CONVERGENCE OF THE MIMETIC FINITE DIFFERENCE METHOD FOR DIFFUSION PROBLEMS ON POLYHEDRAL MESHES*

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Abstract. The stability and convergence properties of the mimetic finite difference method for diffusion-type problems on polyhedral meshes are analyzed. The optimal convergence rates for the scalar and vector variables in the mixed formulation of the problem are proved.

Key words. compatible discretizations, mimetic finite difference method, convergence, polyhedral meshes

AMS subject classifications. 65N06, 65N12, 65N15, 65N22, 65N30

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1. Introduction. The main goal of this paper is to establish convergence of *mimetic discretizations* of the first-order system that describes linear stationary diffusion on unstructured polyhedral meshes. The main idea of the mimetic finite difference (MFD) method is to mimic the underlying properties of the original continuum differential operators, e.g., conservation laws, solution symmetries, and the fundamental identities and theorems of vector and tensor calculus. For the linear diffusion problem, this means that the mimetic discretizations mimic the Gauss divergence theorem needed for the local mass conservation, the symmetry between the continuous gradient and divergence operators needed for proving symmetry and positivity of the resulting discrete operator, and the null spaces of the involved operators needed for stability of the discretizations.

The MFD method has been successfully employed for solving problems of continuum mechanics [19], electromagnetics [14], gas dynamics [8], and linear diffusion on simplicial and quadrilateral meshes in both the Cartesian and polar coordinates [15, 13, 20, 17]. Recent advances in extending the mimetic discretizations to general polygonal meshes [16] have inspired us to develop the rigorous convergence theory for unstructured polygonal and polyhedral meshes.

The polyhedral elements appear naturally in reservoir models simulating thinning or tapering out ("pinching out") of geological layers. The pinchouts are modeled with mixed types of mesh elements, pentahedrons, prisms, and tetrahedrons which

are frequently obtained by collapsing some of the elements in a structured hexahedral or prismatic mesh.

Other sources of polyhedral meshes are the adaptive mesh refinement methods. A necessity to have a conformal mesh results in an abundant mesh refinement, e.g., in the methods using the red-green refinement strategy. However, the locally refined mesh may be considered as the conformal polyhedral mesh with degenerate elements (for instance, when the angle between two faces is 180°). If we know how to discretize a problem on a general polyhedral mesh, the superfluous mesh refinements can be avoided. A similar argument can be applied to nonmatching meshes which frequently may be treated as conformal polyhedral meshes with degenerate elements. This is the way followed, for instance, in [16] for two-dimensional (2D) meshes.

Allowing arbitrary shape for a mesh element provides greater flexibility in the mesh generation process, especially in the regions where the geometry is extremely complex. Even in the case of an unstructured hexahedral mesh, it may be beneficial to split the curvilinear faces into triangles in order to use more accurate discretization methods and to get a smaller number of unknowns relative to a tetrahedral partition. It is obvious that by splitting each face of a hexahedron into four triangles we get a 24-face polyhedron which is frequently nonconvex.

Some of the simulations in the fluid dynamics indicate that the polyhedral meshes may lead to superior convergence rates and accuracy relative to tetrahedral meshes. We refer readers to the CD-adapco group website (www.cd-adapco.com/news/18/newsdev.htm) for more detail. The polyhedral meshes are also used in a number of radiation–hydrodynamics applications [21, 22, 7]. For instance, one of the approaches to increase robustness of arbitrary Lagrangian–Eulerian simulations is to change the mesh connectivity which leads obviously to general polyhedral meshes.

The diffusion-type (elliptic) problems appear in many applications, for instance, the temperature equation in heat diffusion or the pressure equation in flow problems. The necessity to solve such problems arises in numerical methods for radiation transport coupled with hydrodynamics, mesh smoothing algorithms, etc. In this paper, we consider a diffusion problem formulated as a system of two first-order equations, which is suitable for deriving locally conservative discretizations.

The mimetic discretizations have demonstrated excellent robustness and accuracy in simulations; however, a rigorous convergence proof has always been lacking. The original approach to prove the convergence of these discretizations has been based on establishing the relationship between the MFD and mixed finite element methods [2, 3] which is certainly not enough for many interesting applications. In this paper, we developed a novel technique for proving convergence estimates which may be applied to the case of meshes consisting of arbitrary types of elements, e.g., tetrahedrons, pyramids, hexahedrons, degenerate polyhedrons, etc. The restrictions on a polyhedron shape imposed in section 2 still allow extremely complex elements which cover the majority of meshes used in applications. Note that the developed methodology can be applied to 2D diffusion problems on unstructured polygonal meshes with minor modifications.

The paper is organized as follows. In section 2, we describe the problem under consideration and the class of polyhedral meshes used in the convergence analysis. In section 3, we formulate the MFD method. In section 4, we prove the stability result. In section 5, we prove the convergence of mimetic discretizations. One of the key elements used in our technique, the lift property, is discussed in detail in the appendix.
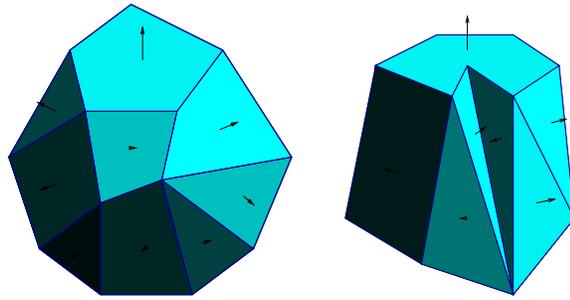
FIG. 2.1. *Two possible elements and the normal to their faces.*

2. The assumptions on the problem and on the mesh. Let us consider a model elliptic boundary value problem

$$(2.1) \qquad \qquad \text{div } \mathbf{F} = b,$$

$$(2.2) \qquad \qquad \mathbf{F} = -\mathbf{K} \text{ grad } p.$$

Here p denotes a scalar function that we refer to as the pressure, $\mathbf{F}$ denotes a vector function that we refer to as the velocity, $\mathbf{K}$ denotes a full symmetric tensor, and b denotes a source function. The problem is posed in a bounded polyhedral domain $\Omega \subset \mathbb{R}^3$, and is subject to appropriate boundary conditions on $\partial \Omega$. For simplicity, we assume that the homogeneous Dirichlet boundary conditions are imposed on $\partial \Omega$. We also assume that $\mathbf{K}$ satisfies the following regularity and ellipticity property.

P1 (*regularity and ellipticity of* $\mathbf{K}$). Every component of $\mathbf{K}$ is in $W_\infty^1(\Omega)$ and $\mathbf{K}$ is strongly elliptic, meaning that there exist two positive constants κ_* and κ^* such that

$$(2.3) \qquad \kappa_* \|\mathbf{v}\|^2 \leq \mathbf{v}^T \mathbf{K}(\mathbf{x}) \, \mathbf{v} \leq \kappa^* \|\mathbf{v}\|^2 \quad \forall \, \mathbf{v} \in \mathbb{R}^3 \quad \forall \, \mathbf{x} \in \Omega.$$

Let $\mathcal{T}_h$ be a nonoverlapping conformal partition of Ω into polyhedral elements E. For every element E, we denote by $|E|$ its volume and by h_E its diameter. Similarly, for each face e we denote by $|e|$ its area and for every edge ℓ we denote by $|\ell|$ its length. Depending on context, we shall use ∂E either for the boundary of E or the union of element faces. We also set as usual

$$h = \sup_E h_E.$$

The elements E are assumed to be closed simply connected polyhedrons, rather general in shape (see, for instance, Figure 2.1). However, we need some basic assumptions of shape regularity. As we shall see, the assumptions are formally complicated sometimes, but they will hold for practically all partitions which are not totally unreasonable.

M1 (*assumptions on the domain* Ω). We assume that Ω is a polyhedron with a Lipschitz continuous boundary.

M2 (*number of faces and edges*). We assume that we have two positive integers N_e and N_ℓ such that every element E has at most N_e faces, and each face e has at most N_ℓ edges.

M3 (*volumes, areas, and lengths*). We assume that there exist three positive constants v_*, a_*, and l_* (for volume, area, and length, respectively) such that for
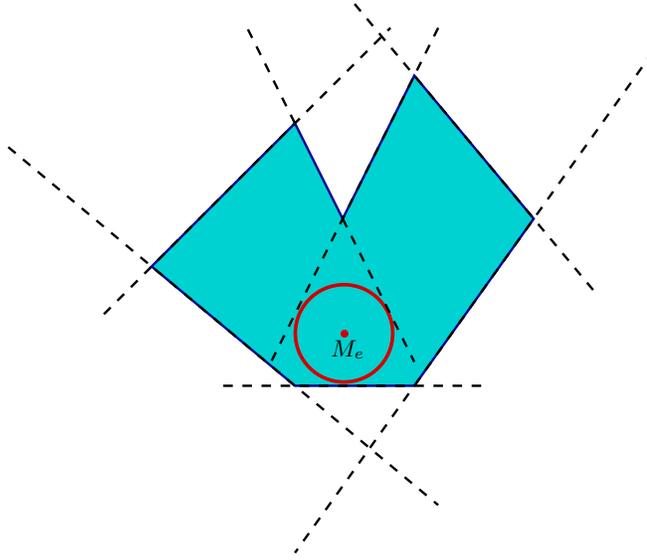
FIG. 2.2. *A star-shaped face with the circle of radius ρ_* centered at M_e.*

every element E we have

$$(2.4) \qquad v_* \, h_E^3 \le |E|, \quad a_* \, h_E^2 \le |e|, \quad l_* \, h_E \le |\ell|$$

for all faces e and edges ℓ of E.

M4 (*star-shaped faces*). We assume that the mesh faces are flat and that there exists a positive number γ_* such that for each element E and for each face $e \in \partial E$ there exists a point $M_e \in e$ such that e is star-shaped with respect to every point in the disk of radius $\gamma_* h_E$ centered at M_e.

We recall that e is star shaped with respect to a point $P \in e$ if every straight ray exiting from P (in the plane of e) intersects ∂e only once. In what follows we shall often use the notation

$$(2.5) \qquad \rho_* = \gamma_* h_E,$$

which is illustrated in Figure 2.2.

M5 (*the pyramid property*). With the notation of Assumption M4, we further assume that for every $E \in \mathcal{T}_h$, and for every $e \in \partial E$, there exists a pyramid P_E^e *contained in E* such that its base equals e, its height equals $\gamma_* h_E$, and the projection of its vertex onto e is M_e.

M6 (*star-shaped elements*). We assume that there exists a positive number τ_* such that for each element E there exists a point $M_E \in E$ such that E is star shaped with respect to every point in the sphere of radius $\tau_* h_E$ centered at M_E.

As before, we say that E is star shaped with respect to a point $P \in E$ if every straight ray exiting from P intersects ∂E only once.

3. MFD method. Let us introduce an operator $\mathcal{G}$, $\mathcal{G} p = -\mathbf{K} \mathrm{grad}\, p$, which we refer to as the flux operator. Furthermore, we introduce the following scalar products:

$$(3.1) \qquad (\mathbf{F}, \mathbf{G})_X = \int_\Omega \mathbf{F} \cdot \mathbf{K}^{-1} \mathbf{G} \, \mathrm{d}V$$

and

(3.2)
$$(p, q)_Q = \int_\Omega pq \, dV$$

in the space X of velocities and in the space Q of pressures, respectively. Using the above notation, we may rewrite the Green's formula

(3.3)
$$\int_\Omega \mathbf{F} \cdot (\mathbf{K}^{-1}\mathcal{G} \, p) \, dV = \int_\Omega p \, \mathrm{div} \, \mathbf{F} \, dV$$

in the equivalent form

$$(\mathbf{F}, \mathcal{G} \, p)_X = (p, \, \mathrm{div} \, \mathbf{F})_Q.$$

The last expression clearly states that the flux and divergence operators are adjoint to each other:

$$\mathcal{G} = \mathrm{div}^*.$$

The MFD method produces discretizations of these operators which are adjoint to each other with respect to scalar products in the discrete velocity and pressure spaces.

The *first* step of the MFD method is to specify the degrees of freedom for physical variables p and $\mathbf{F}$ and their location.

We consider the space Q^d of discrete pressures that are constant on each polyhedron E. For $\mathbf{q} \in Q^d$ we shall denote by q_E (or by $(\mathbf{q})_E$) its (constant) value on E. The dimension, N_Q, of Q^d is obviously equal to the number of polyhedrons in $\mathcal{T}_h$. In what follows, we shall denote by Q^d either the vector space $\mathbb{R}^{N_Q}$ or the space of piecewise constant functions depending on context. The identification will be obvious and no confusion should arise.

The definition of the space of discrete velocities requires some additional considerations. To every element E in $\mathcal{T}_h$ and to every face e of E we associate a number F_E^e and the vector field $F_E^e \, \mathbf{n}_E^e$, where $\mathbf{n}_E^e$ is the unit normal to e that points outside of E. We clearly make the *continuity* assumption that for each face e shared by two polyhedra E_1 and E_2, we have

(3.4)
$$F_{E_1}^e = -F_{E_2}^e.$$

We denote the vector space of face-based velocity unknowns by X^d. The number, N_X, of our discrete velocity unknowns is equal to the number of boundary faces plus *twice* the number of internal faces. In our theoretical discussion, we shall consider X^d as the subspace of $\mathbb{R}^{N_X}$ which verifies (3.4).

For a discrete velocity field $\mathbf{G}$ we will denote by $\mathbf{G}_E$ its restriction to the boundary of E, and by G_E^e (or by $(\mathbf{G}_E)^e$) the restriction of $\mathbf{G}_E \cdot \mathbf{n}_E$ to a face e belonging to the boundary of E. It will be convenient sometimes to use the notation

(3.5)
$$X_E^d := \{\text{restrictions of } X^d \text{ to the element } E\}.$$

It is clear that, in practice, condition (3.4) will make the number of *true independent unknowns* equal the total number of mesh faces. This means that, in a computer program, we shall prescribe one direction for the normal to each internal face e, and assign a single unknown G^e to each face, assuming that each of the two G_E^e coincides either with G^e (when the outward normal $\mathbf{n}_E$ on e coincides with the prescribed direction) or with $-G^e$ (otherwise).

To summarize, one pressure unknown is defined on each polyhedron and the discrete velocities are defined as face-based normal components. Once we get the degrees of freedom in Q^d and in X^d, we can define interpolation operators from the spaces of smooth enough scalar- and vector-valued functions to the discrete spaces Q^d and X^d, respectively. To every function q in $L^1(\Omega)$ we associate the element $\mathbf{q}^I \in Q^d$ defined by

$$(3.6) \qquad (\mathbf{q}^I)_E := \frac{1}{|E|} \int_E q \, dV \quad \forall E \in \mathcal{T}_h.$$

Similarly, for every vector-valued function $\mathbf{G} \in (L^s(\Omega))^3$, $s > 2$, with div $\mathbf{G} \in L^2(\Omega)$, we define $\mathbf{G}^I \in X^d$ by

$$(3.7) \qquad \left(\mathbf{G}_E^I\right)^e := \frac{1}{|e|} \int_e \mathbf{G} \cdot \mathbf{n}_E \, dS \quad \forall E \in \mathcal{T}_h \quad \forall e \in \partial E.$$

In the next section, we shall prove that this interpolation operator is well defined and uniformly bounded. In what follows, we shall use bold capital letters either for vectors from X^d or for continuous vector functions depending on context and leaving no room for confusion.

The *second* step of the MFD method is to equip the spaces of discrete pressures and velocities with scalar products. The scalar product on the vector space Q^d is given by

$$(3.8) \qquad [\mathbf{p}, \mathbf{q}]_{Q^d} = \sum_{E \in \mathcal{T}_h} p_E \, q_E |E| \quad \forall \mathbf{p}, \mathbf{q} \in Q^d.$$

In order to define the scalar product in X^d, we first define a scalar product $[\mathbf{F}, \mathbf{G}]_E$ for every element $E \in \mathcal{T}_h$ in the following way. Let $e_1, e_2, \ldots, e_{k_E}$ be a numbering of the faces of the element E (where k_E is clearly the total number of faces). We assume that we are given (for each E) a symmetric positive definite $k_E \times k_E$ matrix $M_E \equiv \{M_{E,i,j}\}$, and we set

$$(3.9) \qquad [\mathbf{F}, \mathbf{G}]_E = \sum_{i,j=1}^{k_E} M_{E,i,j} \, (\mathbf{F}_E)^{e_i} \, (\mathbf{G}_E)^{e_j} \quad \forall \mathbf{F}, \mathbf{G} \in X^d \quad \forall E \in \mathcal{T}_h.$$

Some minimal approximation properties for the scalar product (3.9) are required. The construction of the matrix M_E is a nontrivial task for a polyhedral element. We shall return to this problem in section 5. For the time being, we just *assume* that the scalar product (3.9) has the following property.

S1   (*stability of* $[\cdot, \cdot]_E$). We assume that there exist two positive constants s_* and S^* independent of h and E such that, for every $\mathbf{G} \in X^d$ and for every $E \in \mathcal{T}_h$, one has

$$(3.10) \qquad s_* \sum_{e \in \partial E} (G_E^e)^2 \, |E| \le [\mathbf{G}, \mathbf{G}]_E \le S^* \sum_{e \in \partial E} (G_E^e)^2 \, |E|.$$

From (3.9) we can easily construct the scalar product in X^d by setting

$$(3.11) \qquad [\mathbf{F}, \mathbf{G}]_{X^d} = \sum_{E \in \mathcal{T}_h} [\mathbf{F}, \mathbf{G}]_E \quad \forall \mathbf{F}, \mathbf{G} \in X^d.$$

The *third* step of the MFD method is to derive an approximation to the divergence operator. The discrete divergence operator, $\mathcal{DIV}^d : X^d \rightarrow Q^d$, naturally arises from the Gauss divergence theorem as

$$(3.12) \qquad (\mathcal{DIV}^d \, \mathbf{F})_E \overset{def}{=} \frac{1}{|E|} \sum_{e \in \partial E} F_E^e \, |e|.$$

We point out that our interpolation operators, in some sense, *commute* with the divergence operator. Indeed, for every vector field $\mathbf{G}$ smooth enough, we can use (3.12), (3.7), the Gauss divergence theorem, and (3.6) to obtain

(3.13)

$$(\mathcal{DIV}^d \, \mathbf{G}^I)_E = \frac{1}{|E|} \sum_{e \in \partial E} \left(\mathbf{G}_E^I \right)^e |e| = \frac{1}{|E|} \int_{\partial E} \mathbf{G} \cdot \mathbf{n}_E \, \mathrm{d}S = \frac{1}{|E|} \int_E \mathrm{div}\, \mathbf{G} \, \mathrm{d}V = (\mathrm{div}\, \mathbf{G})_E^I$$

for every element E in $\mathcal{T}_h$.

The *fourth* step of the MFD method is to define the discrete flux operator, $\mathcal{G}^d : Q^d \rightarrow X^d$, as the adjoint to the discrete divergence operator, $\mathcal{DIV}^d$, with respect to scalar products (3.8) and (3.11), i.e.,

$$(3.14) \qquad [\mathbf{F}, \, \mathcal{G}^d \, \mathbf{p}]_{X^d} = [\mathbf{p}, \, \mathcal{DIV}^d \, \mathbf{F}]_{Q^d} \quad \forall \mathbf{p} \in Q^d \quad \forall \mathbf{F} \in X^d.$$

Using the discrete flux and divergence operators, the continuous problem (2.1), (2.2) is discretized as follows:

$$(3.15) \qquad\qquad\qquad \mathcal{DIV}^d \, \mathbf{F}_d = \mathbf{b},$$

$$(3.16) \qquad\qquad\qquad \mathbf{F}_d = \mathcal{G}^d \, \mathbf{p}_d,$$

where $\mathbf{b} \equiv \mathbf{b}^I$ is the vector of mean values of the source function b.

4. Stability analysis. In this section we analyze the stability of the MFD discretization (3.15)–(3.16) following the well-established theory of saddle-point problems [5]. More precisely, we prove the coercivity condition (4.4) and the inf-sup condition (4.5).

Using the discrete Green's formula (3.14), we rewrite (3.15) and (3.16) in a form more suitable for analysis:

$$(4.1) \qquad\qquad [\mathbf{F}_d, \, \mathbf{G}]_{X^d} - [\mathbf{p}_d, \, \mathcal{DIV}^d \, \mathbf{G}]_{Q^d} = 0 \quad \forall \mathbf{G} \in X^d,$$

$$(4.2) \qquad\qquad [\mathcal{DIV}^d \, \mathbf{F}_d, \, \mathbf{q}]_{Q^d} = [\mathbf{b}, \, \mathbf{q}]_{Q^d} \qquad \forall \mathbf{q} \in Q^d.$$

Let us introduce the following mesh norms on discrete spaces X^d and Q^d:

$$|||\mathbf{p}|||_{Q^d}^2 := [\mathbf{p}, \, \mathbf{p}]_{Q^d}, \quad |||\mathbf{F}|||_{X^d}^2 := [\mathbf{F}, \, \mathbf{F}]_{X^d},$$

and

$$(4.3) \qquad\qquad |||\mathbf{F}|||_{div}^2 := |||\mathbf{F}|||_{X^d}^2 + \sum_{E \in \mathcal{T}_h} h_E^2 \, \|\mathcal{DIV}^d \, \mathbf{F}\|_{L^2(E)}^2.$$

Let V^d be the space of divergence-free discrete fluxes:

$$V^d = \{\mathbf{F} \in X^d : \quad \mathcal{DIV}^d \, \mathbf{F} = 0\}.$$

We begin the stability analysis by noticing that the scalar product (3.11) is continuous. It is also obvious that the scalar product satisfies the V^d-ellipticity condition:

$$(4.4) \qquad\qquad [\mathbf{F},\, \mathbf{F}]_{X^d} \geq |||\mathbf{F}|||^2_{div} \quad \forall \mathbf{F} \in V^d.$$

The analysis of the inf-sup condition is more involved. Following [5], for every $\mathbf{q} \in Q^d$, we have to find a vector $\mathbf{G} \in X^d$ such that

$$(4.5) \qquad\qquad [\mathcal{DIV}^d\, \mathbf{G},\, \mathbf{q}]_{Q^d} \geq \beta_* |||\mathbf{G}|||_{div}\, |||\mathbf{q}|||_{Q^d},$$

where β_* is a positive constant independent of $\mathbf{q}$, $\mathbf{G}$, and $\mathcal{T}_h$. Let us denote by $q^h \in L^2(\Omega)$ the piecewise constant function on $\mathcal{T}_h$ with values given by the entries of the vector $\mathbf{q}$ (so that $(q^h)^I \equiv \mathbf{q}$). It is obvious that $\|q^h\|_{L^2(\Omega)} = |||\mathbf{q}|||_{Q^d}$. Let us consider the homogeneous Dirichlet boundary value problem

$$\Delta \psi = q^h \quad \text{in} \quad \Omega.$$

Since Ω has a Lipschitz-continuous boundary, there exist an $s > 2$ and a constant C^*_Ω such that

$$(4.6) \qquad\qquad \|\psi\|_{W^1_s(\Omega)} \leq C^*_\Omega \|q^h\|_{L^2(\Omega)}.$$

Let $\mathbf{H} = \nabla\psi$, so that we have immediately

$$(4.7) \qquad\qquad \text{div}\,\mathbf{H} = q^h,$$

and from (4.6)

$$(4.8) \qquad \|\mathbf{H}\|_{(L^s(\Omega))^3} + \left(\sum_{E \in \mathcal{T}_h} h_E^2 \|\text{div}\,\mathbf{H}\|^2_{L^2(E)} \right)^{1/2} \leq (C^*_\Omega + h)\|q^h\|_{L^2(\Omega)}.$$

We now set

$$(4.9) \qquad\qquad \mathbf{G} := \mathbf{H}^I \equiv (\nabla\psi)^I,$$

where the interpolation operator is still the one defined in (3.7). Thanks to the commutative property (3.13) and to (4.7), we have

$$(4.10) \qquad\qquad \mathcal{DIV}^d\, \mathbf{G} = (q^h)^I \equiv \mathbf{q}.$$

Thus, inequality (4.5) is reduced to

$$(4.11) \qquad\qquad |||\mathbf{q}|||_{Q^d} \geq \beta_* |||\mathbf{G}|||_{div}.$$

At this point we need the following technical lemma.

LEMMA 4.1. *Under Assumptions* M1–M6 *and* S1, *for every $s > 2$, there exists a positive constant β^*_s such that*

$$(4.12) \qquad |||\mathbf{G}^I|||_{div} \leq \beta^*_s \left\{ \|\mathbf{G}\|_{(L^s(\Omega))^3} + \left(\sum_{E \in \mathcal{T}_h} h_E^2 \|\text{div}\,\mathbf{G}\|^2_{L^2(E)} \right)^{1/2} \right\}$$

for every $\mathbf{G} \in (L^s(\Omega))^3$ with $\text{div}\,\mathbf{G} \in L^2(\Omega)$, and where $\mathbf{G}^I$ is defined in (3.7).

Collecting (4.9) and (4.12), we get

$$|||\mathbf{G}|||_{div} = |||\mathbf{H}^I|||_{div} \le \beta_s^* \left\{ \|\mathbf{H}\|_{(L^s(\Omega))^3} + \left(\sum_{E \in \mathcal{T}_h} h_E^2 \|\mathrm{div}\,\mathbf{H}\|_{L^2(E)}^2 \right)^{1/2} \right\}.$$

This, together with (4.8), implies (4.11), and hence (4.5), with $\beta_* = (\beta_s^* \, (C_\Omega^* + h))^{-1}$. Therefore, we have just to prove Lemma 4.1.

Proof of Lemma 4.1. From (3.13) we immediately have

$$(4.13) \qquad |||\mathcal{DIV}^d\,\mathbf{G}^I|||_{Q^d} = |||(\mathrm{div}\,\mathbf{G})^I|||_{Q^d} \le \|\mathrm{div}\,\mathbf{G}\|_{L^2(\Omega)}.$$

Therefore, in view of (4.3), it is sufficient to prove that there exists a constant $\widetilde{\beta_s^*}$ such that

$$(4.14) \qquad |||\mathbf{G}^I|||_{X^d} \le \widetilde{\beta_s^*} \left\{ \|\mathbf{G}\|_{(L^s(\Omega))^3} + \left(\sum_{E \in \mathcal{T}_h} h_E^2 \|\mathrm{div}\,\mathbf{G}\|_{L^2(E)}^2 \right)^{1/2} \right\}.$$

The desired result (4.12) follows from (4.14) with $\beta_s^* = \widetilde{\beta_s^*} + 1$. In the following discussion, we shall make a wide use of the conjugate exponent t, depending on s through the usual formula

$$(4.15) \qquad \frac{1}{s} + \frac{1}{t} = 1.$$

Assumption (3.10) implies clearly that

$$(4.16) \qquad [\mathbf{G}^I, \mathbf{G}^I]_{X^d} \le S^* \sum_{E \in \mathcal{T}_h} |E| \sum_{e \in \partial E} \left(G_E^e \right)^2,$$

so that we have to estimate the (G_E^e)'s in terms of $\mathbf{G}$, or, rather, in terms of the norm of $\mathbf{G}$ appearing in (4.12). Our basic instrument for that is called the *lift property*. The main difficulty, in various cases, will be to prove that the lift property holds true.

LP (lift property). For every $t < 2$ there exists a constant $\lambda^* = \lambda^*(t)$ such that for every $E \in \mathcal{T}_h$ and for every $e \in \partial E$ there exists a function φ_E^e from E to $\mathbb{R}$ that verifies

$$(4.17) \qquad \varphi_E^e = 1 \quad \text{on} \quad e, \quad \varphi_E^e = 0 \quad \text{on} \quad \partial E \setminus e$$

and

$$(4.18) \qquad \left\|\varphi_E^e\right\|_{L^2(E)} \le \lambda^* h_E^{3/2}, \quad \left\|\nabla \varphi_E^e\right\|_{(L^t(E))^3} \le \lambda^* h_E^{3/t-1}.$$

The lift property LP is proved in the appendix.

Up to an approximation of $\mathbf{G}$ by smooth functions, and passage to the limit, we have, using (3.7), (4.17), the Green's formula,

$$(4.19) \qquad \begin{aligned} G_E^e &= \frac{1}{|e|} \int_e \mathbf{G} \cdot \mathbf{n}_E \, dS = \frac{1}{|e|} \int_{\partial E} \varphi_E^e \mathbf{G} \cdot \mathbf{n}_E \, dS \\ &= \frac{1}{|e|} \int_E \mathbf{G} \cdot \nabla \varphi_E^e \, dV + \frac{1}{|e|} \int_E \varphi_E^e \, \mathrm{div}\,\mathbf{G} \, dV. \end{aligned}$$

Using the Hölder inequality and (4.18) in (4.19), we then have

$$|e|\, G_E^e \leq \|\mathbf{G}\|_{L^s(E)} \left\|\nabla\varphi_E^e\right\|_{L^t(E)} + \|\text{div}\,\mathbf{G}\|_{L^2(E)} \left\|\varphi_E^e\right\|_{L^2(E)}$$
$$\leq \lambda^* \left\{(h_E)^{3/t-1}\|\mathbf{G}\|_{L^s(E)} + (h_E)^{3/2}\,\|\text{div}\,\mathbf{G}\|_{L^2(E)}\right\}.$$

Taking the squares and remembering that $(a+b)^2 \leq 2(a^2+b^2)$, we have

$$(4.20) \qquad |e|^2\, \left(G_E^e\right)^2 \leq 2\,(\lambda^*)^2 \left\{(h_E)^{6/t-2}\|\mathbf{G}\|_{L^s(E)}^2 + (h_E)^3\,\|\text{div}\,\mathbf{G}\|_{L^2(E)}^2\right\}.$$

On the other hand, using conditions (2.4), we easily obtain

$$(4.21) \qquad |E| \leq h_E^3 = h_E^{-1}\left(h_E^2\right)^2 \leq h_E^{-1}\,(a^*)^{-2}|e|^2.$$

We can now join (4.21) with (4.20) to deduce that

$$(4.22) \qquad \begin{aligned} |E|\left(G_E^e\right)^2 &\leq h_E^{-1}\,(a^*)^{-2}|e|^2\left(G_E^e\right)^2 \\ &\leq \sigma^*\left\{(h_E)^{6/t-3}\|\mathbf{G}\|_{L^s(E)}^2 + (h_E)^2\,\|\text{div}\,\mathbf{G}\|_{L^2(E)}^2\right\}, \end{aligned}$$

where $\sigma^* = 2\,(\lambda^*)^2\,(a^*)^{-2}$. Now we can sum (4.22) over all faces e of E and then over all elements E of $\mathcal{T}_h$. We use (4.16) and Assumption M2 on the number of faces per element to get

$$(4.23) \qquad \begin{aligned} \||\mathbf{G}^I\||_{X^d}^2 &\leq N_e\, S^*\, \sigma^* \left\{\sum_{E\in\mathcal{T}_h}(h_E)^{6/t-3}\|\mathbf{G}\|_{L^s(E)}^2 + \sum_{E\in\mathcal{T}_h}h_E^2\,\|\text{div}\,\mathbf{G}\|_{L^2(E)}^2\right\} \\ &\leq N_e\, S^*\, \sigma^* \left\{\left(\sum_{E\in\mathcal{T}_h}\left\{(h_E)^{6/t-3}\right\}^r\right)^{1/r}\left(\sum_{E\in\mathcal{T}_h}\|\mathbf{G}\|_{L^s(E)}^s\right)^{2/s}\right. \\ &\qquad\qquad \left. + \sum_{E\in\mathcal{T}_h}h_E^2\|\text{div}\,\mathbf{G}\|_{L^2(E)}^2\right\}, \end{aligned}$$

where in the last step we applied the Hölder inequality with r, the conjugate exponent of $s/2$,

$$(4.24) \qquad \frac{1}{r} + \frac{2}{s} = 1.$$

A simple algebraic manipulation using (4.15) and (4.24) gives

$$(4.25) \qquad \sum_{E\in\mathcal{T}_h}\left\{(h_E)^{6/t-3}\right\}^r = \sum_{E\in\mathcal{T}_h}h_E^3 \leq v_*^{-1}|\Omega|,$$

where we have also used (2.4) in the last step. Inserting (4.25) into (4.23), we finally get

$$(4.26) \qquad \||\mathbf{G}^I\||_{X^d} \leq \widetilde{\beta_s^*}\left\{\|\mathbf{G}\|_{(L^s(\Omega))^3} + \left(\sum_{E\in\mathcal{T}_h}h_E^2\|\text{div}\,\mathbf{G}\|_{L^2(E)}^2\right)^{1/2}\right\},$$

where $\widetilde{\beta_s^*}$ depends only on $\lambda^*(t)$, S^*, v_*, a_*, and N_e. This proves the assertion of the lemma. $\square$

5. Convergence analysis.

5.1. Consistency assumption. In order to prove error estimates, we need some assumptions on the scalar product (3.11), and more precisely on the relationships between the continuous scalar product (3.1) and its discrete counterpart (3.11). Our basic assumption will be the following one.

S2 (*consistency of* $[\cdot, \cdot]_E$). For every element E, every linear function q^1 on E, and every $\mathbf{G} \in X^d$, we have

$$(5.1) \qquad \left[(\tilde{\mathbf{K}} \nabla q^1)^I, \mathbf{G} \right]_E = \int_{\partial E} q^1 \, \mathbf{G}_E \cdot \mathbf{n}_E \, \mathrm{d}S - \int_E q^1 \, (\mathcal{DIV}^d \, \mathbf{G})_E \, \mathrm{d}V,$$

where $(\cdot)^I$ is the interpolation operator (3.7) and $\tilde{\mathbf{K}}$ is a constant tensor on E such that

$$(5.2) \qquad \sup_{x \in E} \sup_{i,j} |\{\mathbf{K}(\mathbf{x})\}_{i,j} - \{\tilde{\mathbf{K}}\}_{i,j}| \le C_K^* \, h_E,$$

where C_K^* is a constant independent of E.

Note that $\tilde{\mathbf{K}}$ may be any reasonable piecewise constant approximation of $\mathbf{K}$. In practice, we use either the value of $\mathbf{K}$ at the polyhedron mass center or its mean value.

Condition (5.1) is rather new and requires some comments. First, we point out that for divergence-free vectors, $\mathbf{G} \in V^d$, it reads

$$(5.3) \qquad [(\tilde{\mathbf{K}} \nabla q^1)^I, \mathbf{G}]_E = \int_{\partial E} q^1 \, \mathbf{G}_E \cdot \mathbf{n}_E \, \mathrm{d}S$$

showing the remarkable property of using only *boundary integrals*. However, as $\mathcal{DIV}^d \, \mathbf{G}$ is constant in each E and q^1 is supposed to be linear, the volume integral appearing in (5.1) is not difficult to compute. Taking $\mathbf{G} = (\tilde{\mathbf{K}} \nabla \tilde{q}^1)^I$ (with $\tilde{q}^1$ another polynomial of degree ≤ 1) in (5.3), we conclude that Assumption S2 implies that *the scalar product* (3.11) *gives an exact value for the integral of two constant velocities.*

In the context of the local MFD method [13], and taking for simplicity $\tilde{\mathbf{K}} = \mathbf{I}$, condition (5.1) means that the discrete gradient operator is exact for linear functions, i.e., $\mathcal{G}^d (\mathbf{q}^1)^I$ is a constant vector whose entries are equal to ∇q^1. This property has been used in [18] to build a one-parameter family of symmetric positive definite matrices M_E for a triangle. As a particular case, this family includes the mass matrix appearing in the finite element discretizations with the Raviart–Thomas finite elements.

What is still remarkable in (5.1) is that *it does not require the construction of a lifting operator* from the values G_E^e on ∂E to the interior of E. It is not difficult to show, however, that *if* we have any reasonable lifting operator R_E, then the choice

$$[\mathbf{F}, \mathbf{G}]_E := \int_E \tilde{\mathbf{K}}^{-1} R_E(\mathbf{F}_E) \cdot R_E(\mathbf{G}_E) \, \mathrm{d}V$$

will automatically satisfy (5.1) as well as (3.10). We have indeed the following theorem.

THEOREM 5.1. *Assume that for every element $E \in \mathcal{T}_h$ we have a lifting operator R_E acting on X_E^d (the restriction of X^d to E) and with values in $(L^2(E))^3$ such that*

$$(5.4) \qquad \begin{aligned} R_E(\mathbf{G}_E) \cdot \mathbf{n}_E &\equiv \mathbf{G}_E \cdot \mathbf{n}_E &&\text{on} \quad \partial E, \\ \mathrm{div} \, R_E(\mathbf{G}_E) &\equiv (\mathcal{DIV}^d \, \mathbf{G})_E &&\text{in} \quad E \end{aligned}$$

for all $\mathbf{G} \in X^d$, *and*

(5.5)
$$R_E(\mathbf{G}_E^I) = \mathbf{G}$$

for all $\mathbf{G}$ *constant on* E. *Then the choices*

(5.6)
$$\{\tilde{\mathbf{K}}\}_{i,j} := \frac{1}{|E|} \int_E \{\mathbf{K}\}_{i,j} \, dV$$

and

(5.7)
$$[\mathbf{F}, \mathbf{G}]_E := \int_E \tilde{\mathbf{K}}^{-1} R_E(\mathbf{F}_E) \cdot R_E(\mathbf{G}_E) \, dV$$

will automatically satisfy (5.2) *and* (5.1). *If, moreover, there exist two positive constants* c_R^* *and* C_R^*, *independent of* E *such that*

(5.8) $c_R^* \left(|E| \sum_{e \in \partial E} \left(G_E^e \right)^2 \right)^{1/2} \leq \|R_E(\mathbf{G})\|_{(L^2(E))^3} \leq C_R^* \left(|E| \sum_{e \in \partial E} \left(G_E^e \right)^2 \right)^{1/2}$

for all $\mathbf{G} \in X^d$, *then* (3.10) *will also hold with constants* s_* *and* S^* *depending only on* c_R^*, C_R^*, *and on the constants* κ_*, κ^* *from* (2.3).

Proof. The validity of (5.2) is immediate. The validity of (5.1) is also easily checked:

$$\begin{aligned}
[(\tilde{\mathbf{K}} \nabla q^1)^I, \mathbf{G}]_E &= \int_E \tilde{\mathbf{K}}^{-1} R_E((\tilde{\mathbf{K}} \nabla q^1)_E^I) \cdot R_E(\mathbf{G}_E) \, dV \quad \text{(use (5.5) and } \nabla q^1 = \text{const)} \\
&= \int_E \tilde{\mathbf{K}}^{-1} \tilde{\mathbf{K}} \nabla q^1 \cdot R_E(\mathbf{G}_E) \, dV \qquad \qquad \text{(use } \tilde{\mathbf{K}}^{-1} \tilde{\mathbf{K}} = Id) \\
&= \int_E \nabla q^1 \cdot R_E(\mathbf{G}_E) \, dV \qquad \qquad \quad \text{(integrate by parts)} \\
&= \int_{\partial E} q^1 R_E(\mathbf{G}_E) \cdot \mathbf{n}_E \, dS - \int_E q^1 \, \mathrm{div} R_E(\mathbf{G}_E) \, dV \quad \text{(use (5.4))} \\
&= \int_{\partial E} q^1 \, \mathbf{G}_E \cdot \mathbf{n}_E \, dS - \int_E q^1 \, (\mathcal{DIV}^d \, \mathbf{G})_E \, dV.
\end{aligned}$$

Finally, (3.10) follows immediately from (5.7), (2.3), and (5.8) after noting that (2.3) is equivalent to

(5.9) $(\kappa^*)^{-1} \|\mathbf{v}\|^2 \leq \mathbf{v}^T \mathbf{K}^{-1}(\mathbf{x}) \mathbf{v} \leq (\kappa_*)^{-1} \|\mathbf{v}\|^2 \quad \forall \mathbf{v} \in \mathbb{R}^3 \quad \forall \mathbf{x} \in \Omega.$

This ends the proof of the theorem.   □

A possible way of getting (5.1) is, therefore, to construct a lifting operator R_E satisfying (5.4), (5.5), and (5.8), and then define M_E following (5.7). For instance, the way followed in [16] for polygonal domains can be interpreted as the construction of a lifting operator satisfying (5.4) and (5.5).

In general, we may consider assumption (5.1) as a system of linear equations where the unknowns are the coefficients of M_E, and use it, in each element E, to construct the matrix M_E. Since the matrix M_E should be symmetric and positive definite, this is a problem with nonlinear constraints. An analytical solution has been found only for triangular elements [18].

Let us see this in more detail. We consider an element E having k_E faces. Equation (5.1) should then hold for k_E different possible choices of $\mathbf{G}_E$ and three possible choices of q^1 corresponding to $q^1 = x$, $q^1 = y$, and $q^1 = z$. Note that for $q^1 = 1$, (5.1) is automatically satisfied as it is reduced to our definition of the operator $\mathcal{DIV}^d$. We have, therefore, $3k_E$ equations. It can be shown that only $3k_E - 3$ equations are linearly independent. Since $\tilde{\mathbf{K}}$, and hence M_E, is symmetric, the number of unknown coefficients of M_E is $k_E(k_E + 1)/2$, that is, bigger than $3k_E - 3$ as soon as $k_E \geq 4$. The system will always be compatible, since we could always define a lifting R_E first by solving, for each $\mathbf{G}_E$, the Neumann problem

$$\Delta \chi = \mathcal{DIV}^d \mathbf{G}_E \qquad \text{in} \quad E,$$
$$\partial \chi / \partial \mathbf{n}_E = \mathbf{G}_E \cdot \mathbf{n}_E \quad \text{on} \quad \partial E,$$

then by taking $R_E(\mathbf{G}_E) := \nabla \chi$, and finally by defining M_E through (5.7). This would be totally impractical but shows that at least a solution M_E of (5.1), symmetric and positive definite, exists (although, in general, the solution will not be unique).

A sparsity structure could be imposed on M_E in order to reduce the number of unknowns. For instance, we can require that each face interacts only with a few neighboring faces, reducing the number of unknowns to $3k_E - 3$, which equals the number of equations and makes the linear system much easier to solve on the computer (see [6] for more detail).

An advantage of this approach is that it can be rather easily extended to faces that are not flat. This is a case in which the construction of an explicit lifting operator might prove to be very difficult. We shall consider meshes with curved faces in the future publications.

5.2. Error estimate for the vector variable. Using Assumption S2, we are going to prove error estimates for our discretization. Let $(p, \mathbf{F})$ be the exact solution of (2.1) and (2.2), let $(\mathbf{p}_d, \mathbf{F}_d)$ be the discrete solution (see (3.15) and (3.16)), and let $\mathbf{p}^I$ and $\mathbf{F}^I$ be the interpolants of the exact solution. Finally, for every element E, we denote by p_E^1 a suitable polynomial of degree ≤ 1 that approximates p, and that will be decided later on. We notice first that from (2.1), (3.13), and (3.15), we easily have

$$(5.10) \qquad \mathcal{DIV}^d (\mathbf{F}^I - \mathbf{F}_d) = \mathbf{b} - \mathbf{b} = 0.$$

Using (2.2) and (3.16), then (3.14), and finally (5.10), we get

$$[\mathbf{F}^I - \mathbf{F}_d, \mathbf{F}^I - \mathbf{F}_d]_{X^d} = [(-\mathbf{K}\nabla p)^I, \mathbf{F}^I - \mathbf{F}_d]_{X^d} - [\mathcal{G}^d \mathbf{p}_d, \mathbf{F}^I - \mathbf{F}_d]_{X^d}$$
$$= [(-\mathbf{K}\nabla p)^I, \mathbf{F}^I - \mathbf{F}_d]_{X^d} - [\mathbf{p}_d, \mathcal{DIV}^d (\mathbf{F}^I - \mathbf{F}_d)]_{Q^d}$$
$$(5.11) \qquad = [(-\mathbf{K}\nabla p)^I, \mathbf{F}^I - \mathbf{F}_d]_{X^d}.$$

Then, adding and subtracting the terms, we have

$$|||\mathbf{F}^I - \mathbf{F}_d|||^2_{X^d} = [(-\mathbf{K}\nabla p)^I + (\mathbf{K}\nabla p^1)^I, \mathbf{F}^I - \mathbf{F}_d]_{X^d} + [(-\mathbf{K}\nabla p^1)^I, \mathbf{F}^I - \mathbf{F}_d]_{X^d}$$
$$= \mathbf{I}_1 + [(-\mathbf{K}\nabla p^1 + \tilde{\mathbf{K}}\nabla p^1)^I, \mathbf{F}^I - \mathbf{F}_d]_{X^d} + [(-\tilde{\mathbf{K}}\nabla p^1)^I, \mathbf{F}^I - \mathbf{F}_d]_{X^d}$$
$$= \mathbf{I}_1 + \mathbf{I}_2 + [(-\tilde{\mathbf{K}}\nabla p^1)^I, \mathbf{F}^I - \mathbf{F}_d]_{X^d}$$
$$(5.12) \qquad = \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3.$$

Using (5.1) and (5.10), the third term reads

$$\mathbf{I}_3 = \sum_{E \in \mathcal{T}_h} \left\{ \int_{\partial E} p_E^1 (\mathbf{F}^I - \mathbf{F}_d)_E \cdot \mathbf{n}_E \, \mathrm{d}S - \int_E p_E^1 (\mathcal{DIV}^d (\mathbf{F}^I - \mathbf{F}_d))_E \, \mathrm{d}V \right\}$$

$$(5.13) \quad = \sum_{E \in \mathcal{T}_h} \int_{\partial E} p_E^1 (\mathbf{F}^I - \mathbf{F}_d)_E \cdot \mathbf{n}_E \, \mathrm{d}S.$$

We are, therefore, left with the problem of estimating $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_3$. A first estimate of $\mathbf{I}_2$ is trivial. From (5.2) we immediately have

$$(5.14) \quad \mathbf{I}_2 \equiv [(-\mathbf{K}\nabla p^1 + \tilde{\mathbf{K}}\nabla p^1)^I, \mathbf{F}^I - \mathbf{F}_d]_{X^d} \le C_K^* h \, |||(\nabla p^1)^I|||_{X^d} \, |||\mathbf{F}^I - \mathbf{F}_d|||_{X^d},$$

where p^1 still has to be defined.

Let us recall some known properties of the approximation theory. For the sake of simplicity, we assume that our solution p is in $H^2(\Omega)$. Note that with a little additional effort we could use a weaker regularity and get a lower order of convergence.

We first recall that, under Assumption M6 (*star-shaped elements*), it is possible to find a constant C_{app}^*, depending only on τ_*, such that for every element E and for every $p \in H^2(E)$ there exist a constant p_E^0 and a polynomial p_E^1 of degree ≤ 1 such that

$$(5.15) \qquad\qquad \left\| p - p_E^0 \right\|_{L^2(E)} \le C_{app}^* \, h_E \, \|p\|_{H^1(E)},$$

$$(5.16) \quad \left\| p - p_E^1 \right\|_{L^2(E)} \le C_{app}^* \, h_E^2 \, \|p\|_{H^2(E)}, \quad \left\| p - p_E^1 \right\|_{H^1(E)} \le C_{app}^* \, h_E \, \|p\|_{H^2(E)}$$

(see [4, Lemma 4.3.8]). Concerning the error on faces, we can use a result due to Agmon made popular in the numerical analysis community by Arnold [1]. Applied to our case, it says that there exists a constant C_{agm}^*, depending only on the constant γ_* of Assumption M4, such that for every pyramid P_E^e (as described in Assumption M5), and for every function $\chi \in H^1(P_E^e)$, we have

$$(5.17) \qquad\qquad \|\chi\|_{L^2(e)}^2 \le C_{agm}^* \left(h_E^{-1} \|\chi\|_{L^2(P_E^e)}^2 + h_E \|\chi\|_{H^1(P_E^e)}^2 \right).$$

It is then immediate to derive from (5.17) that

$$(5.18) \qquad\qquad \|\nabla\chi\|_{L^2(e)}^2 \le C_{agm}^* \left(h_E^{-1} \|\chi\|_{H^1(P_E^e)}^2 + h_E \|\chi\|_{H^2(P_E^e)}^2 \right)$$

for every $\chi \in H^2(E)$. Applying this to the difference $p - p_E^1$, and using (5.16), we get

$$(5.19) \qquad \left\| p - p_E^1 \right\|_{L^2(e)}^2 + h_E^2 \left\| \nabla (p - p_E^1) \right\|_{L^2(e)}^2 \le C_{face}^* \, h_E^3 \, \|p\|_{H^2(E)}^2,$$

where C_{face}^* depends only on τ_* and γ_*.

Now, we can finish the estimate of $\mathbf{I}_2$. Note that ∇p^1 is a constant vector. Then, (5.16) and the triangle inequality give

$$|||(\nabla p_E^1)^I|||_{X^d} = \left\| \nabla p_E^1 \right\|_{L^2(E)} \le \|\nabla p\|_{L^2(E)} + \left\| \nabla (p - p_E^1) \right\|_{L^2(E)} \le (1 + h_E C_{app}^*) \|p\|_{H^2(E)}.$$

Thus, we obtain immediately from (5.14) that

$$(5.20) \qquad\qquad \mathbf{I}_2 \le C_{I_2}^* \, h \, \|p\|_{H^2(\Omega)} \, |||\mathbf{F}^I - \mathbf{F}_d|||_{X^d},$$

where $C_{I_2}^*$ equals $(1 + h_E\, C_{app}^*)C_K^*$ with C_K^* given in (5.2).

The estimate of $\mathbf{I}_1$ is obtained in the following lemma.

LEMMA 5.2. *Let $p \in H^2(\Omega)$ and let, in each $E \in \mathcal{T}_h$, p^1 be such that (5.16) holds. Let $(\cdot)^I$ be the interpolation operator defined in (3.7), and let finally $\mathbf{G} \in X^d$. Then*

$$(5.21) \qquad [(-\mathbf{K}\,\nabla p)^I + (\mathbf{K}\,\nabla p^1)^I, \, \mathbf{G}]_{X^d} \le C_{I_1}^*\, h\, \|p\|_{H^2(\Omega)}\, |||\mathbf{G}|||_{X^d},$$

where the constant $C_{I_1}^*$ is independent of p, $\mathbf{G}$, and h.

Proof. The proof follows immediately from (3.10), the definition of the interpolation operator (3.7), the Cauchy–Schwarz inequality, and the approximation results quoted above. Indeed, we have

$$|||(-\mathbf{K}\,\nabla p)^I + (\mathbf{K}\,\nabla p^1)^I|||_{X^d}^2 \le S^* \sum_{E \in \mathcal{T}_h} \sum_{e \in \partial E} \left(((-\mathbf{K}\,\nabla p)^I + (\mathbf{K}\,\nabla p^1)^I)_E^e\right)^2 |E|$$

$$\le S^* \sum_{E \in \mathcal{T}_h} \sum_{e \in \partial E} \left(\frac{1}{|e|} \int_e \mathbf{K}\,\nabla(p - p_E^1) \cdot \mathbf{n}_E\, \mathrm{d}S\right)^2 |E|$$

$$\le S^* \sum_{E \in \mathcal{T}_h} \sum_{e \in \partial E} \frac{1}{|e|} \left\|\mathbf{K}\,\nabla(p - p_E^1)\right\|_{L^2(e)}^2 |E|$$

$$\le C_{I_1}^*\, h^2\, \|p\|_{H^2(\Omega)}^2,$$

where $C_{I_1}^*$ depends only on a_* given in (2.4), S^* given in (3.10), κ^* given in (2.3), N_e from Assumption M2, and C_{face}^* obtained in (5.19). □

The following lemma gives an estimate for $\mathbf{I}_3$.

LEMMA 5.3. *Let $p \in H^2(\Omega)$ and let, in each $E \in \mathcal{T}_h$, p^1 be such that (5.16) holds. Moreover, let $\mathbf{G} \in X^d$. Then*

$$(5.22) \qquad \sum_{E \in \mathcal{T}_h} \int_{\partial E} p^1\, \mathbf{G}_E \cdot \mathbf{n}_E\, \mathrm{d}S \le C_{I_3}^*\, h\, \|p\|_{H^2(\Omega)}\, |||\mathbf{G}|||_{X^d},$$

where the constant $C_{I_3}^*$ is independent of p, $\mathbf{G}$, and h.

Proof. The first (crucial) step of the proof uses the continuity of p and the fact that $\mathbf{G}_E \cdot \mathbf{n}_E$ takes opposite values for the two elements sharing a common internal face. Then, the result follows with usual instruments such as the Cauchy–Schwarz inequality and approximation results (5.16):

$$\sum_{E \in \mathcal{T}_h} \int_{\partial E} p_E^1\, \mathbf{G}_E \cdot \mathbf{n}_E\, \mathrm{d}S = \sum_{E \in \mathcal{T}_h} \int_{\partial E} (p_E^1 - p)\, \mathbf{G}_E \cdot \mathbf{n}_E\, \mathrm{d}S$$

$$\le \sum_{E \in \mathcal{T}_h} \sum_{e \in \partial E} \left\|p - p_E^1\right\|_{L^2(e)} \left\|G_E^e\right\|_{L^2(e)}$$

$$= \sum_{E \in \mathcal{T}_h} \sum_{e \in \partial E} \left\|p - p_E^1\right\|_{L^2(e)} |G_E^e|\, |e|^{1/2}$$

$$\le v_*^{-1/2}(C_{face}^*)^{1/2} \sum_{E \in \mathcal{T}_h} h_E \|p\|_{H_2(E)} \sum_{e \in \partial E} |G_E^e|\, |E|^{1/2}$$

$$\le C_{I_3}^*\, h\|p\|_{H^2(\Omega)}\, |||\mathbf{G}|||_{X^d},$$

where $C_{I_3}^* = (v_*^{-1}\, s_*^{-1}\, C_{face}^*)^{1/2} N_e$. This proves the assertion of the lemma. □

Combining (5.12) with (5.20), (5.21), and (5.22), we finally get the main convergence result.

THEOREM 5.4. *Under Assumptions* P1, *M1–M6, and* S1–S2, *let* $(p, \mathbf{F})$ *be the solution of* (2.1)–(2.2), *and let* $(p_d, \mathbf{F}_d)$ *be the discrete solution, given by* (3.15)–(3.16). *Moreover, let* $\mathbf{F}^I$ *be the interpolant of* $\mathbf{F}$, *introduced in* (3.7). *Then, we have*

$$(5.23) \qquad\qquad |||\mathbf{F}^I - \mathbf{F}_d|||_{X^d} \leq C^* \, h \, \|p\|_{H^2(\Omega)},$$

where C^* *depends only upon the various constants appearing in Assumptions* P1, *M1–M6, and* S1–S2.

5.3. Error estimates for the scalar variable. In order to derive estimates on the scalar variable $\mathbf{p}_d$, we shall go back to the proof of inf-sup condition (4.5). For the sake of simplicity, we assume that Ω is convex. Let ψ be the solution of

$$\begin{aligned} -\mathrm{div}(\mathbf{K}\nabla\psi) &= \mathbf{p}^I - \mathbf{p}_d \quad &\text{in } \Omega, \\ \psi &= 0 \quad &\text{on } \partial\Omega, \end{aligned}$$

where, for simplicity, we identified $\mathbf{p}_d - \mathbf{p}^I$ with the corresponding piecewise constant function. The convexity of Ω implies that there exists a constant C_Ω^*, depending only on Ω, such that

$$(5.24) \qquad\qquad \|\psi\|_{H^2(\Omega)} \leq C_\Omega^* \, |||\mathbf{p}_d - \mathbf{p}^I|||_{Q^d}.$$

We now set

$$(5.25) \qquad\qquad \mathbf{H} = \mathbf{K}\nabla\psi$$

and define $\mathbf{G} \in X^d$ as $\mathbf{G} = \mathbf{H}^I$, so that

$$(5.26) \qquad\qquad \mathcal{DIV}^d \, \mathbf{G} = \mathbf{p}_d - \mathbf{p}^I.$$

Finally, we denote by ψ^1 a piecewise linear approximation of ψ that satisfies (5.16) for each $E \in \mathcal{T}_h$. Using (5.26), then (4.1), then (3.6) and (3.13), then integrating by parts, and finally integrating once again by parts and using (2.1) and (2.2), we get

$$\begin{aligned} |||\mathbf{p}_d - \mathbf{p}^I|||_{Q^d}^2 &= [\mathcal{DIV}^d \, \mathbf{G}, \, \mathbf{p}_d - \mathbf{p}^I]_{Q^d} \\ &= [\mathbf{F}_d, \, \mathbf{G}]_{X^d} - [\mathcal{DIV}^d \, \mathbf{G}, \, \mathbf{p}^I]_{Q^d} = [\mathbf{F}_d, \, \mathbf{G}]_{X^d} - \int_\Omega p \, \mathrm{div}(\mathbf{K}\nabla\psi) \, \mathrm{d}V \\ &= [\mathbf{F}_d, \, \mathbf{G}]_{X^d} + \int_\Omega \mathbf{K}\nabla \, p \cdot \nabla\psi \, \mathrm{d}V \\ &= [\mathbf{F}_d, \, \mathbf{G}]_{X^d} + \int_\Omega b \, \psi \, \mathrm{d}V. \end{aligned}$$

Now, using the definition of $\mathbf{G}$ and adding and subtracting the terms, we have

$$\begin{aligned} |||\mathbf{p}_d - \mathbf{p}^I|||_{Q^d}^2 &= [\mathbf{F}_d, \, (\mathbf{K}\nabla\psi)^I - (\mathbf{K}\nabla\psi^1)^I]_{X^d} + [\mathbf{F}_d, \, (\mathbf{K}\nabla\psi^1)^I]_{X^d} + \int_\Omega b \, \psi \, \mathrm{d}V \\ &= J_1 + [\mathbf{F}_d, \, ((\mathbf{K} - \tilde{\mathbf{K}})\nabla\psi^1)^I]_{X^d} + [\mathbf{F}_d, \, (\tilde{\mathbf{K}}\nabla\psi^1)^I]_{X^d} + \int_\Omega b \, \psi \, \mathrm{d}V \\ (5.27) \qquad &= J_1 + J_2 + [\mathbf{F}_d, \, (\tilde{\mathbf{K}}\nabla\psi^1)^I]_{X^d} + \int_\Omega b \, \psi \, \mathrm{d}V. \end{aligned}$$

Using (5.21), the term J_1 can be easily bounded by

$$(5.28) \qquad J_1 \equiv [\mathbf{F}_d, (\mathbf{K}\nabla\psi)^I - (\mathbf{K}\nabla\psi^1)^I]_{X^d} \leq C^*_{I_1} h\, |||\mathbf{F}_d|||_{X^d}\, \|\psi\|_{H^2(\Omega)}.$$

The term J_2 is bounded as in (5.14), (5.20) by

$$(5.29) \qquad J_2 \equiv [\mathbf{F}_d, ((\mathbf{K} - \tilde{\mathbf{K}})\nabla\psi^1)^I]_{X^d} \leq C^*_{I_2} h\, |||\mathbf{F}_d|||_{X^d}\, \|\psi\|_{H^2(\Omega)}.$$

For the third term in the last line of (5.27), we can use (5.1) to obtain

$$(5.30) \qquad [\mathbf{F}_d, (\mathbf{K}\nabla\psi^1)^I]_{X^d} = \sum_{E\in\mathcal{T}_h} \int_{\partial E} \psi^1 (\mathbf{F}_d)_E \cdot \mathbf{n}_E\, \mathrm{d}S - \int_\Omega \mathbf{b}\,\psi^1\,\mathrm{d}V.$$

With the help of (5.22), we then get

$$(5.31)$$
$$\left| [\mathbf{F}_d, (\tilde{\mathbf{K}}\nabla\psi^1)^I]_{X^d} + \int_\Omega b\,\psi\,\mathrm{d}V \right| \leq C^*_{I_3} h\, |||\mathbf{F}_d|||_{X^d}\, \|\psi\|_{H^2(\Omega)} + \left| \int_\Omega (b\,\psi - \mathbf{b}\,\psi^1)\,\mathrm{d}V \right|,$$

where the last term is easily bounded by $2\,C^*_{app}\, h\, \|b\|_{H^1(\Omega)}\, \|\psi\|_{H^1(\Omega)}$. Collecting inequalities (5.27)–(5.31), we obtain

$$(5.32) \qquad |||\mathbf{p}_d - \mathbf{p}^I|||^2_{Q^d} \leq C^*\, h\, \left\{ |||\mathbf{F}_d|||_{X^d} + \|b\|_{H^1(\Omega)} \right\} \|\psi\|_{H^2(\Omega)},$$

which, combined with estimates (5.24), Theorem 5.4, and Lemma 4.1, gives the proof of the second convergence result.

THEOREM 5.5. *Under assumptions of Theorem 5.4, plus the convexity of Ω, we have*

$$(5.33) \qquad |||\mathbf{p}_d - \mathbf{p}^I|||_{Q^d} \leq C^*\, h\, (\|p\|_{H^2(\Omega)} + \|b\|_{H^1(\Omega)}),$$

where the constant C^* depends only on the constants appearing in Assumptions P1, M1–M6, *and* S1–S2, *on C^*_Ω appearing in* (5.24), *and on β^*_s appearing in* (4.12).

It is interesting to note that, assuming that in each element E we had a suitable lifting R_E, a better estimate for the scalar variable could be obtained. We have indeed the following theorem.

THEOREM 5.6. *Together with the assumptions of Theorem 5.5, assume, moreover, that for each element E we have a lifting operator R_E with properties* (5.4), (5.5), *and* (5.8) *such that*

$$(5.34) \quad \|R_E(\mathbf{G}^I) - \mathbf{G}\|_{L^2(E)} \leq C^*_{Ra}\, h_E\, \|\mathbf{G}\|_{(H^1(E))^3} \quad \forall \mathbf{G} \in (H^1(E))^3 \quad \forall E \in \mathcal{T}_h,$$

where C^*_{Ra} is a constant independent of $\mathbf{G}$ and h_E. Then, the choice

$$(5.35) \qquad [\mathbf{F}, \mathbf{G}]_E := \int_E \mathbf{K}^{-1} R_E(\mathbf{F}_E) \cdot R_E(\mathbf{G}_E)\, \mathrm{d}V$$

will give

$$(5.36) \qquad |||\mathbf{p}_d - \mathbf{p}^I|||_{Q^d} \leq C^*\, h^2\, \left(\|p\|_{H^2(\Omega)} + \|b\|_{H^1(\Omega)} \right),$$

where the constant C^* depends only on the constants appearing in Assumptions P1, M1–M6, *and* S1–S2, *on C^*_Ω appearing in* (5.24), *on β^*_s appearing in* (4.12), *and on C^*_{Ra} from* (5.34).

Proof. Let $R(\mathbf{G})$ be such that $R(\mathbf{G})|_E = R_E(\mathbf{G}_E)$. Following essentially [11] and using (5.26), then (4.1), (3.6), and (3.13) (as in the previous proof) with (5.4), then integrating by parts, and finally using (2.2) and (5.35), we get

$$|||\mathbf{p}_d - \mathbf{p}^I|||^2_{Q^d} = [\mathcal{DIV}^d\,\mathbf{G},\,\mathbf{p}_d - \mathbf{p}^I]_{Q^d}$$

$$= [\mathbf{F}_d,\,\mathbf{G}]_{X^d} - \int_\Omega p\,\mathrm{div}\,R(\mathbf{G})\,\mathrm{d}V$$

$$= [\mathbf{F}_d,\,\mathbf{G}]_{X^d} + \int_\Omega \nabla p\cdot R(\mathbf{G})\,\mathrm{d}V = [\mathbf{F}_d,\,\mathbf{G}]_{X^d} + \int_\Omega \mathbf{K}^{-1}\mathbf{K}\nabla p\cdot R(\mathbf{G})\,\mathrm{d}V$$

$$= \int_\Omega \mathbf{K}^{-1}(R(\mathbf{F}_d) - \mathbf{F})\,R(\mathbf{G})\,\mathrm{d}V.$$

Adding and subtracting $\mathbf{H}$ defined in (5.25), we get

$$|||\mathbf{p}_d - \mathbf{p}^I|||^2_{Q^d} = \int_\Omega \mathbf{K}^{-1}(R(\mathbf{F}_d) - \mathbf{F})\,(R(\mathbf{G}) - \mathbf{H})\,\mathrm{d}V + \int_\Omega \mathbf{K}^{-1}(R(\mathbf{F}_d) - \mathbf{F})\,\mathbf{H}\,\mathrm{d}V$$

$$= J_3 + \int_\Omega (R(\mathbf{F}_d) - \mathbf{F})\,\nabla\psi\,\mathrm{d}V = J_3 - \int_\Omega \psi\,\mathrm{div}(R(\mathbf{F}_d) - \mathbf{F})\,\mathrm{d}V$$

$$= J_3 - \int_\Omega (\mathbf{b}^I - b)\psi\,\mathrm{d}V$$

(5.37)
$$= J_3 - \int_\Omega (\mathbf{b}^I - b)(\psi - \boldsymbol\psi^I)\,\mathrm{d}V = J_3 + J_4.$$

In their turn, J_3 and J_4 can be easily bounded using the previous estimates and the usual arguments. Indeed, the triangle inequality, then (3.10) and (5.8), and finally (5.23) and (5.34) imply that

$$\|R(\mathbf{F}_d) - \mathbf{F}\|_{(L^2(\Omega))^3} \le \|R(\mathbf{F}_d - \mathbf{F}^I)\|_{(L^2(\Omega))^3} + \|R(\mathbf{F}^I) - \mathbf{F}\|_{(L^2(\Omega))^3}$$

$$\le C_R^* s_*^{-1/2}|||\mathbf{F}_d - \mathbf{F}^I|||_{X^d} + \|R(\mathbf{F}^I) - \mathbf{F}\|_{(L^2(\Omega))^3}$$

(5.38)
$$\le C\,h\,\|p\|_{H^2(\Omega)}.$$

Using assumption (5.34) and (5.24), we get

(5.39)
$$\|R(\mathbf{G}) - \mathbf{H}\|_{(L^2(\Omega))^3} = \|R(\mathbf{H}^I) - \mathbf{H}\|_{(L^2(\Omega))^3} \le C_{Ra}^* h\|\mathbf{H}\|_{(H^1(\Omega))^3} \le Ch|||\mathbf{p}_d - \mathbf{p}^I|||_{Q^d}.$$

The approximation property (5.15) gives the following estimates:

(5.40)
$$|||\mathbf{b}^I - b\|_{L^2(\Omega)} \le C_{app}^*\,h\|b\|_{H^1(\Omega)}$$

and

(5.41)
$$\|\psi - \boldsymbol\psi^I\|_{L^2(\Omega)} \le C_{app}^*\,h\|\psi\|_{H^1(\Omega)} \le C_{app}^* C_\Omega^*\,h\,|||\mathbf{p}_d - \mathbf{p}^I|||_{Q^d}.$$

Inserting estimates (5.38)–(5.41) into (5.37), we immediately get the result. □

REMARK 5.1. *It is very likely that our additional assumption (5.34) is not needed, as it should be possible to deduce it from (5.4), (5.5), possibly with minor additional assumptions on the geometry. However, in essentially all cases in which R_E can be explicitly built, it is easy to prove directly that (5.34) holds true. Therefore, we decided that it would be simpler to just assume it.*

6. Conclusion. In this paper, we have considered the MFD method for the mixed formulation of the diffusion problem on polyhedral meshes. We have proved the stability of the mimetic discretizations and the optimal convergence rates for the scalar and vector variables. The key elements of our methodology are the consistency Assumption S2 and the lift property LP. In future work, we plan to extend the convergence results to polyhedral meshes with curved faces.

Appendix. Proof of the list property.　　The purpose of this appendix is to prove the lift property (4.17)–(4.18), which we recall for convenience of the reader.

LP (lift property). For every $t < 2$ there exists a constant $\lambda^* = \lambda^*(t)$ such that for every $E \in \mathcal{T}_h$ and for every $e \in \partial E$ there exists a function φ_E^e from E to $\mathbb{R}$ that verifies

$$(\text{A.1}) \qquad \varphi_E^e = 1 \quad \text{on} \quad e, \quad \varphi_E^e = 0 \quad \text{on} \quad \partial E \backslash e$$

and

$$(\text{A.2}) \qquad \left\| \varphi_E^e \right\|_{L^2(E)} \le \lambda^* h_E^{3/2}, \quad \left\| \nabla \varphi_E^e \right\|_{(L^t(E))^3} \le \lambda^* h_E^{3/t - 1}.$$

A traditional way would be to assume that there exist a finite number of reference elements $\hat{E}_1, \ldots, \hat{E}_1$ and a positive constant L^* such that for each $E \in \mathcal{T}_h$ there is an $\hat{E}_k$ and a bi-Lipschitz map Φ_k^E from $\hat{E}_k$ to E such that

$$(\text{A.3}) \qquad \left| \Phi_k^E \right|_{W_\infty^1(\hat{E}_k)} \le L^*, \qquad \left\| \Phi_k^E \right\|_{L_\infty(\hat{E}_k)} \le L^* h_E$$

and

$$(\text{A.4}) \qquad \left| (\Phi_k^E)^{-1} \right|_{W_\infty^1(E)} \le L^*, \qquad \left\| (\Phi_k^E)^{-1} \right\|_{L_\infty(E)} \le L^* h_E^{-1}.$$

Then, for each reference element $\hat{E}_k$ and for each face $\hat{e}$ of $\hat{E}_k$ we could construct the harmonic function $\hat{\varphi}_{\hat{E}_k}^{\hat{e}}$ with boundary value 1 on $\hat{e}$ and zero on the other faces, and verify that it belongs to $W_t^1(\hat{E}_k)$ for every $t < 2$. Finally each function φ_E^e could be constructed by combining one of the reference functions $\hat{\varphi}_{\hat{E}_k}^{\hat{e}}$ with the corresponding map Φ_k^E. This is surely feasible but will become rather cumbersome if we want to consider a big variety of possible shapes for our elements.

We decided here to follow a different path that requires only the fact that the faces are star shaped (M4) and the pyramid property (M5), which are possibly more difficult to explain but much easier to check and to enforce. The general idea is first to build a function $\hat{\varphi}_1$ on the unit cone $\mathcal{C}_1$; then, for every h, to build a function φ_h on a cone $\mathcal{C}_h$ obtained by scaling the unit cone; and finally, for each element E and for each face e, to map the cone $\mathcal{C}_{\gamma_* h_E}$ (where γ_* is given in Assumption M4) into the pyramid P_E^e described in Assumption M5 with a Lipschitz continuous mapping. This will give us a function $\varphi = \varphi_E^e$ on the pyramid, having the right norms. This function will finally be extended by zero to the whole element E, and still it will have the right norms. But let us look at the procedure in more detail.

For each element E and for each face e of E we want to build a function $\varphi = \varphi_E^e$ with the following properties.

- The support of φ is contained in the pyramid $P = P_E^e$ satisfying Assumption M5.
- $\varphi \equiv 1$ on e and $\varphi \equiv 0$ on the other faces of P_E^e.

- φ satisfies the following estimates:

(A.5) $\|\varphi\|_{L^2(P)} \leq \lambda^* h_E^{3/2}$ and $\|\nabla\varphi\|_{(L^t(P))^3} \leq \lambda^* h_E^{3/t-1},$

where the constant λ^* is independent of E and e.

As we said before, we start our work on cones: for $\rho > 0$ we shall refer to the solid

$$\mathcal{C}_\rho \equiv \{(x, y, z): \ 0 \leq z \leq \rho \quad \text{and} \quad x^2 + y^2 \leq (\rho - z)^2\}$$

as the *circular cone of radius ρ*.

LEMMA A.1. *Let $\mathcal{C}_1$ be the circular cone of radius 1, and let $\hat{\varphi}_1$ be the harmonic function that takes value 1 on the base and 0 on the lateral boundary. Then $\hat{\varphi}_1$ belongs to $L_\infty(\mathcal{C}_1)$ and $\nabla\hat{\varphi}_1$ belongs to $(L^t(\mathcal{C}_1))^3$ for all $t < 2$.*

Proof. The first part of the statement follows from the maximum principle, which gives $0 \leq \hat{\varphi}_1 \leq 1$. The second part of the statement follows immediately from the known results concerning domains with corners (see, e.g., [12] or [10]).   □

In view of the previous lemma, we set

(A.6) $\hat{C}_t := \|\nabla\varphi_1\|_{(L^t(\mathcal{C}_1))^3}.$

It is clear that $\hat{C}_t$ depends on t and hence on s through (4.15).

LEMMA A.2. *For every positive real number h, let $\mathcal{C}_h$ be a circular cone of radius h. Then, there exists a function φ_h taking value 1 on the base, value zero on the lateral surface, and satisfying*

(A.7) $\|\varphi_h\|_{L^2(\mathcal{C}_h)} \leq |\mathcal{C}_h|^{1/2}$ and $\|\nabla\varphi_h\|_{(L^t(\mathcal{C}_h))^3} \leq h^{3/t-1}\hat{C}_t,$

where $|\mathcal{C}_h|$ is the volume of $\mathcal{C}_h$.

Proof. The proof follows with the usual scaling arguments (see, e.g., [9, Theorem 3.1.2]).   □

Consider now a face e of E. For convenience, we assume that (a) the face e lies in the plane $z = 0$, (b) M_e, defined in Assumption M4 (*star-shaped faces*), is the origin of the axes, and (c) the polyhedron E is locally in the half-space $z > 0$. By Assumptions M4 and M5 (*the pyramid property*), there exists a γ_* such that the circular cone $\mathcal{C}_h$ having its base on the face e (with center in M_e), and radius $h = \rho_* = \gamma_* h_E$, is strictly contained in the pyramid P_E^e having the same vertex and base equal to e. Hence, $\mathcal{C}_h$ is contained in E.

Let us show that Assumption M4 implies the existence of a radial mapping in the plane $z = 0$ which maps the disk D_{ρ_*} with center in M_e and radius ρ_* onto the face e, is one-to-one, Lipschitz continuous together with its inverse, and with W_∞^1 norms bounded in terms of γ_* and the number of edges of e.

LEMMA A.3. *Under Assumption M4 there exists a map Φ_2, mapping the disk D_{ρ_*} onto the face e, which is Lipschitz continuous together with the inverse map Φ_2^{-1}. Moreover,*

(A.8) $\|\Phi_2\|_{W_\infty^1(D_{\rho_*})} \leq C_e^*$ and $\|\Phi_2^{-1}\|_{W_\infty^1(e)} \leq C_e^*,$

where C_e^* depends only on the constant γ_* from Assumption M4.

Proof. To show this, we note that the plane $z = 0$ can be split in a finite number of sectors by the vertices of e. Each sector corresponds to the straight rays coming
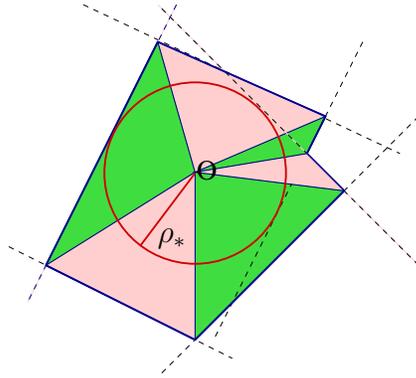
FIG. A.1. *The splitting of e in sectors.*

out of the origin M_e and intersecting the edge ℓ_k (see Figure A.1). For each point $\mathbf{P} \in D_{\rho_*}$, we first consider the ray emanating from the origin and passing through $\mathbf{P}$. This ray intersects ∂e at a point $\mathbf{V}(\mathbf{P})$. Our mapping is defined as follows:

$$(A.9) \qquad \tilde{\mathbf{P}} \equiv \Phi_2(\mathbf{P}) := \frac{|\mathbf{V}(\mathbf{P})|}{\rho_*} \mathbf{P}.$$

It is clear that Φ_2 maps every point $\mathbf{P}$ onto a point $\tilde{\mathbf{P}}$ on the same ray so that

$$(A.10) \qquad \mathbf{V}(\mathbf{P}) = \mathbf{V}(\tilde{\mathbf{P}}) \quad \forall \mathbf{P} \in D_{\rho^*}.$$

It is immediate to check that, on each ray, the map is continuous and monotone, and that it maps the points of the circumference of radius ρ_* onto the corresponding points of ∂e on the same ray. Hence it maps D_{ρ_*} onto e in a one-to-one way. It is also clear that the map is globally continuous, invertible, and the inverse map

$$(A.11) \qquad \mathbf{P} \equiv \Phi_2^{-1}(\tilde{\mathbf{P}}) := \frac{\rho_*}{|\mathbf{V}(\mathbf{P})|} \tilde{\mathbf{P}} \equiv \frac{\rho_*}{|\mathbf{V}(\tilde{\mathbf{P}})|} \tilde{\mathbf{P}}$$

is also continuous and maps e onto D_{ρ_*}. Note that we used (A.10) in the last step.

In order to show the Lipschitz continuity, we have to bound the distance between the images $|\tilde{\mathbf{P}} - \tilde{\mathbf{Q}}|$ by a constant time the distance $|\mathbf{P} - \mathbf{Q}|$. For this, we note that Assumption M4 implies

$$(A.12) \qquad 1 \leq \frac{|\mathbf{V}|}{\rho_*} \leq \frac{h_E}{\gamma_* h_E} = \frac{1}{\gamma_*} \qquad \text{for every } \mathbf{V} \in \partial e.$$

As shown in Figure A.2, it also implies that for every point $\mathbf{V}$ on an edge ℓ of ∂e, the angle α_V between ℓ and the ray passing through $\mathbf{V}$ verifies

$$(A.13) \qquad |\sin \alpha_V| = \frac{|\mathbf{H}_\ell|}{|\mathbf{V}|} \geq \frac{\rho_*}{|\mathbf{V}|} \geq \gamma_*,$$

where $\mathbf{H}_\ell$ is the orthogonal projection of the origin M_e on the line containing ℓ, and we used (A.12) in the last step.

The Lipschitz continuity is obvious when $\mathbf{P}$ and $\mathbf{Q}$ are on the same ray:

$$(A.14) \qquad |\tilde{\mathbf{P}} - \tilde{\mathbf{Q}}| = \frac{|\mathbf{V}(\mathbf{P})|}{\rho_*} |\mathbf{P} - \mathbf{Q}| \leq \frac{1}{\gamma_*} |\mathbf{P} - \mathbf{Q}|.$$
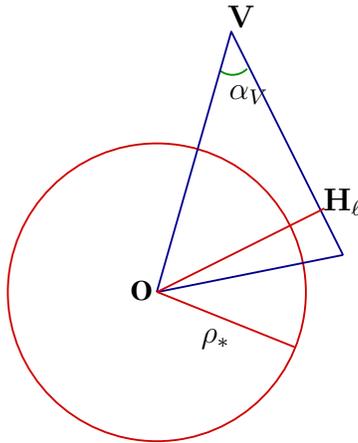
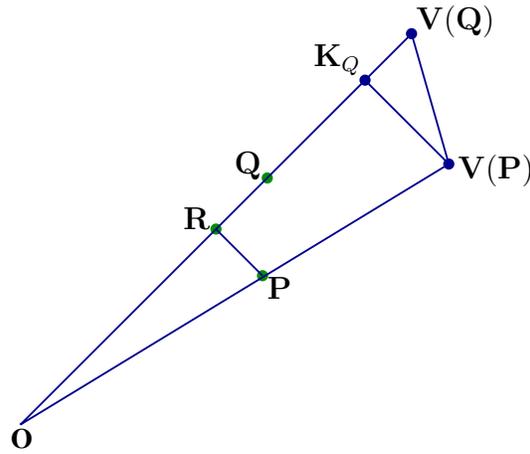FIG. A.2. *Lower bound on* $|\sin \alpha_V|$.

FIG. A.3. *Lipschitz continuity within a sector.*

If $\mathbf{P}$ and $\mathbf{Q}$ are on two different rays in the same sector, we first denote by $\mathbf{K}_Q$ and $\mathbf{R}$ (respectively) the orthogonal projections of $\mathbf{V(P)}$ (respectively, of $\mathbf{P}$) on the ray containing $\mathbf{Q}$ (see Figure A.3). Then, applying the Thaletes theorem, we get

$$
(A.15) \qquad \frac{|\mathbf{V(P)} - \mathbf{K}_Q|}{|\mathbf{V(P)}|} = \frac{|\mathbf{P} - \mathbf{R}|}{|\mathbf{P}|} \leq \frac{|\mathbf{P} - \mathbf{Q}|}{|\mathbf{P}|}.
$$

Collecting (A.15), (A.13), and (A.12), we have

$$
(A.16) \qquad |\mathbf{V(P)} - \mathbf{V(Q)}| = \frac{|\mathbf{V(P)} - \mathbf{K}_Q|}{\left|\sin\left(\alpha_{V(Q)}\right)\right|} \leq \frac{|\mathbf{P} - \mathbf{Q}|}{\gamma_* |\mathbf{P}|} |\mathbf{V(P)}| \leq \frac{|\mathbf{P} - \mathbf{Q}|}{(\gamma_*)^2 |\mathbf{P}|} \rho_*,
$$

where obviously the role of $\mathbf{P}$ and $\mathbf{Q}$ can be interchanged. Finally, the triangle inequality together with (A.9) and (A.16) gives

(A.17)
$$|\tilde{\mathbf{P}} - \tilde{\mathbf{Q}}| = \left| \frac{|\mathbf{V}(\mathbf{P})|\,\mathbf{P} - |\mathbf{V}(\mathbf{Q})|\,\mathbf{Q}}{\rho_*} \right| \le \frac{|\mathbf{V}(\mathbf{P}) - \mathbf{V}(\mathbf{Q})|}{\rho_*} |\mathbf{P}| + \frac{|\mathbf{V}(\mathbf{Q})|}{\rho_*} |\mathbf{P} - \mathbf{Q}|$$
$$\le \frac{|\mathbf{P} - \mathbf{Q}|}{(\gamma_*)^2} + \frac{1}{\gamma_*} |\mathbf{P} - \mathbf{Q}| = \frac{1 + \gamma_*}{(\gamma_*)^2} |\mathbf{P} - \mathbf{Q}|.$$

The case of $\mathbf{P}$ and $\mathbf{Q}$ belonging to different sectors can be easily deduced by inserting suitable intermediate points at the boundaries of the sectors and then using the triangle inequality.

In a similar way, we can show that the inverse mapping is also Lipschitz continuous. For instance, using (A.11) we get

(A.18) $$|\mathbf{P} - \mathbf{Q}| = \left| \frac{\rho_*}{|\mathbf{V}(\tilde{\mathbf{P}})|} \tilde{\mathbf{P}} - \frac{\rho_*}{|\mathbf{V}(\tilde{\mathbf{Q}})|} \tilde{\mathbf{Q}} \right| = \frac{\rho_*}{|\mathbf{V}(\tilde{\mathbf{P}})|\,|\mathbf{V}(\tilde{\mathbf{Q}})|} \left||\mathbf{V}(\tilde{\mathbf{Q}})|\tilde{\mathbf{P}} - |\mathbf{V}(\tilde{\mathbf{P}})|\tilde{\mathbf{Q}}\right|.$$

Then, adding and subtracting $|\mathbf{V}(\mathbf{P})|\mathbf{P}$ and using the triangle inequality, we have

(A.19) $$\left||\mathbf{V}(\tilde{\mathbf{Q}})|\tilde{\mathbf{P}} - |\mathbf{V}(\tilde{\mathbf{P}})|\tilde{\mathbf{Q}}\right| \le |\mathbf{V}(\tilde{\mathbf{P}}) - \mathbf{V}(\tilde{\mathbf{Q}})|\,|\tilde{\mathbf{P}}| + |\mathbf{V}(\tilde{\mathbf{P}})|\,|\tilde{\mathbf{P}} - \tilde{\mathbf{Q}}|.$$

On the other hand, we can apply the argument of (A.16) to obtain

(A.20) $$|\mathbf{V}(\tilde{\mathbf{P}}) - \mathbf{V}(\tilde{\mathbf{Q}})| \le \frac{|\tilde{\mathbf{P}} - \tilde{\mathbf{Q}}|}{(\gamma_*)^2|\tilde{\mathbf{P}}|} \rho_*.$$

Collecting (A.18), (A.19), and (A.20), and using (A.12) (this time as $\rho_*/|\mathbf{V}| \le 1$), we finally obtain

(A.21) $$|\mathbf{P} - \mathbf{Q}| \le \frac{1}{(\gamma_*)^2}|\tilde{\mathbf{P}} - \tilde{\mathbf{Q}}| + |\tilde{\mathbf{P}} - \tilde{\mathbf{Q}}| = \frac{1 + (\gamma_*)^2}{(\gamma_*)^2} |\tilde{\mathbf{P}} - \tilde{\mathbf{Q}}|.$$

This proves the assertion of the lemma.   □

Now, we can construct a mapping Φ_3 from the cone $\mathcal{C}_h$ (having D_{ρ_*} as the base and with height equal to ρ_*) onto the pyramid P_E^e (having e as the base and with the same vertex as $\mathcal{C}_h$), which is Lipschitz continuous with its inverse, by taking

(A.22) $$(\tilde{x}, \tilde{y}) = \Phi_2(x, y), \quad \tilde{z} = z.$$

Again, the Lipschitz norms of the map Φ_3 and of its inverse depend only on γ_*. This proves the following lemma.

LEMMA A.4. *Under Assumption* M4 *there exists a map* Φ_3, *mapping the cone* $\mathcal{C}_h$ *onto the pyramid* P_E^e, *which is Lipschitz continuous together with the inverse map* Φ_3^{-1}. *Moreover,*

(A.23) $$\left\|\Phi_3\right\|_{W_\infty^1(\mathcal{C}_h)} \le C_{pyr}^* \quad and \quad \left\|\Phi_3^{-1}\right\|_{W_\infty^1(P_E^e)} \le C_{pyr}^*,$$

where C_{pyr}^* *depends only on the constant* γ_* *of Assumption* M4.

The last step is to construct, for each element E and for each face $e \in \partial E$, the function φ_E^e satisfying (A.5) (with the right boundary conditions). Let

$$\varphi_E^e(x, y, z) = \varphi_h\big(\Phi_3^{-1}(x, y, z)\big),$$

where φ_h is the function from Lemma A.2 defined for the circular cone of radius $h = \rho_* = \gamma_* h_E$. It is clear that φ_E^e will be in $L^2(P_E^e)$, that $\nabla\varphi_E^e$ will be in $(L^t(P_E^e))^3$, and that their norms will be bounded by

$$(A.24) \qquad \left\|\varphi_E^e\right\|_{L^2(P_E^e)} \leq C_{pyr}^* \, h_E^{3/2} \quad \text{and} \quad \left\|\nabla\varphi_E^e\right\|_{(L^t(P_E^e))^3} \leq \hat{C}_t \, C_{pyr}^* h_E^{3/t-1},$$

where $\hat{C}_t$ is given in (A.6) and C_{pyr}^* depends only on γ_*. Hence φ_E^e satisfies (A.5) as required. Finally, we take the prolongation of φ_E^e (that we call again φ_E^e) by zero in $E\backslash P_E^e$.

This ends the proof of the lift property (A.1)–(A.2).

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