Traffic flows in one lane roadways

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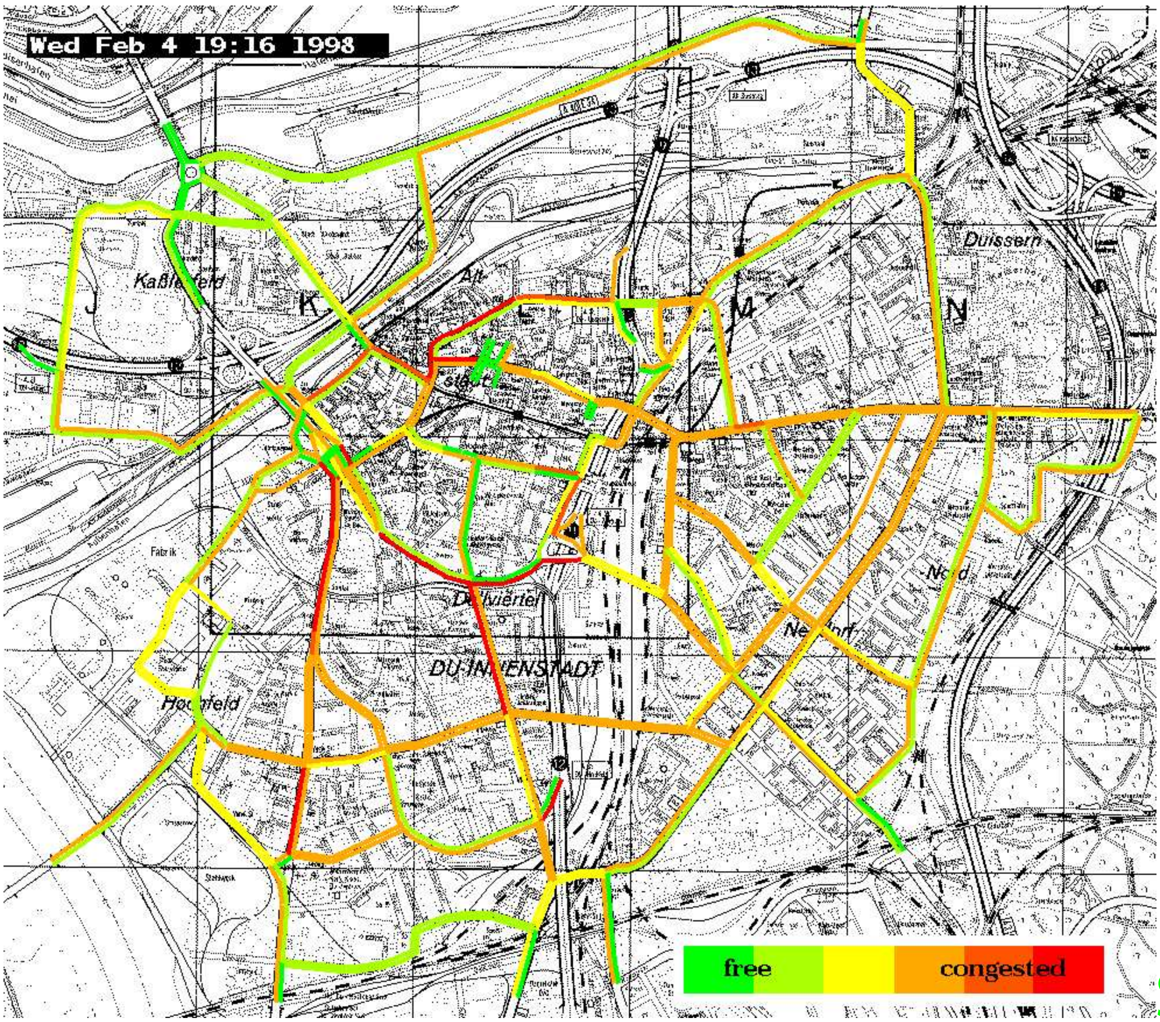
Traffic

- strongly interacting many body systems
- low dimensional systems:
  - rural traffic is 1D
  - urban grid traffic is 2D-like
- rich phenomenology:
  - shock waves
  - jamming — Nagel J Physique 92
  - clustering — EB PRE 94
  - phase transitions — Kerner PRL 97

can we use gas/fluid theories and concepts (kinetic theory/fluid dynamics)?

Cellular automata microsimulation



Schreck

A variety of approaches

		cars	x	t
	Cellular Automata (CA)	D	D	D
	Hopping Models (MC)	D	D	C
$\longrightarrow$	Event Driven (MD)	D	C	C
	Fluid Dynamics (FD)	C	C	C

CA - Nagel, Schreckenberg

MC - Derrida, Lebowitz, Schmitmann, Zia

MD - Helbing, Nagatani, Progogine

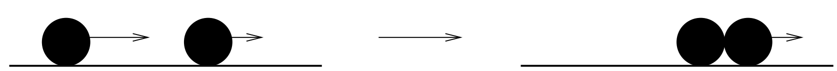
FD - Herman, Kerner, Williams

so far, mainly numerical results

The Model

simplifications:

- uniform roads
- infinite roads ($L = \infty$)
- collision time $= 0$
- sizeless cars
- purely ballistic motion (constant velocity)
- $P_0(v) =$ intrinsic velocity distribution

collision rule: 

goals: given $P_0(v) = P(v, t = 0)$, find:

- $P(v, t) =$ distribution of clusters with velocity v at time t.
- $P_k(v, t) =$ distribution of clusters of k cars with velocity v at time t.

No passing zones

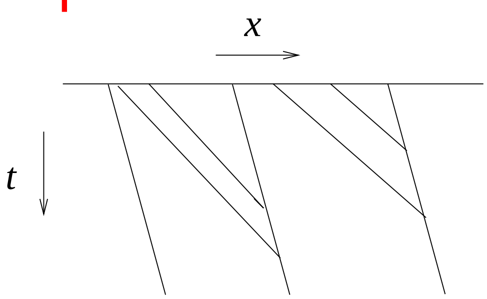
preliminary: discrete velocity distributions

$$P_0(v) = \sum_{i=1}^{N} P_i \delta(v - v_i) \qquad v_1 < v_2 < \cdots$$

$$P(v,t) = \sum_{i=1}^{N} P_i S_i(t) \delta(v - v_i)$$

$S_i(t)$ = survival probability of v_i

special case: bimodal distribution $(N = 2)$



$$P(v,t) = P_1 S_1(t) + P_2 S_2(t)$$

$$S_1(t) = 1$$

$S_2(t) =$ probability interval $[x, x + (v_2 - v_1)t]$ does not contain any v_1 at time $t = 0$

$$S_2(t) = [1 - P_1 \Delta x]^{\frac{(v_2 - v_1)t}{\Delta x}} \underset{\Delta x \to 0}{\longrightarrow} \exp[-t(v_2 - v_1)P_1]$$

$$\boxed{S_2(t) = \exp\left[-t(v_2 - v_1)P_1\right]}$$

trimodal distributions $v_1 < v_2 < v_3$

$S_3(t) =$ probability interval $[0,(v_3 - v_1)t]$ has no v_1 **and** interval $[0,(v_3 - v_2)t]$ has no v_2

$$S_3(t) = \exp\left[-t(v_3 - v_1)P_1 - t(v_3 - v_2)P_2\right]$$

generally,

$$S_k(t) = \exp\left[-t\sum_{j=1}^{k-1}(v_k - v_j)P_j\right]$$

continuous velocity distributions

$$v_i \to v \qquad \sum_{j=1}^{k-1} \to \int_0^v dv' \qquad P_j \to P_0(v)$$

survival probability

$$S(v,t) = \exp\left[-t\int_0^v dv'(v - v')P_0(v')\right]$$

velocity distribution of clusters $P = P_0 S$

$$P(v,t) = P_0(v)\exp\left[-t\int_0^v dv'(v - v')P_0(v')\right]$$

$$\boxed{\textbf{exact solution for arbitrary } P_0(v)}$$

The Boltzmann equation

$$P(v,t) = P_0(v)\exp\left[-t\int_0^v dv'(v-v')P_0(v')\right]$$

evaluate $\frac{\partial}{\partial t}\ln P(v,t)$

$$\frac{\partial P(v,t)}{\partial t} = -P(v,t)\int_0^v dv'(v-v')P_0(v')$$

- integration limits: v affected only by $v' < v$
- collision rate = velocity difference $(v-v')$
- two point density $= P(v_>,t)P(v_<,0)$

- two point density "remembers" initial state
- slow cars mildly affected
- fast cars exponentially suppressed

Scaling behavior

cluster density $c(t) = \int dv P(v,t)$ Nagatani PRE 96

$$c(t) \sim t^{-\alpha} \qquad \alpha = \frac{\mu+1}{\mu+2}$$

average cluster velocity $\langle v(t)\rangle = \int dv\, v\, P(v,t)/c(t)$

$$\langle v(t)\rangle \sim t^{-\beta} \qquad \beta = 1-\alpha = \frac{1}{\mu+2}$$

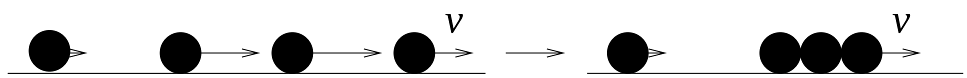
extremal properties of $P_0(v)$ determine the long time behavior ($P_0(v) \sim v^{\mu}$ as $v \to 0$)

velocity distribution scales

$$P(v,t) \sim t^{\beta-\alpha} F(vt^{\beta}) \qquad F(x) \sim x^{\mu} \exp\left[-x^{\mu+2}\right]$$

- self-similar velocity distribution
- late time behavior gives μ
- slow car statistics dominate the flow

Why? extreme statistics



$$P_-(v) = \int_0^v dv' P_0(v') \quad P_+(v) = \int_v^\infty dv' P_0(v')$$

probability final cluster has k cars

$$P_k(v) = P_- P_+^k \quad \Rightarrow \quad \langle m \rangle = \sum_k k P_- P_+^k = \frac{P_+}{P_-}$$

using this heuristic picture, it is possible to reproduce the scaling behavior

$$P_0(v) \sim v^\mu \;\Rightarrow\; P_-(v) \sim v^{\mu+1} \;\Rightarrow\; \langle m \rangle \sim v^{-(\mu+1)} \quad (*)$$

dimensional analysis suggests

$$m \sim vt \quad (**)$$

combining $(*), (**)$

$$c \sim m^{-1} \sim t^{-\frac{\mu+1}{\mu+2}} \qquad v \sim t^{-\frac{1}{\mu+2}}$$

Cluster mass-velocity distribution

$Q_m(v,t)$ = distribution of clusters with <u>at least</u> m cars moving at velocity v

$$\begin{aligned}
Q_1(v,t) &= P(v,t) \\
Q_2(v,t) &= P(v,t)\int_v^\infty dv_1 P_0(v_1) \int\limits_{x_1<(v_1-v)t} dx_1 \exp(-x_1) \\
Q_m(v,t) &= P(v,t)\prod_{i=1}^{m-1}\int_v^\infty dv_i P_0(v_i) \int\limits_{x_1+\cdots+x_i<(v_i-v)t} dx_i \exp(-x_i)
\end{aligned}$$

$P_m(v,t) = Q_m(v,t) - Q_{m+1}(v,t) =$ distribution of clusters with <u>exactly</u> m cars moving at v

$P_m(v,t)$ scales: $P_m(v,t) \sim t^{\beta-2\alpha}F(mt^{-\alpha}, vt^{\beta})$

$$F(x,y) = (x+y)^{\mu+1}y^{\mu}\exp[-(x+y)^{\mu+2}]$$

$P(v,t)$ and $P_m(t)$ are Gaussian when $\mu = 0$

Conclusions I

- exact Boltzmann equation derived

$$\frac{\partial P(v,t)}{\partial t} = -P(v,t)\int_0^v dv'(v-v')P_0(v')$$

- scaling asymptotic behavior

$$c \sim m^{-1} \sim t^{-\frac{\mu+1}{\mu+2}} \qquad v \sim t^{-\frac{1}{\mu+2}}$$

- self similar distributions
- slow car distribution - the only relevant parameter in long time limit

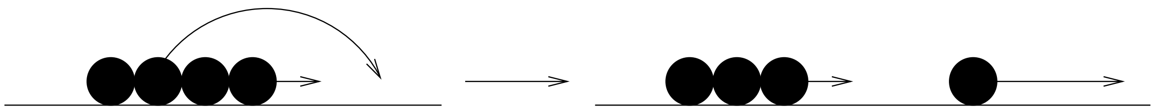
$$\mu = \lim_{v\to 0} v\frac{\partial}{\partial v}\ln P_0(v)$$

- spatial inhomogeneities

$$P(x,v,t) = P_0(x-vt,v)\exp\left[-\int_0^v dv' \int_{x-vt}^{x-v't} dx' P_0(x',v')\right]$$

- complete exact solution

Passing zones



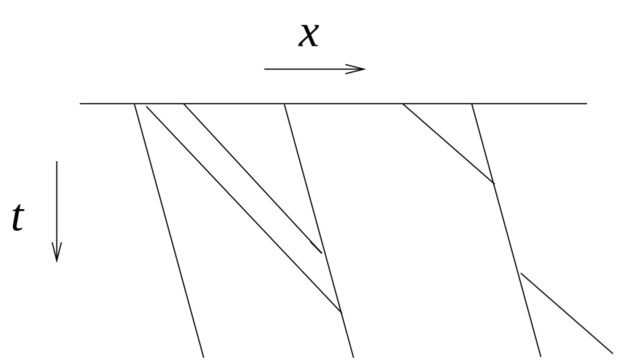
introduce possibility of escape/passing

simplifications:

- every car can escape
- constant escape rate $= t_0^{-1}$
- escape time $= 0$

goals: given $P_0(v) = P(v, t = 0)$, find:

- steady state distribution $P(v) \equiv P(v, t = \infty)$
- flux - J, average cluster size $\langle m \rangle$



Heuristic picture

let

$$
\begin{aligned}
t_0^{-1} &= \text{escape rate} \\
v_0 &= \text{typical velocity} \\
c_0 &= \text{initial car concentration} \\
c &= \text{steady state cluster concentration}
\end{aligned}
$$

then $m = c_0/c =$ average cluster size

$$
\begin{aligned}
\langle m \rangle \, t_0^{-1} &\sim \text{total escape rate} \\
c \, v_0 &\sim \text{total collision rate}
\end{aligned}
$$

in steady state, # of cars entering and leaving clusters must balance!

$$\langle m \rangle \, t_0^{-1} \sim c \, v_0 \quad \Rightarrow \quad c \sim (c_0/v_0 t_0)^{1/2}$$

$$t_{\rm esc} = t_0 \quad t_{\rm col} = (c_0 v_0)^{-1}$$

$$R = c_0 v_0 t_0 = \frac{t_{\rm esc}}{t_{\rm col}} \quad \langle m \rangle \sim R^{1/2}$$

single dimensionless number

Modified Boltzmann equation

preliminary: use dimensionless variables

$$c_0 x \to x \quad v/v_0 \to v \quad c_0 v_0 t \to t$$

escape rate renormalizes

$$t_0^{-1} \to (c_0 v_0 t_0)^{-1} = R^{-1}$$

$R =$ the "collision" number

mean-field Boltzmann equation

$$\frac{\partial P(v)}{\partial t} = R^{-1}\left[P_0(v) - P(v)\right] - P(v)\int_0^v dv'(v - v')P(v')$$

exact escape term: # of slowed down cars with intrinsic velocity v

approximate collision term: assumes no correlations

what is the steady state distribution $P(v)$?

($t \to \infty$ or $\partial/\partial t \equiv 0$)

Formal steady state solution

let $P(v) \equiv P(v, t = \infty)$

$$0 = R^{-1}\left[P_0(v) - P(v)\right] - P(v)\int_0^v dv'(v-v')P(v')$$

then

$$P(v)\left[R^{-1} + \int_0^v dv'(v-v')P(v')\right] = R^{-1}P_0(v)$$

reduction to ordinary differential equation

$$\begin{aligned} Q(v) &= R^{-1} + \int_0^v dv'(v-v')P(v') \\ Q(0) &= R^{-1} \qquad Q'(0) = 0 \end{aligned}$$

$$Q(v)Q''(v) = R^{-1}P_0(v)$$

auxiliary function $Q(v)$ gives $P(v)$

$$P(v) = Q''(v)$$

this gives only **cluster** properties

$$c = \int dv P(v) \qquad \langle v \rangle = \int dv v P(v)$$

Conditional velocity distribution

$P(v, v')$ =distribution of cars with intrinsic velocity v and actual velocity $v' < v$

normalization

$$P_0(v) = P(v) + \int_0^v dv' P(v, v')$$

master equation

$$\begin{aligned} \frac{\partial P(v, v')}{\partial t} = & - R^{-1} P(v, v') + (v - v') P(v) P(v') \\ & - P(v, v') \int_0^{v'} dv'' (v' - v'') P(v'') \\ & + P(v') \int_{v'}^{v} dv'' (v'' - v') P(v, v''). \end{aligned}$$

steady state solution

$$P(v, v') = \frac{P_0(v) P_0(v')}{Q(v')} \int_{v'}^{v} \frac{du}{[RQ(u)]^2}.$$

$P(v,v')$ gives flux and more

actual velocity distribution $G(v)$, flux J

$$G(v) = P(v) + \int_v^\infty dw P(w,v) \qquad J = \int dv\, v\, G(v)$$

summary

solve one second order nonlinear ODE

$$Q(v)Q''(v) = R^{-1}P_0(v)$$

get most steady state properties

$$\begin{aligned}
P(v) &= Q''(v) \\
P(v,v') &= \frac{P_0(v)P_0(v')}{Q(v')}\int_{v'}^{v}\frac{du}{[RQ(u)]^2} \\
J &= \int_0^\infty dv P_0(v)\int_0^v \frac{du}{[RQ(u)]^2} \\
G(v) &= P(v)\left[1 + R\int_v^\infty dw P_0(w)\int_v^w \frac{du}{[RQ(u)]^2}\right]
\end{aligned}$$

Low collision numbers $R \ll 1$

steady state equation

$$P(v) = P_0(v)\left[1 + R\int_0^v dv'(v-v')P(v')\right]^{-1}$$

perturbation series solution

$$\begin{aligned} P(v) &= P^{(0)}(v) + RP^{(1)}(v) + R^2P^{(2)}(v) + \cdots \\ P(v,v') &= RP^{(1)}(v,v') + R^2P^{(2)}(v,v') + \cdots \end{aligned}$$

for example

$$\begin{aligned} P^{(0)}(v) &= P_0(v) \\ P^{(1)}(v) &= -P_0(v)\int_0^v dv'(v-v')P_0(v') \\ P^{(1)}(v,v') &= (v-v')P_0(v)P_0(v') \\ c &= 1 - c_1R \\ J &= J_0 - J_1R \qquad J_1 = \langle v^2\rangle_0 - \langle v\rangle_0^2 \end{aligned}$$

perturbation series in R is possible

“laminar” uninterrupted flow

Large collision numbers $R \gg 1$

small velocities unaffected

$$P(v) \cong P_0(v) \left[1 - R \int_0^v dv'(v - v')P_0(v')\right] \qquad v \ll v^*$$

assume algebraic intrinsic distribution

$$P_0(v) \sim v^\mu \qquad v \to 0$$

threshold velocity

$$v^* \sim R^{-\frac{1}{\mu+2}}$$

faster than v^* cars are rare

$$P(v) \sim \begin{cases} (v^*)^\mu (v/v^*)^{\mu-1} & \mu < 0 \\ (v^*)^\mu (v/v^*)^{\mu/2-1} & \mu > 0 \end{cases} \qquad v \gg v^*$$

inner & outer solutions match at boundary

$$P(v^*) \sim P_0(v^*)$$

$\boxed{\textbf{boundary layer structure}}$

Scaling properties

scaling with respect to R

$$\langle m \rangle \sim \begin{cases} R^{(\mu+1)/(\mu+2)} & \mu < 0 \\ R^{1/2} & \mu > 0 \end{cases}$$

heuristic picture holds only when $\mu < 0$

$$\langle v \rangle \sim \begin{cases} R^{\mu/(\mu+2)} & \mu < 0 \\ \text{const.} & \mu > 0 \end{cases}$$

flux is significantly reduced

$$J \sim v^* \sim R^{-\frac{1}{\mu+2}}$$

again slow car statistics relevant - μ

slowing down due to clustering

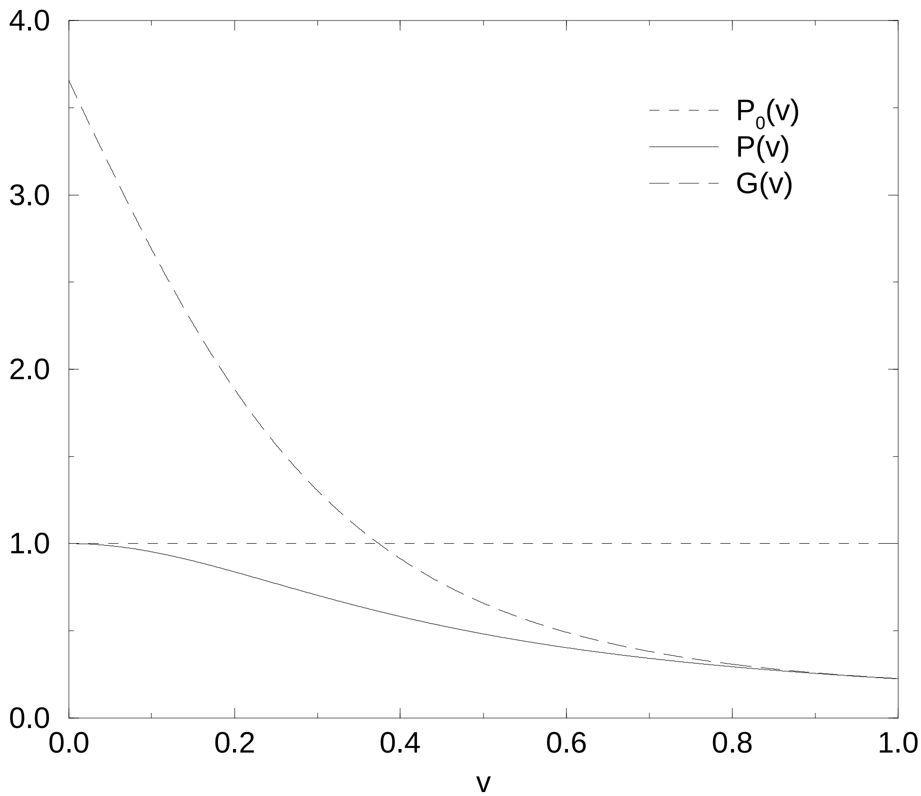
high collision numbers: "turbulent" flow

I flat intrinsic distribution

$P_0(v) = 1$ in $[0:1]$

$R = 10$ - behavior consistent with predictions

$$G(0) \sim R^{1/2} \quad v^* \sim R^{-1/2} \quad P(1) \sim R^{-1/2}$$



ODE $Q(v)Q''(v) = R^{-1}$ is integrable here

$$\int_1^{RQ} \frac{dq}{\sqrt{2\ln q}} = v\sqrt{R}$$

II discrete velocity distributions

$$P(v) = \frac{P_0(v)}{1 + \int_0^v dv'(v - v')P(v)} \qquad (R = 1)$$

$$P_0(v) = \sum_i p_i \delta(v - v_i) \qquad P(v) = \sum_i c_i \delta(v - v_i)$$

$$c_i = \frac{p_i}{1 + \sum_{j=1}^{i-1} (v_i - v_j) c_j}$$

$$\begin{aligned}
c_1 &= p_1 \\
c_2 &= \frac{p_2}{1 + (v_2 - v_1)p_1} \\
c_3 &= \frac{p_3}{1 + (v_3 - v_1)p_1 + \frac{(v_3 - v_2)p_2}{1 + (v_2 - v_1)p_1}}
\end{aligned}$$

explicit continued fraction solution

Conclusions II

- a single dimensionless number $R = c_0 v_0 t_0$ underlies the behavior

- exact solution of MFT possible for $P(v)$, $P(v, v')$, $G(v)$, J

- $R \ll 1$ laminar uninterrupted flow:

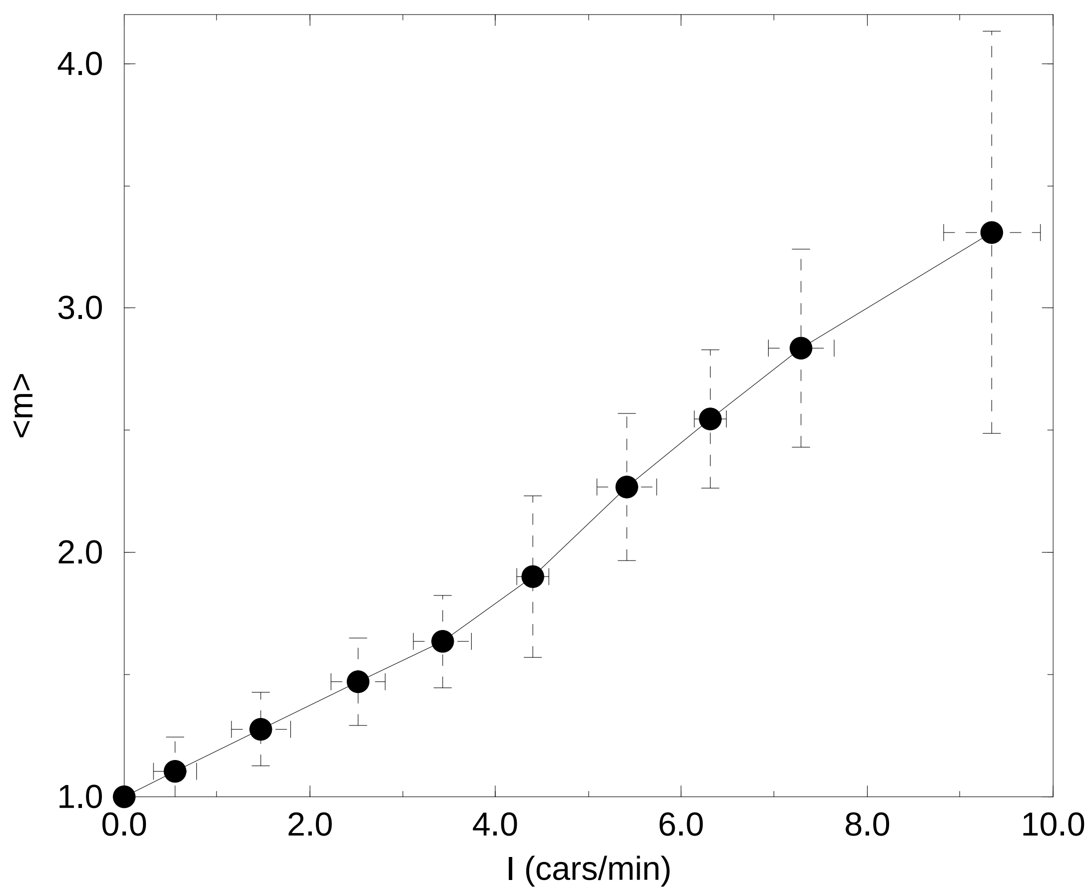
  — perturbative solution
  — exponential mass distribution $P_k \sim R^k$
  — each collision generates a factor R

- $R \gg 1$ collisions dominate:

  — scaling in R
  — boundary layer structure in v
  — strong flux reduction
  — slow car statistics plays crucial role

Rural traffic observations

$$
\begin{aligned}
I &= c_0(v_{\min} + v_0 J) \propto c_0 v_{\min} \propto R \\
\langle m \rangle &\cong 1 + \text{const.} \times R \cong 1 + \text{const.} \times I
\end{aligned}
$$



Linear growth of cluster size confirmed in laminar case ($R \ll 1$)

Outlook

- compare velocity distributions with real data
- solve for steady state mass distribution
- solve when only first car can escape
- examine role of escape mechanism
- generalize to multilane flows - escape naturally couples neighboring lanes
- role of other dimensionless numbers?

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