_i + u_i)/2}{\delta m_{i+1,i} + \delta m_{i,i+1}}$$

and the similar equation for $\tilde{u}_{i,i-1}$, we obtain the next relation

for internal energy:

$$\Delta E_I = \frac{\delta m_{i,i+1}(\delta m_{i+1,i})^2}{(\delta m_{i,i+1}+\delta m_{i+1,i})^2}(u_{i+1}-u_i)\left(\frac{\hat{u}_{i+1}+u_{i+1}}{2}-\frac{\hat{u}_i+u_i}{2}\right)+$$

$$+\frac{\delta m_{i,i-1}(\delta m_{i-1,i})^2}{(\delta m_{i,i-1}+\delta m_{i-1,i})^2}(u_{i-1}-u_i)\left(\frac{\hat{u}_{i-1}+u_{i-1}}{2}-\frac{\hat{u}_i+u_i}{2}\right).$$

It is easy to show $\Delta E_I>0$ for a small step of time.

Let us investigate how to choose the δm_{ik}. We may use the relation $\delta m_{ik} = \rho_i D_{i,k} \Delta t$, where $D_{i,k}$ some characteristic velocity, frequently we use $D_{i,k}=C_i$, where C_i is the adiabatic velocity of sound in particle I. This choice corresponds to linear artificial viscosity. To provide the computation with the shock waves of "cold" background we usually use the relation $D_{i,k}=abs(u_i-u_k)$. This choice corresponds to quadratic viscosity. It possible to use some combination of linear and quadratic viscosity.

The above idea can be extended to the two-dimensional case as follows. First, a particle has inelastic collisions only with its neighbors. Second, the collision happens on the common side of Dirichlet particle I and its neighbor K. The direction of collision is perpendicular to this side. Each collision is the same as in the one-dimensional case, the role of velocity u is played by the normal with respect to the perpendicular component of the velocity vector W. The details for the two-dimensional case are presented in [16]. The difference equations for velocity and for internal energy are

$$\rho\frac{(W^{n+1}-W^n)}{\Delta t} = -\,\mathrm{GRAD}\ p + L(W)$$

$$\rho\frac{(\varepsilon^{n+1}-\varepsilon^n)}{\Delta t} = -\,p\ \mathrm{DIW}\ W + \delta\varepsilon$$

where

$$(LW)_i = (\Sigma\, e_{ik} F_{ik})/(\Delta t\, V_i), \qquad (\delta\varepsilon)_i = (\Delta E_I)_i/(\Delta t\, V_i)$$

e_{ik} is a unit vector directed from point I to point K, and

$$F_{i,k} = \frac{\delta m_{i,k}\,\delta m_{k,i}}{\delta m_{i,k}+\delta m_{k,i}}(W_k-W_i, e_{ik})$$

The expression for $\delta\varepsilon$ is similar to that of the one-dimensional case.

There are many approaches to construct the quantities δm. The main condition is $\delta m_{ik} \sim l_{ik}$, where l_{ik} is the common side of cell I and all K. This condition ensures the continuous variation of the viscosity term when the structure of the grid changes.

Treatment of boundary conditions.

The treatment of boundary conditions is one of the main parts of each free-Lagrangian approach.

First we consider the condition (W,n) = 0 on non-deformable walls, and convex computational domains. In this case all particles are placed inside the computational domain. The side of certain Dirichlet cells coincide with part of the boundary of the computational domain. For each particle of this type we introduce a fictitious neighbor placed mirror symmetric with respect to the boundary. For this fictitious particle we use the same pressure as for the original particle, and a velocity whose normal component has reversed sign, and whose tangential one is the same as in the original particle. Then we use the general equations where the fictitious particles are included. It is possible for particle to leave the computational domain for the reason of discretized time. In this case particles are returned into the computational domain in in accordance with the law of elastic collision between the particle and the boundary.

If the computational domain is not convex then, in conformity with to the definition of Dirichlet cells (Sect.1), we have to place some particles upon the vertices of the polygon approximating the boundary. For the particles lying on the boundary, the equations are written in the natural way. For those particles whose Dirichlet cells tuch the boundary, we may use the same formulas as in the case of a convex domain.

The main difficulties arize when we consider the free boundary problems. The approach we have to use, depends on the character of the physical problem.

If the boundary of the domain moves smoothly without any self-intersections and if the connectivity does not change, we will use the following approach. First we place particles upon the vertices of the polygon approximating the boundary. The operator DIV is constructed directly as shown in Sect.2. There is a difference in the formulas for $\partial V_i/\partial x_k$ when particle I and K are lying on the boundary. The operator GRAD is derived from the difference analogue of the equation

$$\int_V p \operatorname{div} \mathbf{A}\, dv + \int_V (\mathbf{A}, \operatorname{grad} p)\, dv = \oint_S p\,(\mathbf{A}, \mathbf{n})\, ds$$

This corresponds to the method of support operators. From this relation we obtain an expression for the operator GRAD on the boundary and inside V.

This method of constructing the difference scheme allows us to derive conservative schemes. The main difficulty of constructing the analogue of the integral equation lies in building the analogue of the boundary integral in such a way that at least the zero-order approximation of GRAD is guaranteed. The details are described in [22].

If strong deformation, self-intersection, and other singularities are present when the boundaryis moving, then we use the following method. For describing of the domain, we use particles placed inside the domain only. At the start point in time we place all particles into some sufficiently big rectangle. That space of the rectangle that is free from particles, we fill with fictitious particles. The distribution of these particles has to satisfy the following conditions. First the particles have to cover the free space in rectangule. Second, parts boundaries between fictitious and real particles have to approximate the free boundary. At the next moments of time we use a special algorithm for the movement of the fictitious particles in order to fulfillthe second condition. This algorithm is based on considering the equation of motion for the sequence of lines approximating the free boundary.

Let us demonstrate the main idea for the one-dimensional case. Assume that, at the first moment, the free boundary lies on point X_b on Fig.12 , and all real particles are placed between X_{min}, X_{max}.

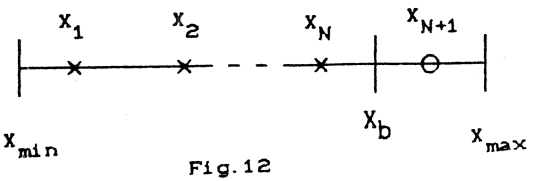

Fig.12

The real particles are numbered from 1 to N and marked by the symbol * in Fig 12. In order to describe the free boundary, we introduce fictitious particle N+1 such that $X_b=(x_N + x_{N+1})/2$. Consequently point X_b coincides with the common boundary of the Dirichlet cells for particles N and N+1. If we consider the free boundary condition in point X_b then the pressure P_b in this point is given. Now we can specify the equation of motion for free boundary

$$\frac{U_b^{n+1} - U_b^n}{\Delta t} = \frac{P_N^n - P_b^n}{\rho_N^n (x_b^n - x_N^n)}$$

In accordance with our method we want to preserve the relation $X_b=(x_N + x_{N+1})/2$ for all moments of time. From the last two equations we obtain the expression of velocity in the fictional point N+1: $u_{N+1}^{n+1} = 2U_b^{n+1} - u_N^{n+1}$. The velocity in the point N is computed by the common formula because particle N has the real neighbor N-1 and the fictitious neighbor N+1 and the pressure in particle N+1 is equal to P_b. The velocity u_{N+1}^{n+1} is used for computing the internal energy in particle N.

Let us briefly discuss the two-dimensional case. If a real particle has only one fictitious neighbor then the situation is similar to the one-dimensional case. Note that the role of u in the above consideration is played by the normal component of the velocity vector with respect to the boundary. In order to restore the full vector in a fictitious particle, we use the tangential component of velocity in the real particle. If a real particle has several fictitious neighbors then we subject to each fictitious neighborthe above procedure and then the obtained results are smoothed. All details are described in [22].

There are problems when the normal component of the velocity vector has a discontinuity on the boundary. An example would be the task of penetration of one body to another. For the problems of this type the above approaches do not give satisfactory results. It is impossible to use the first approach because the connectivity may change in problems of this type. The second approach gives bad results because smoothing used. This procedure elimination the physically essential discontinuity of the normal component of the velocity vector. One possible way of solving a problem of this type is described below.

As for second approach, at the first moment of time, we place all particles into some sufficiently big rectangle, that is free from particles we fill the free space in the rectangle with particles. These particles represente some light "gas". First this gas must be light enough not to prevent the moves of real particles. Second, the equation of state must satisfy two conditions: We require weak dependence on density and that the pressure, given by equation of state, must be approximately equal to the pressure on the free boundary. For all moments of time we assume that the free boundary is a boundary between real particles and some "gas". The details and examples of computation are presented in [23].

A difference scheme in cylindrical coordinates that preserves symmetry of gas dynamic flow.

For many practical two-dimensional problems it is more convenient to use cylindrical coordinates, but this raises new questions. For example a difference scheme can not preserve all symmetries of the original physical problem. Symmetry is very important when the evolution of a small perturbation is investigated. One problem of this type is related to the stability of the compression of spherical laser targets [26]. The evolution of a small distortion of the spherical form of a target shall be investigated as a sample computation. The difference scheme being used has to preserve the symmetry of the original physical problem, and must not introduce additional distortions. In this section, one possible scheme is described preserving symmetry.

Let us rewrite the difference scheme in Cartesian coordinates from Sect.2:

$$\rho_i = V_i/m_i \quad , \quad \rho_i \frac{dW_i}{dt} = - (\mathrm{GRAD}\ p)_i \ , \qquad \rho_i \frac{d\varepsilon_i}{dt} = - p_i\ (\mathrm{DIV}\ W)_i \ , \tag{28}$$

where

$$(\mathrm{DIV}\ W)_i = \frac{1}{V_i} \sum_{k \in S_i} \left(\frac{\partial V_i}{\partial x_k} Wx_k + \frac{\partial V_i}{\partial y_k} Wy_k \right). \tag{29}$$

$$(\mathrm{GRAD}_x\ p)_i = - \frac{1}{V_i} \sum_{k \in S_i} \frac{\partial V_k}{\partial x_i} p_k, \quad (\mathrm{GRAD}_y\ p)_i = - \frac{1}{V_i} \sum_{k \in S_i} \frac{\partial V_k}{\partial y_i} p_k.$$

For cylindrical coordinates (r,z) the difference scheme has the similar form after replacing (x,y) by (r,z). For the cylindrical case, the volume V_i has to correspond to the rotation figure of the Dirichlet cell on the plane (r,z).

From relations (28),(29), we see that all properties of the difference scheme depend on the choice of the formula of the volume V_i. For Cartesian coordinates, we have only one formula that represents an exact volume of a Dirichlet cell on the plane. For the cylindrical coordinates, we have many different

possibilities.

Let us try to describe a scheme in cylindrical coordinates that preserves plane, cylindrical, and spherical types of symmetry.

In the discrete case, the preservation of symmetry means that it holds on the special grid. For example, spherical symmetry holds on a radial-circular grid over the plane (r,z). The scheme will preserve symmetry if we use the following approximating formula for the volume: $V_i^{rz} = r_i V_i$, where V_i is volume of the Dirichlet cell of particle I on the plane (r,z) and r_i is r-coordinate of particle I. If we use this volume then we obtain the following relationsfor difference operators DIV^{rz}, and GRAD^{rz}

$$(\mathrm{DIV}^{rz}\ W)_i = (\mathrm{DIV}^{xy}\ W)_i + (Wr)_i/r_i$$

$$(\mathrm{GRAD}_r^{rz}\ p)_i = (1/r_i)(\mathrm{GRAD}_r^{xy}\ \tilde{p})_i - p_i/r_i$$

$$(\mathrm{GRAD}_z^{rz}\ p)_i = (1/r_i)(\mathrm{GRAD}_z^{xy}\ \tilde{p})_i$$

where $\tilde{p}_k = r_k p_k$. Let us briefly describe the situation of spherical symmetry. We introduce the radial-circular grid, equidistant with respect to angle counted from the z-axis. The coordinates of particles given by

$$r_{ij} = R_j \sin\theta_i, \ z_{ij} = R_j \cos\theta_i, \ \theta_i = (i-0.5)\Delta\theta, \ \Delta\theta = 0.5\ \pi/N$$

where R_j is the spherical radius, and θ_i - is the angle, whence indices vary in the ranges $1 \leqslant i \leqslant N$, $1 \leqslant j \leqslant M$. Let us assume that, at thefirst moment, the following equations are valid for the mesh functions

$$p_{ij} = p_j, \ \varepsilon_{ij} = \varepsilon_j, \ \rho_{ij} = \rho_j,$$

$$W\theta_{ij} = - \cos\theta_i Wr_{ij} + \sin\theta_i Wz_{ij},$$

$$WR_{ij} = \sin\theta_i Wr_{ij} + \cos\theta_i Wz_{ij} = U_j$$

We should prove that these conditions are valid for all moments of time, i.e.

$$- \cos\theta_i (\mathrm{GRAD}_r^{rz}\ p)_{ij} + \sin\theta_i (\mathrm{GRAD}_z^{rz}\ p)_{ij} = 0,$$

$$\sin\theta_i (GRAD_r^{rz}\ p)_{ij} + \cos\theta_i (GRAD_z^{rz}\ p)_{ij} = G_j,$$

$$(DIV^{rz}\ W)_{ij} = D_j.$$

The detailed proofs of these relations are presented in [24].

A Lagrangian scheme of Godunov type on the Dirichlet grid.

In this section we use Godunov's ideas [27,28] for the construction of a Lagrangian difference scheme on the Dirichlet grid. First we write the Lagrangian hydrodynamics equations in conservative integral form

$$m\frac{dWx}{dt} = \oint_{\partial D(t)} p\ dy, \qquad m\frac{dWy}{dt} = -\oint_{\partial D(t)} p\ dx,$$

$$m\frac{d\ E}{d\ t} = -\oint_{\partial D(t)} p(W,n)\ ds$$

$$\frac{dV(D)}{d\ t} = \oint_{\partial D(t)} (W,n)\ ds$$

where D(t) is a liquid particle, V(D) is the volume of D, m is the mass of the particle, not being varied in time, Wx, Wy are the Cartesian components of the velocity vector, E is the full specific energy.

When we construct a difference scheme we have to choose the domain D (particularely, the boundary of this domain) and to approximate the boundary integrals on the right hand side of the equations. We use the boundary of the Dirichlet cell in the capacity of $\partial D(t)$. We apply Godunov's idea in order to approximate the integrands. In accordance with Godunov's idea we make use of the solution of the one-dimensional Riemann problem [29] to calculate the pressure p and the normal component of the velocity vector on a side of the Dirichlet cell. Under this approach we do not need a special artificial viscosity [27]. The detailed description of this approach and computational examples are presented in [25].

The scheme based on Godunov's idea is also presented in [28].

Computational examples.

As a first sample computational experiment, consider the task of interaction of a shock with some heavy obstacle [6]. The problem is stated in Fig.13, all medium are the ideal gas with γ = 5/3. A plane shock wave (with parameters shown on Fig.13) moves with constant velocity and runs against the obstacle given by a semi-infinite right angle.

If we use Cartesian coordinates with origin in the vertex of the right angle and count the time from that moment when the shock meets the obstacle, then all functions will depend on the variables ξ = x/t and η = y/t. The qualitative features of the process are described as follows. At the moment when the shock meets the obstacle on lower the ,boundary we have a Riemann problem. As the result, the pressure on the front of the shock against a heavy material equals 3.95, and the front velocity equals 0.59. At the same time, the front velocity of the shock, going on light gas, equals 4/3. As a result on the right hand side of the obstacle, we have the segment placed below the front of shock in heavy material where the pressure in heavy material is bigger then in light material. This distribution of pressure leads to deformation the obstacle. The resulting configuration is shown in Fig.14.

In the computational experiment, the problem is considered in a bounded domain with moving lower boundary: $0<x<1.6$, $y_1(t)<y<2$, where $y_1(t) = t$. At the first moment the vertex of the angle has coordinates x=1, y=1. In Fig.14, 15 we present the distribution of theparticlesand the boundary of the heavy material for t = 0.6, t = 0.7 in coordinates x,y. The total number of particles equals 800. In Fig.16, 17 we present the same moments in coordinates ξ, η. The form of heavy material in Fig.16, 17 is almost the same. Hence the numerical solution has a self similar behaviour. Fig.18 illustrates the Dirichlet cells for t = 0.7. In Fig.19 we present the trajectories for the particles with coordinates x,y in range [0.8,1.2]x[0.8,1.2] at the first moment,in coordinates ξ, η during the period $0.1< t < 0.7$. The form of the trajectories shows that a vorticex appeared in the domain near the vertex of angle

As a second computational experiment, let us consider the problem of the interaction of a shock with some bubble [30] - (see Fig.20). This problem has axial symmetry. The bubble is filled

with helium and surrounded by neon. Helium and neon are used with the equation of state $p=(\gamma-1)\rho\varepsilon$, γ = 5/3. Gases ahead of the shock and in the bubble initially have a pressure zero, they are in the rest. The density of helium is equal to 1 and the density of neon is equal to 5. The velocity of the shock front equals 4/3. The origin of the reference system is placed in the center of the bubble. The radius of the bubble equals 10. The shock is going along the z-axis.

The computation is carried out in cylindrical coordinates r,z. At the first moment the shock front is placed on z = -12. The boundary conditions are similar to the previous problem. Gas parameters of in front of the shock are ρ = 5, ε = 0, v = 0, behind the front - ρ = 20, ε = 0.5, v = 1. Gas parameters inside the the bubble are ρ = 1, ε = 0, v = 0. The distribution of particles are presented on Fig.20 for several moments of time. The maximal compression of the bubble is obtained at t = 16.2. This result is in good agreement with results from [29].

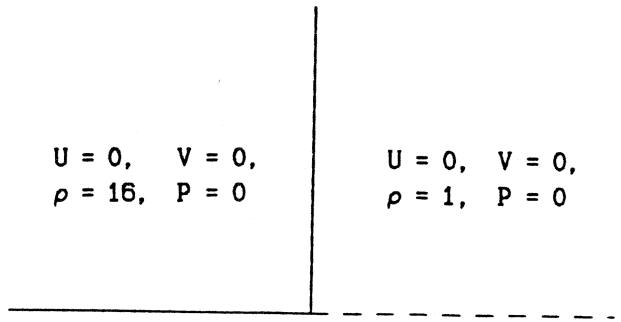

Fig.13

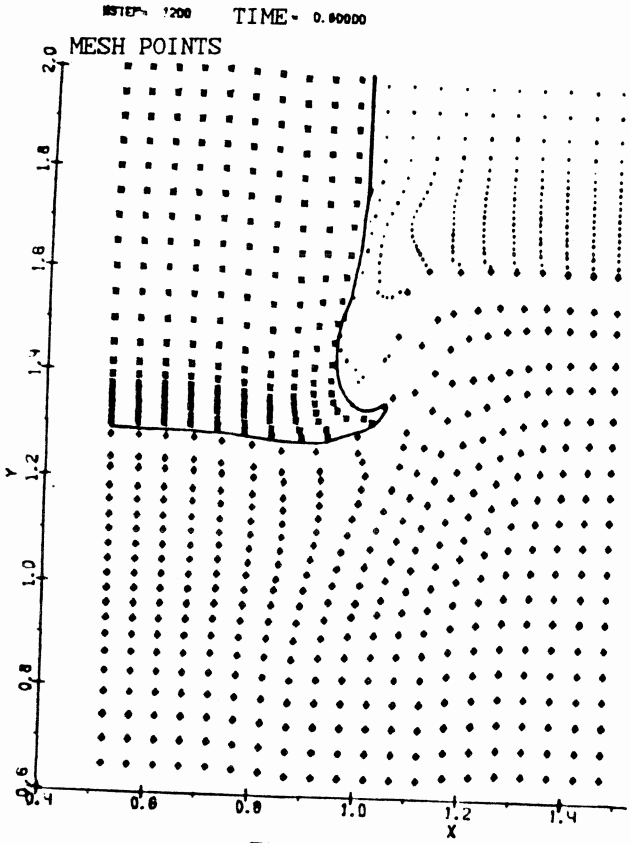

Fig.14

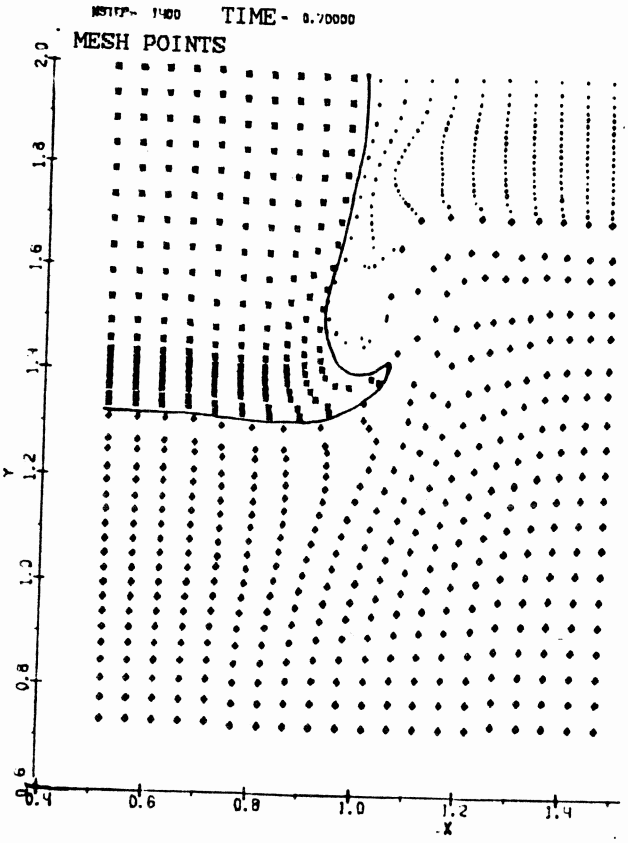

Fig.15

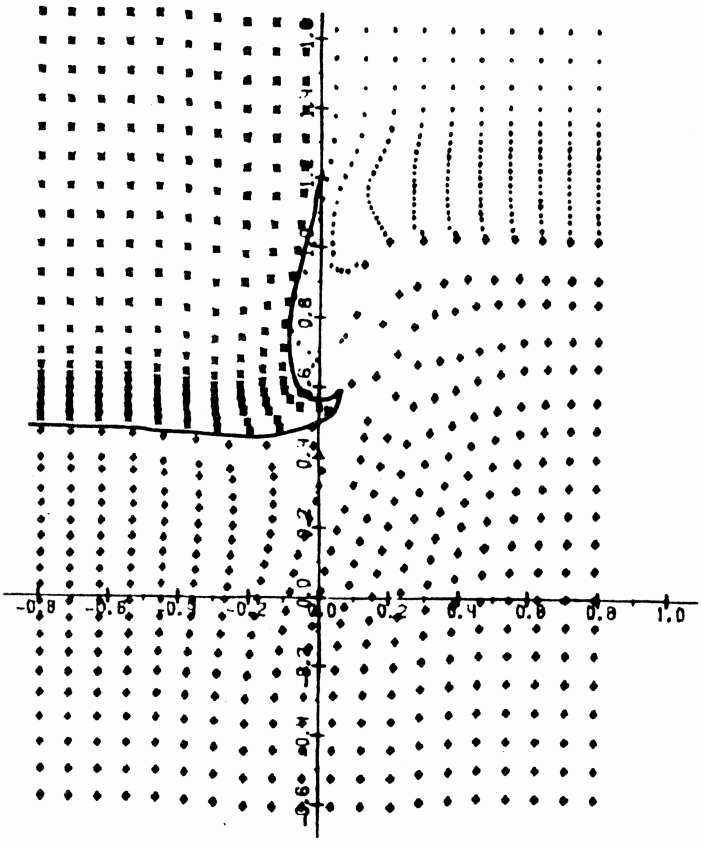

Fig.16

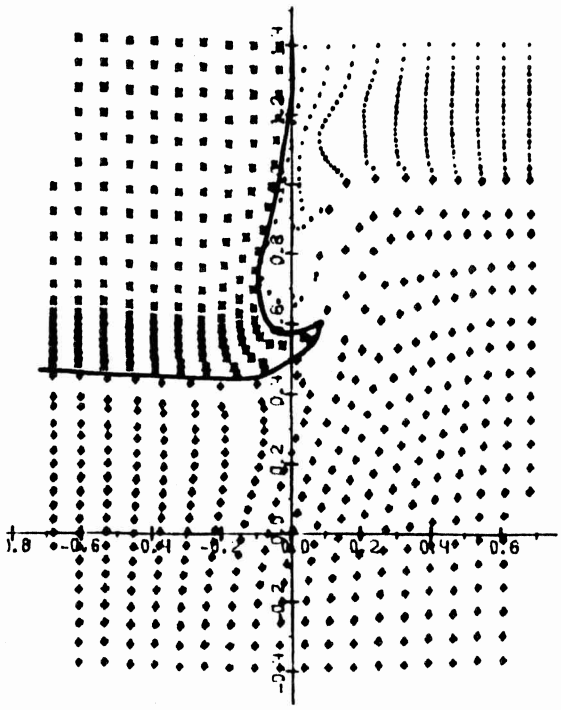

Fig.17

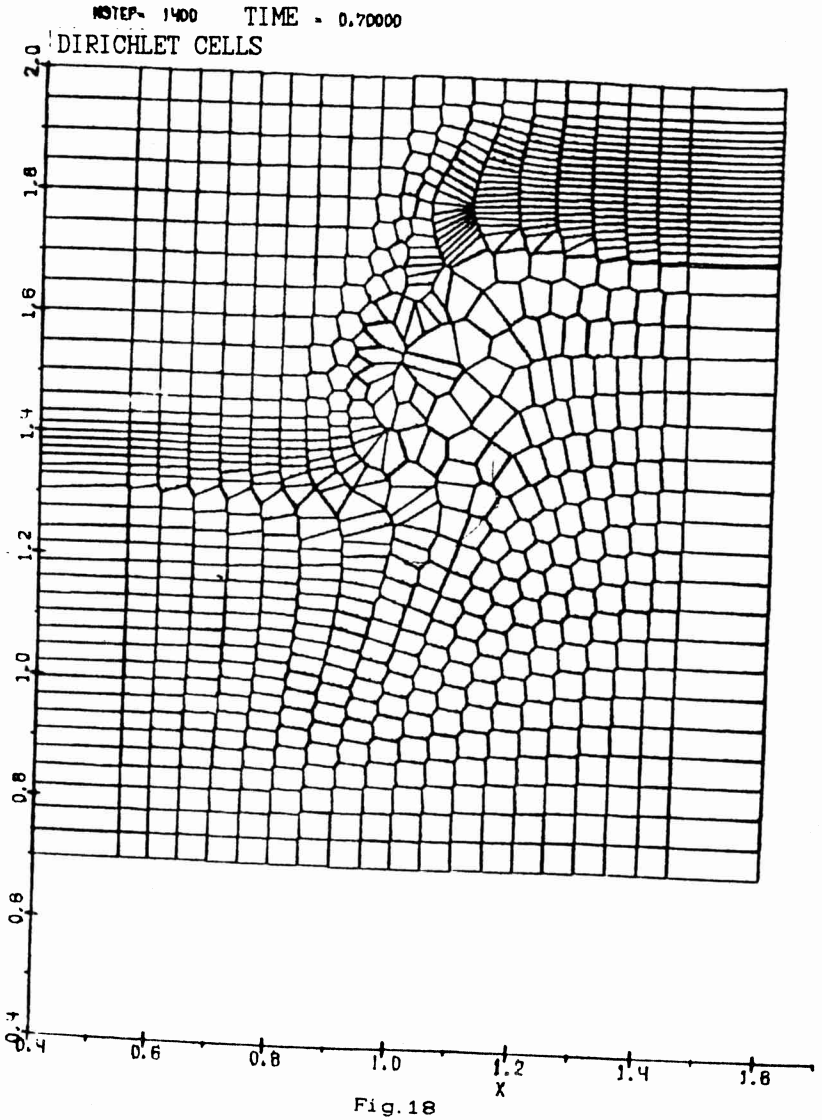
NSTEP= 1400 TIME = 0.70000
DIRICHLET CELLS
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.4
0.6
0.8
1.0
1.2
1.4
1.6
X

Fig.18

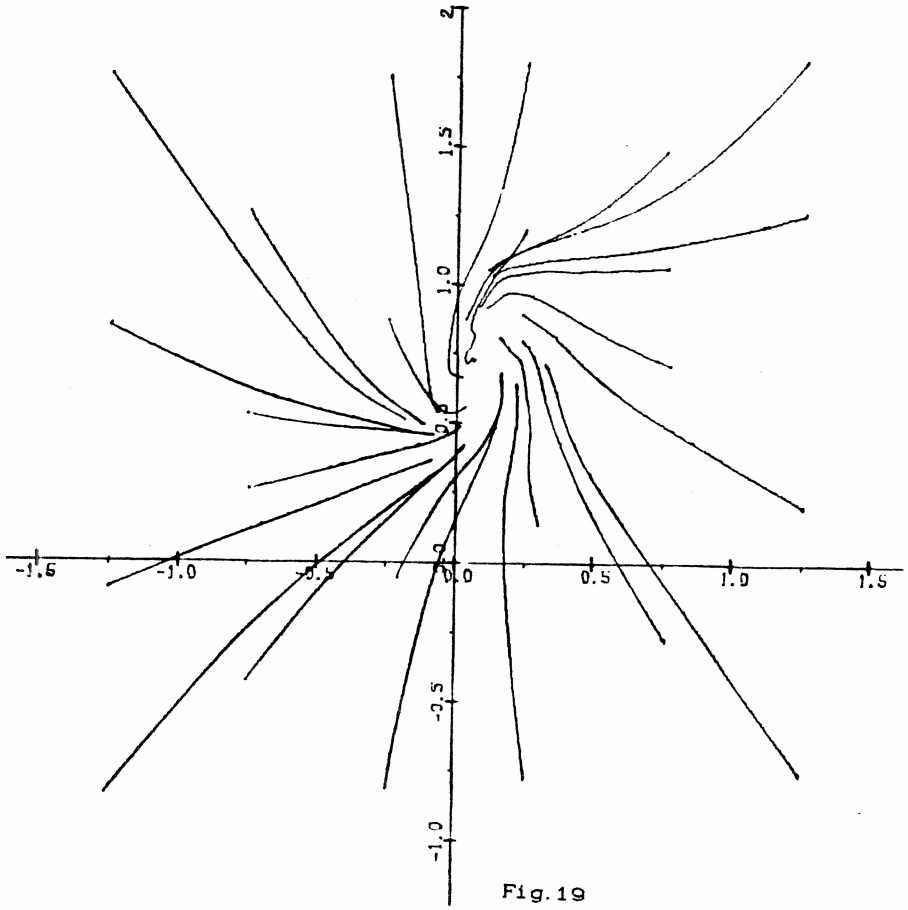
2
1.5
1.0
0.5
-1.5
-1.0
0.0
0.5
1.0
1.5
-2.5
-1.0

Fig.19

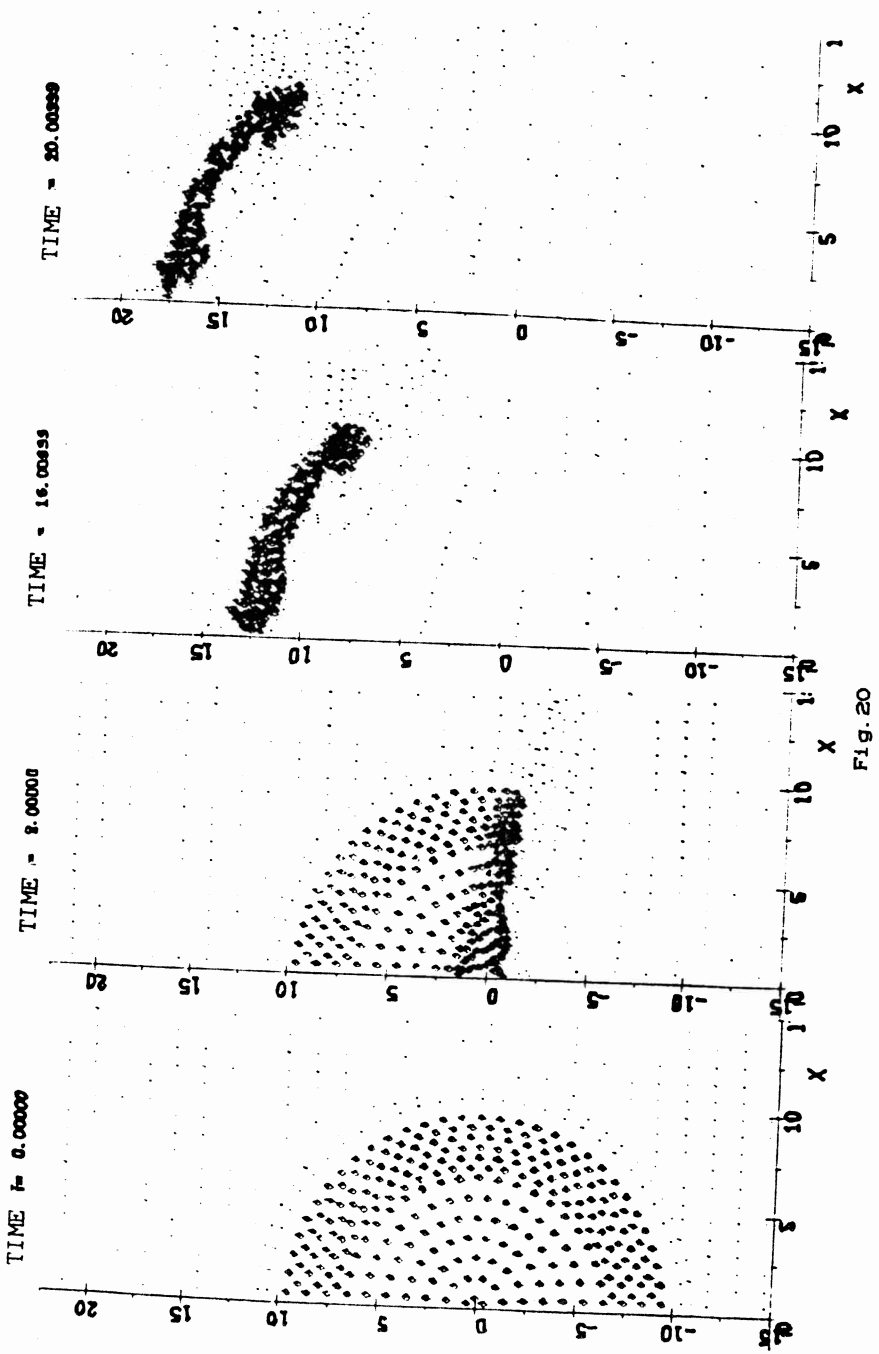

Fig.20

Conclusion.

In this paper we surveyed the purely Lagrangian free-Lagrangian method which can be characterized as follows. The the mass of particles is not changed, there is no mass, momentum and energy exchange between particles, no interpolation or smoothing procedures are used; the cell, surrounding a particle, contains only one material, there are no " mixed " cells. This properties are very important for practical problems that deal with several materials of essentially different physical properties.

By using Dirichlet cells we avoid singularities when the grid structure changes. The support-operators method gives us conservative difference schemes. The local approximation of the equation is shown to be of first order. A natural way of introducing viscosity as the consequence of inellastic collision of particles is proposed. Some approaches to the treatment of free boundaries are described. Applicability is demonstrated by computational experiments.

We believe that this method makes it possible to treat a new class of physical problems, tasks with strong deformations.

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