Jamming and tiling of rectangles

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Talk, publications available from: http://cnls.lanl.gov/~ebn

50 years of stochastic processes at UCSD:
a symposium in honor of Katja Lindenberg
San Diego, CA, August 16, 2019

Plan

I Linear dynamical tiling:

Stochastic fragmentation of rectangles

II Nonlinear dynamical tiling:

Stochastic aggregation of rectangles

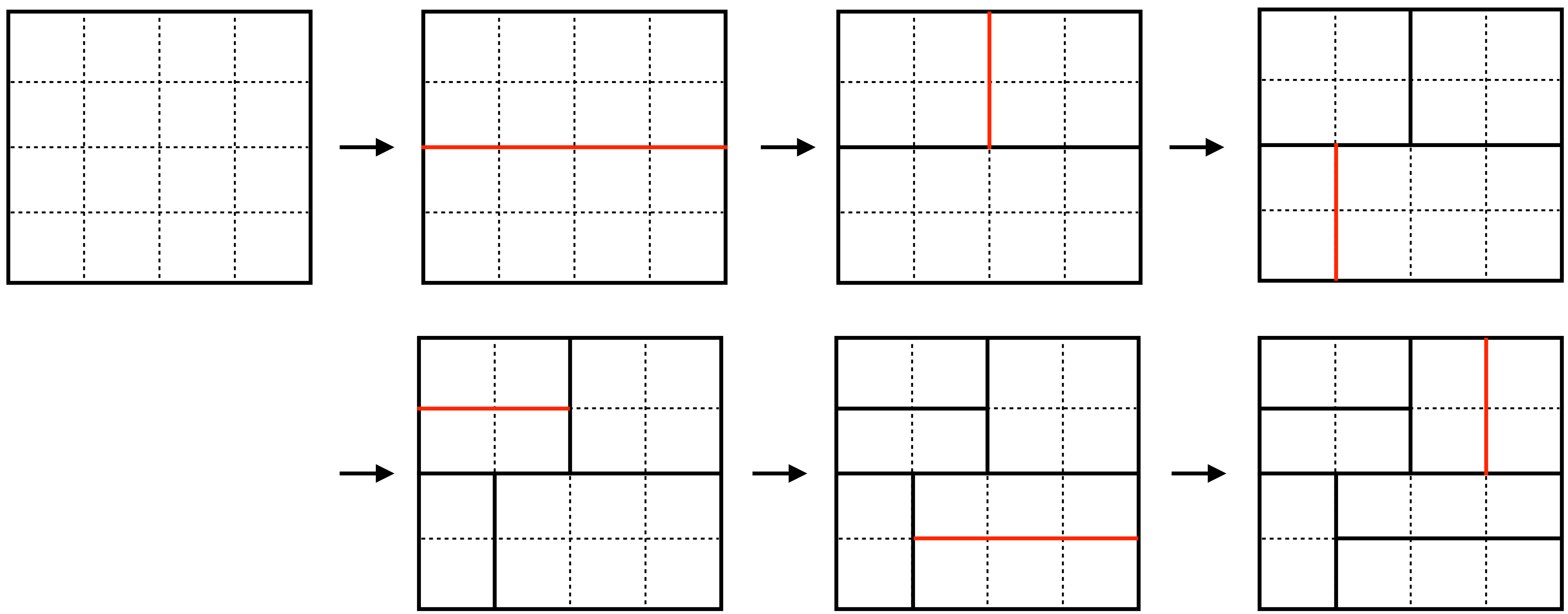
Straightforward generalizations of classical 1D fragmentation and aggregation to 2D Results anything but

Fragmentation of rectangles

Start with a perfect grid

Pick (i) random grid point (ii) random direction

Fragment rectangle into two smaller rectangles



System reaches a jammed state

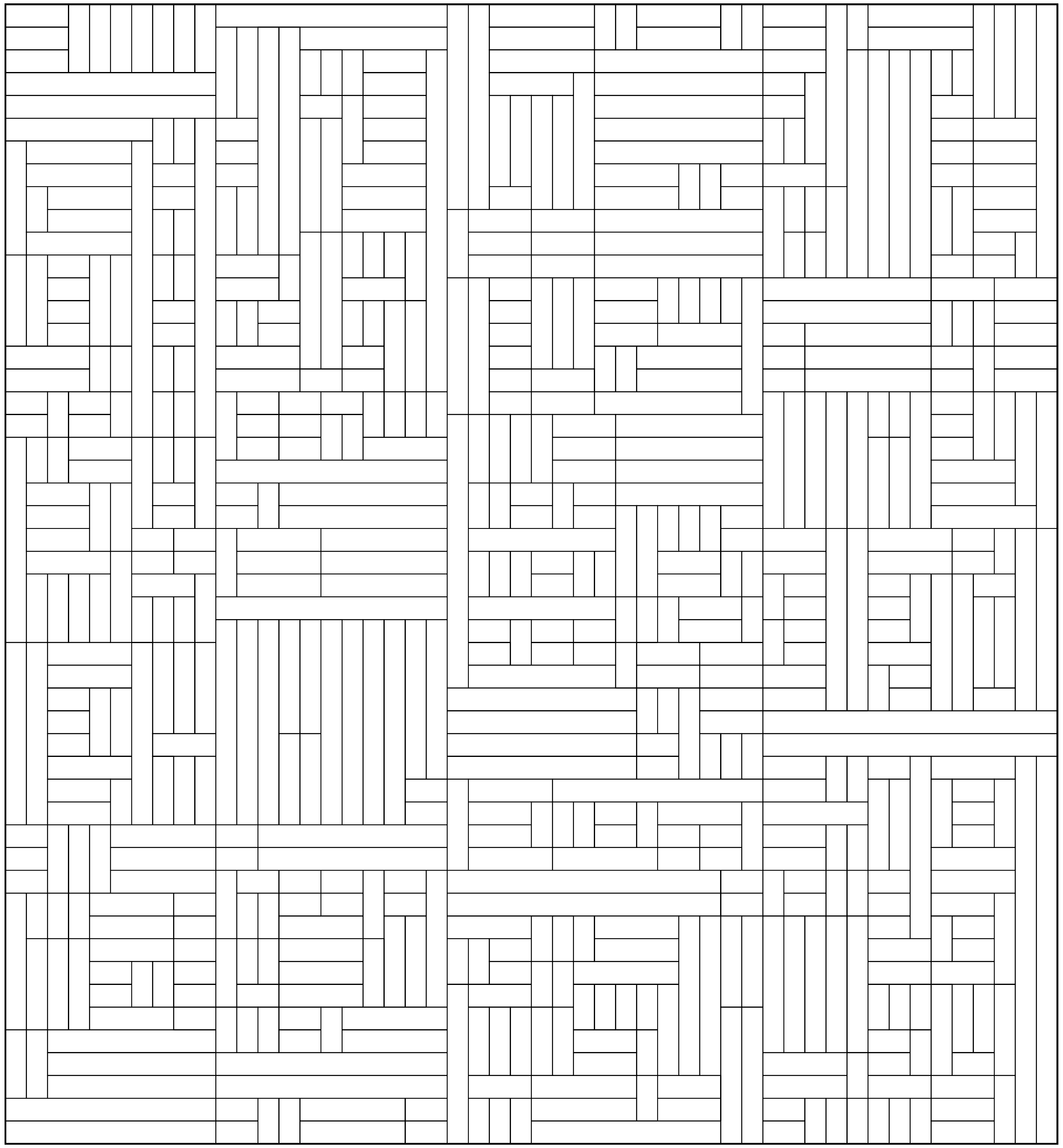
All rectangles are sticks (1 x k or k x 1)

The jammed state

Tiling by sticks

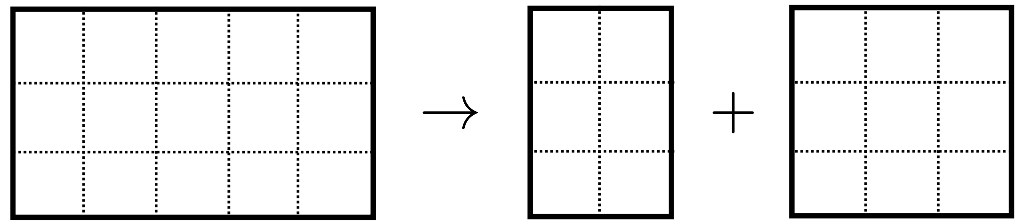
Tiling is:

- Polydisperse
- Dynamical



How many sticks? How long? How many jammed states?

Theoretical approach: recursion equations



1D
Filippov 61
Spouge 84
Ziff, McGrady 85

- Random fragmentation process

$$(m,n) \to \begin{cases} (i,n) + (m-i,n) & \text{with prob. } 1/2 \\ (m,j) + (m,n-j) & \text{with prob. } 1/2 \end{cases}$$

- Average number of sticks $S(m,n)$ in an $m \times n$ rectangle

- Recursion: sum over all possible (i) grid points (ii) directions

$$S(m,n) = \frac{1}{2} \times \frac{1}{m-1} \sum_{i=1}^{m-1} \Big[S(i,n) + S(m-i,n)\Big] + \frac{1}{2} \times \frac{1}{n-1} \sum_{j=1}^{n-1} \Big[S(m,j) + S(m,n-j)\Big]$$

- Linear recursion equations for number of jammed sticks

$$S(m,n) = \frac{1}{m-1} \sum_{i=1}^{m-1} S(i,n) + \frac{1}{n-1} \sum_{j=1}^{n-1} S(m,j)$$

2D
Torrents, Illa, Vives, Planes
PRE 2017

Theory: (i) linear (ii) bypasses dynamics (iii) 2d

Asymptotic analysis

1. Continuum limit (very large rectangles)

$$S(m,n) = \frac{1}{m}\int_1^m di\, S(i,n) + \frac{1}{n}\int_1^n dj\, S(m,j)$$

2. Convert integral equation into partial differential equation

$$\partial_\mu \partial_\nu S(\mu,\nu) = S(\mu,\nu) \qquad \begin{aligned} \mu &= \ln m \\ \nu &= \ln n \end{aligned}$$

3. Introduce double Laplace transform

$$\widehat{S}(p,q) = \int_0^\infty d\mu\, e^{-p\mu} \int_0^\infty d\nu\, e^{-q\nu}\, S(\mu,\nu)$$

4. Obtain Laplace transform in compact form

$$\widehat{S}(p,q) = \frac{1}{pq-1}$$

5. Invert double Laplace transform (saddle point analysis)

$$S(\mu,\nu) = \int_{-i\infty}^{i\infty} \frac{dp}{2\pi i} \int_{-i\infty}^{i\infty} \frac{dq}{2\pi i}\, \frac{e^{p\mu+q\nu}}{pq-1} \quad \to \quad S(\mu,\nu) \simeq \frac{e^{2\sqrt{\mu\nu}}}{\sqrt{4\pi\sqrt{\mu\nu}}}$$

Average number of jammed sticks

- Asymptotic behavior

$$S(m,n) \simeq \frac{e^{2\sqrt{(\ln m)(\ln n)}}}{\sqrt{4\pi\sqrt{(\ln m)(\ln n)}}}$$

- Focus on very large rectangles with finite aspect ratio

$$m \to \infty \quad \text{and} \quad n \to \infty \quad \text{with} \quad m/n = \text{constant}$$

- Universal behavior for all rectangles with same area

$$S(A) \simeq \frac{A}{\sqrt{2\pi \ln A}} \qquad\qquad A = mn$$

- Average stick length $\langle k \rangle = A/S$ grows slowly with area

$$\langle k \rangle \simeq \sqrt{2\pi \ln A}$$

Behavior is independent of aspect ratio

Distribution of stick length

- Number of sticks of given length obeys same recursion

$$S_k(m,n) = \frac{1}{m-1}\sum_{i=1}^{m-1} S_k(i,n) + \frac{1}{n-1}\sum_{j=1}^{n-1} S_k(m,j)$$

- Leading asymptotic behavior

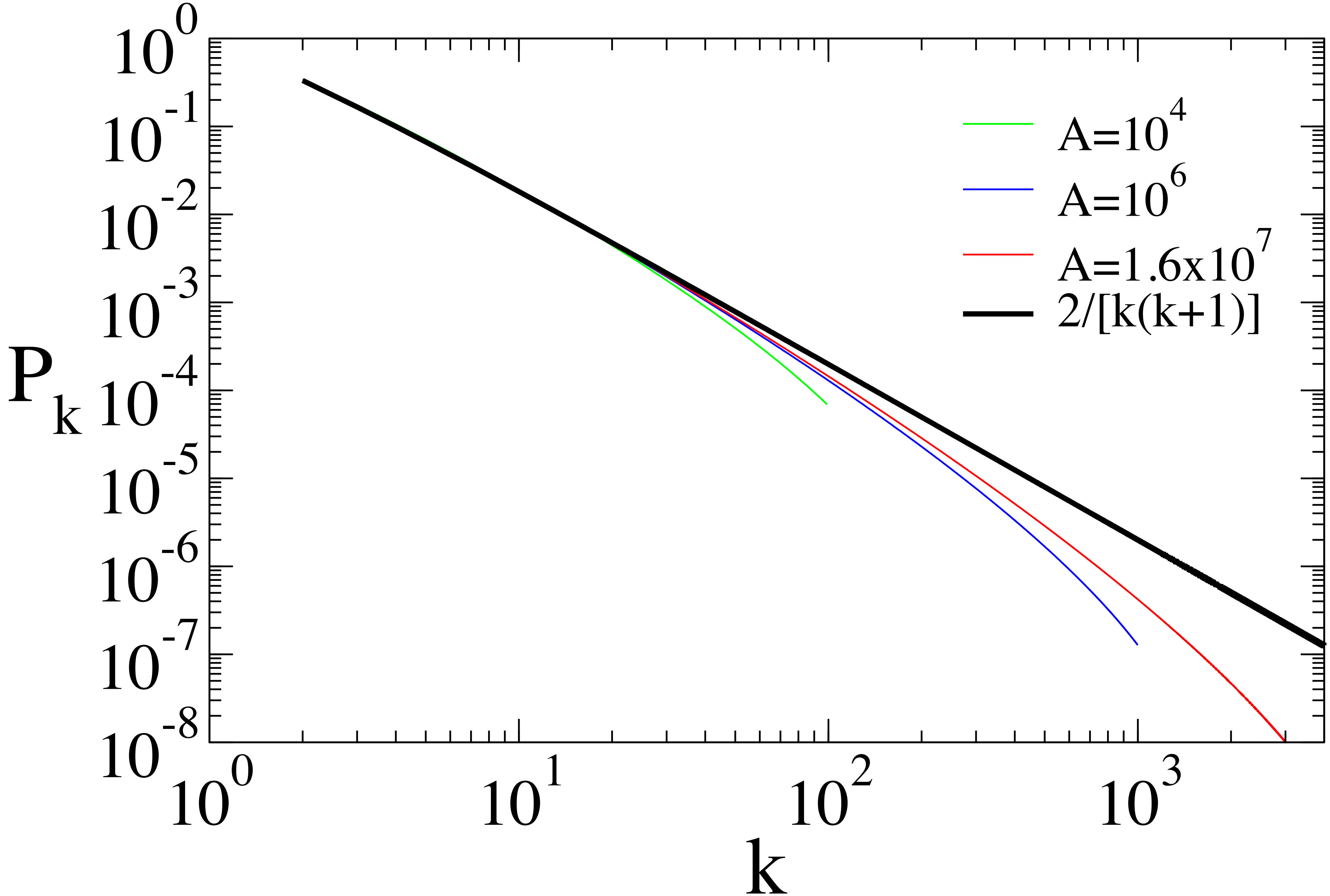
$$P_k \simeq 2k^{-2}\, \exp\left[-\frac{(\ln k)^2}{2\ln A}\right]$$

- Infinite-area limit: exact result

$$P_k = \frac{2}{k(k+1)}$$

Below average length: power law tail
Above average length: log-normal decay

Numerical validation



perfect agreement for small length (within 0.1%)
convergence is very slow

Moments of length distribution

- Normalized moments

$$M_h = \frac{\langle k^h \rangle}{\langle k \rangle} \qquad \langle k^h \rangle = \sum_{k\geq 2} k^h P_k$$

- Multiscaling asymptotic behavior

$$M_h \sim A^{\mu(h)} \quad \text{with} \quad \mu(h) = \frac{(h-1)^2}{h}$$

- Different spectrum than continuum version

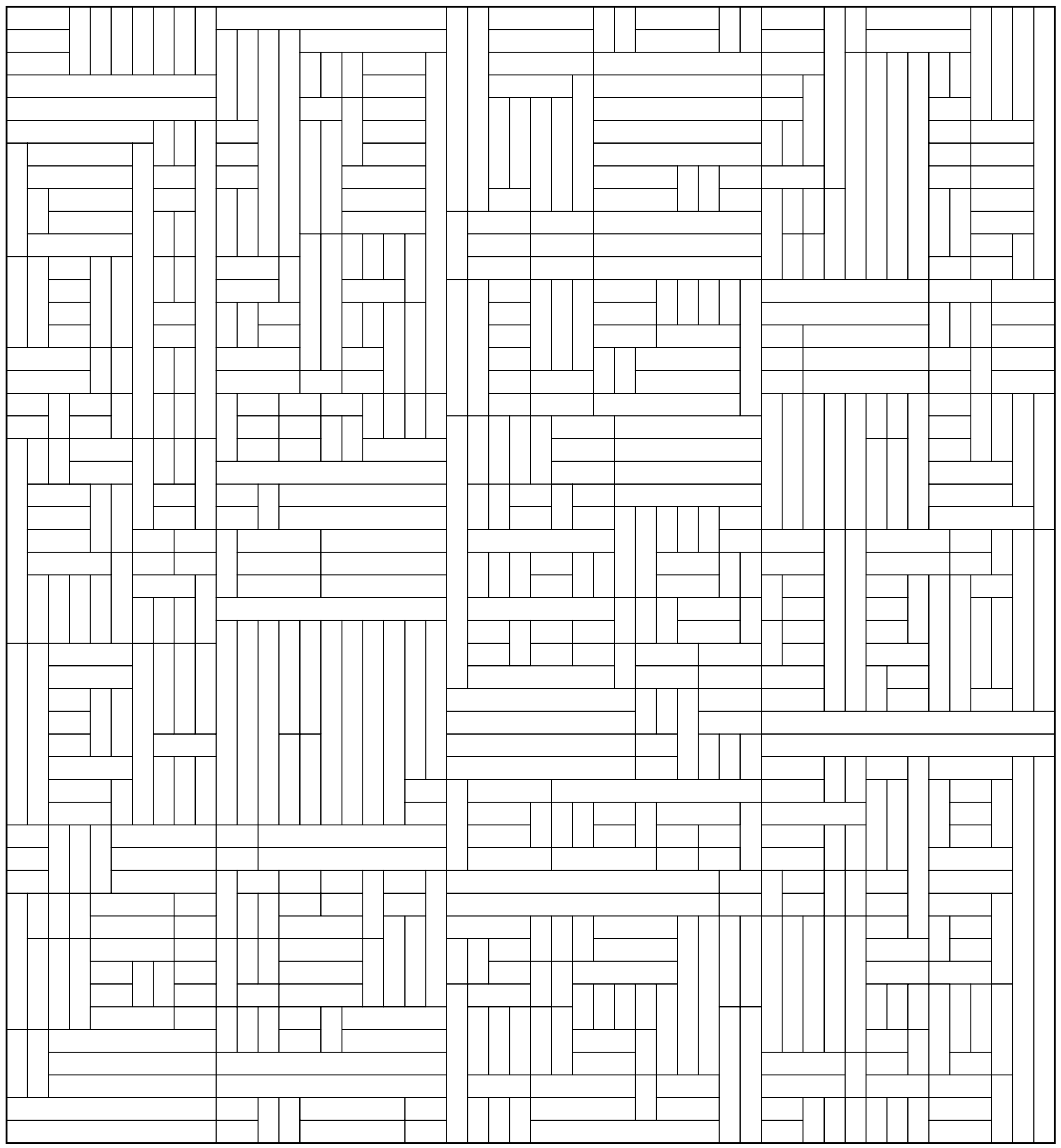
EB, Krapivsky 96

$$M_h \sim A^{\mu_{\rm nojam}(h)} \quad \text{with} \quad \mu_{\rm nojam}(h) = \sqrt{h^2+1} - \sqrt{2}$$

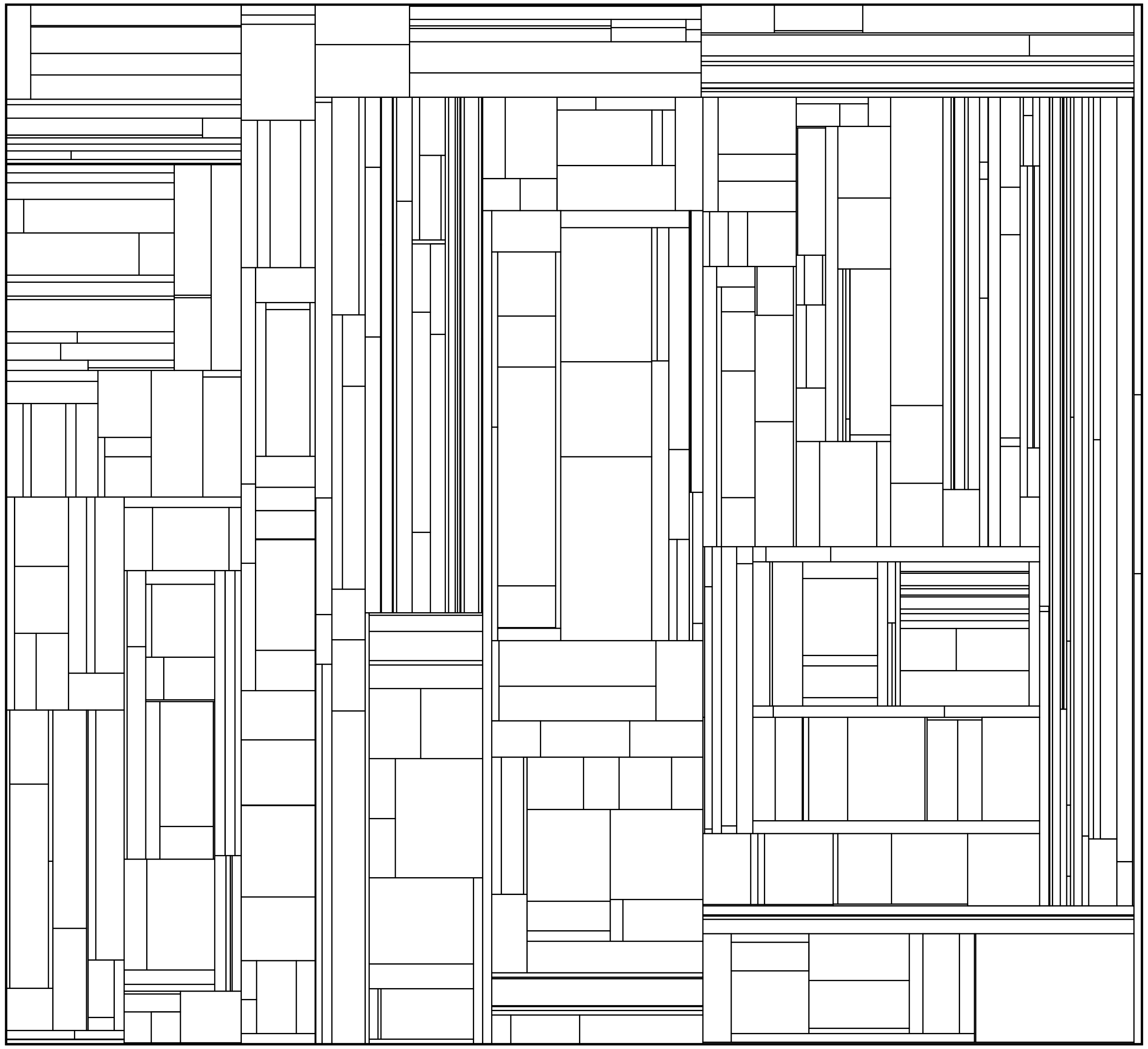
Nonlinear spectrum of scaling exponents
Discrete and continuous versions differ!!!

Discrete versus continuous fragmentation

discrete version
process stops



continuous version
process never stops



Asymmetric fragmentation

- Two fragmentation events realized with different probabilities

$$(m,n) \to \begin{cases} (i,n)+(m-i,n) & \text{with prob.} \quad (1-\alpha)/2 \\ (m,j)+(m,n-j) & \text{with prob.} \quad (1+\alpha)/2 \end{cases}$$

- Discrepancy between two extreme cases

$$S = \sqrt{A} \qquad \alpha = 1 \quad \text{(perfectly asymmetric)}$$

$$S \simeq A/\sqrt{2\pi \ln A} \qquad \alpha = 0 \quad \text{(perfectly symmetric)}$$

- Strongly asymmetric phase: purely power law

$$S \sim A^{\sqrt{1-\alpha^2}} \qquad \alpha > \frac{1}{\sqrt{2}}$$

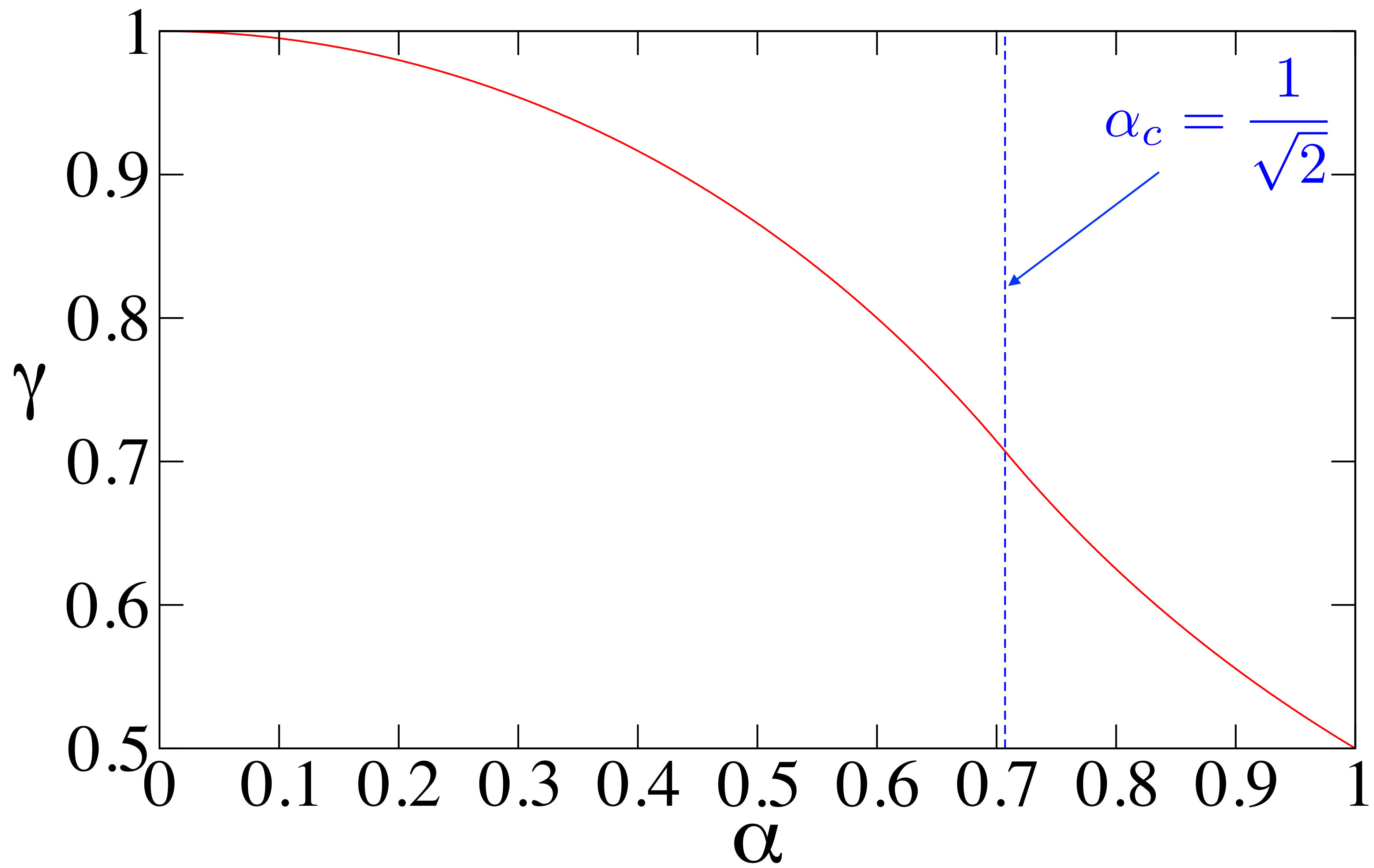
- Weakly asymmetric phase: power law + logarithmic correction

$$S \sim (\ln A)^{-1/2} A^{1/(2\alpha)} \qquad \alpha < \frac{1}{\sqrt{2}}$$

Phase transition at finite asymmetry strength

The growth exponent

$$S \sim A^{\gamma} \quad \text{with} \quad \gamma = \begin{cases} \sqrt{1-\alpha^2} & \alpha \le \alpha_c \\ 1/(2\alpha) & \alpha \ge \alpha_c \end{cases}$$

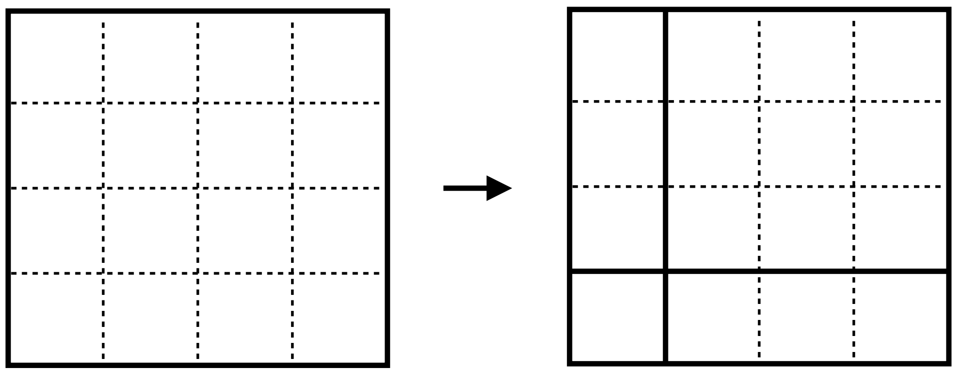


Sub-linear growth with area

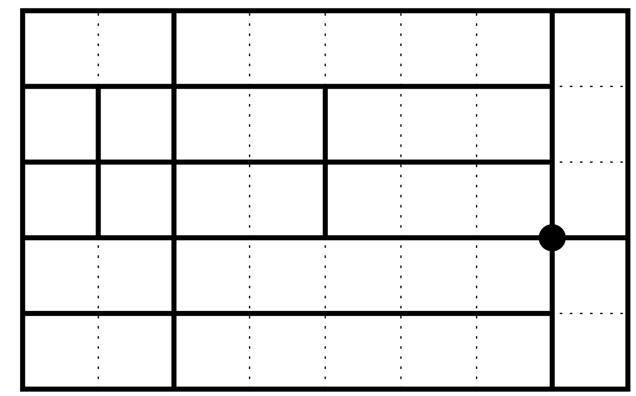
Growth exponent has two distinct forms

Number of jammed configurations

"deterministic" fragmentation into four rectangles



first fragmentation point can be uniquely identified



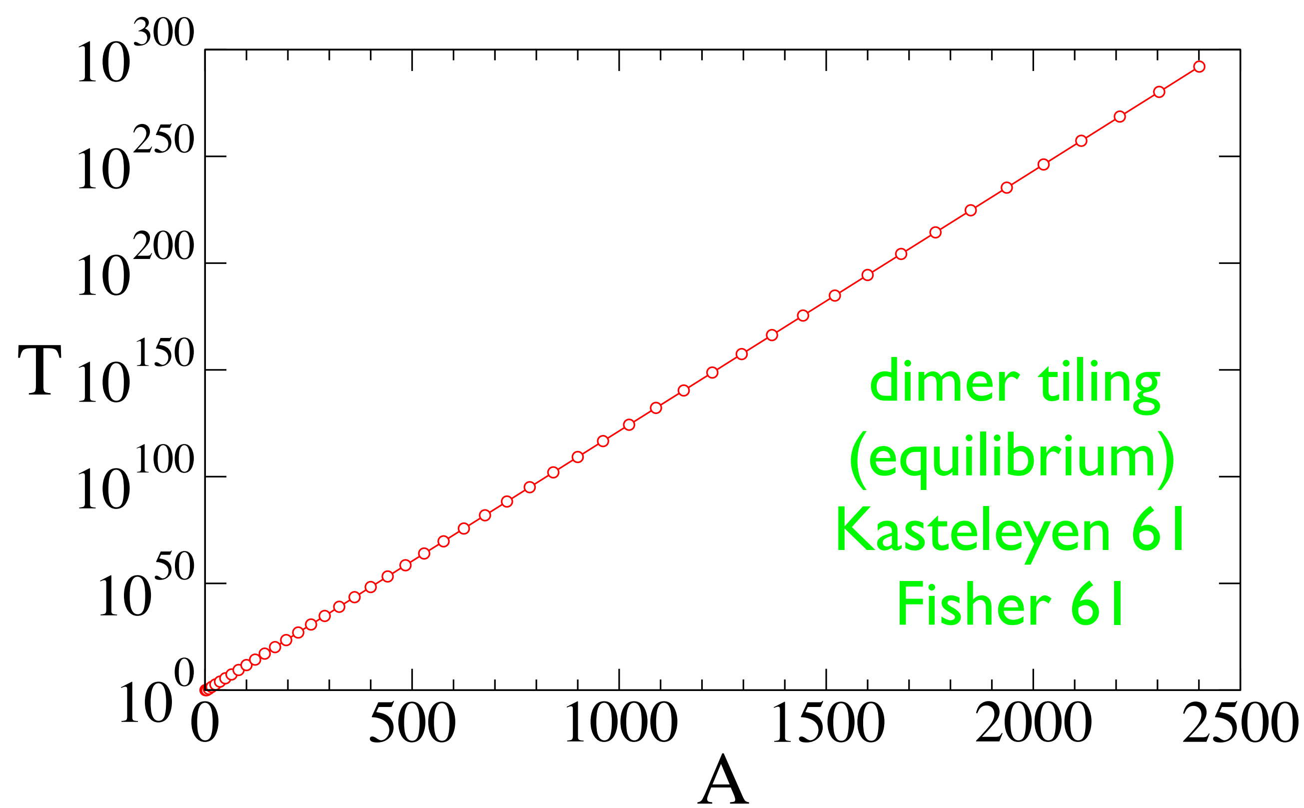
recursion equation for the total number of jammed states

$$T(m,n) = \sum_{\substack{1\leq i\leq m-1 \\ 1\leq j\leq n-1}} T(i,j)T(m-i,j)T(i,n-j)T(m-i,n-j)$$

exponential growth with area

$$T \sim e^{\lambda A}$$

$$\lambda = 0.2805$$



Aspect ratio dependence?

Conclusions I

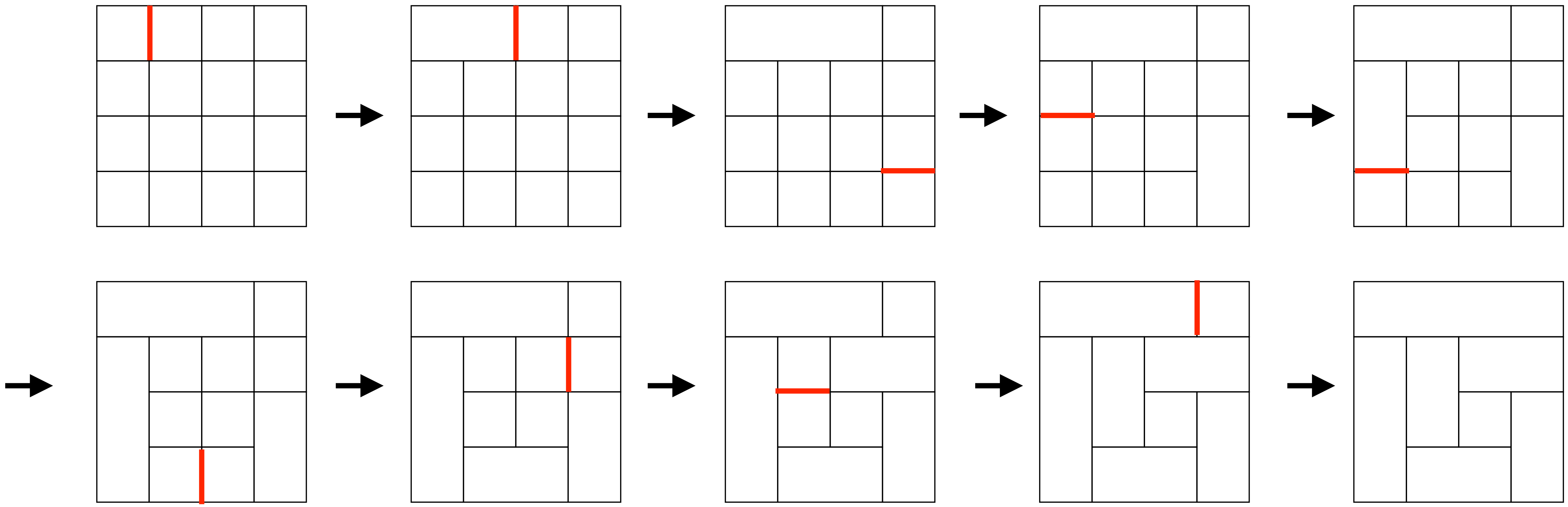
- Random fragmentation of rectangles
- Process reaches a jammed state where all rectangles are sticks
- Recursion equations give statistical property of jammed state
- Number of jammed sticks is independent of aspect ratio
- Distribution of stick length decays as a power law
- Multiscaling: nonlinear spectrum of exponents for moments
- Asymmetric fragmentation: phase transition for growth exponent
- Generally, number of sticks grows sub-linearly with area
- Number of jammed states grows exponentially with area
- Abundance of exact analytic results

Aggregation of rectangles

Start with a perfect grid

Pick two neighboring rectangles at random

Merge the two if compatible



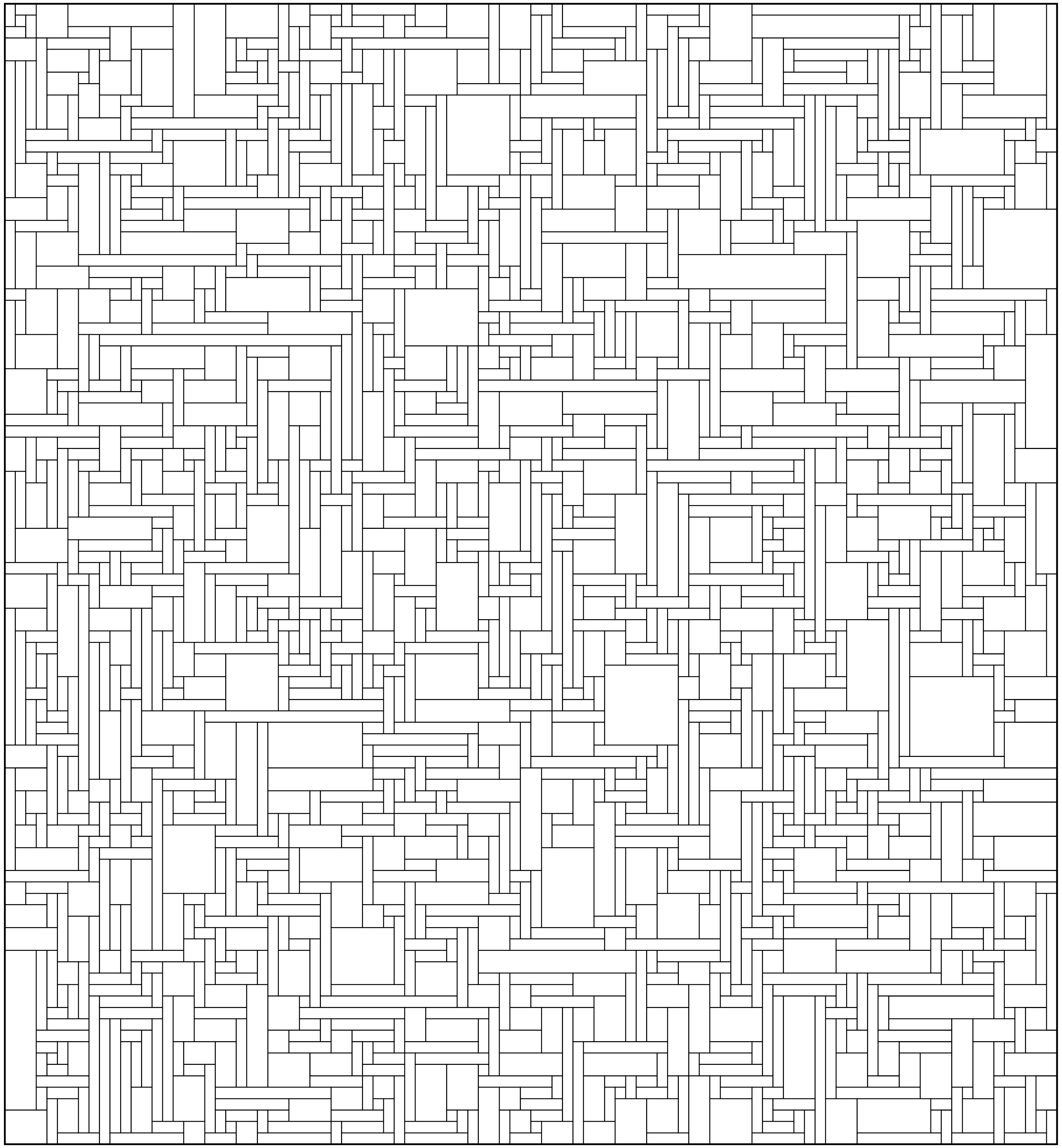
System reaches a jammed state

No two neighboring rectangles are compatible

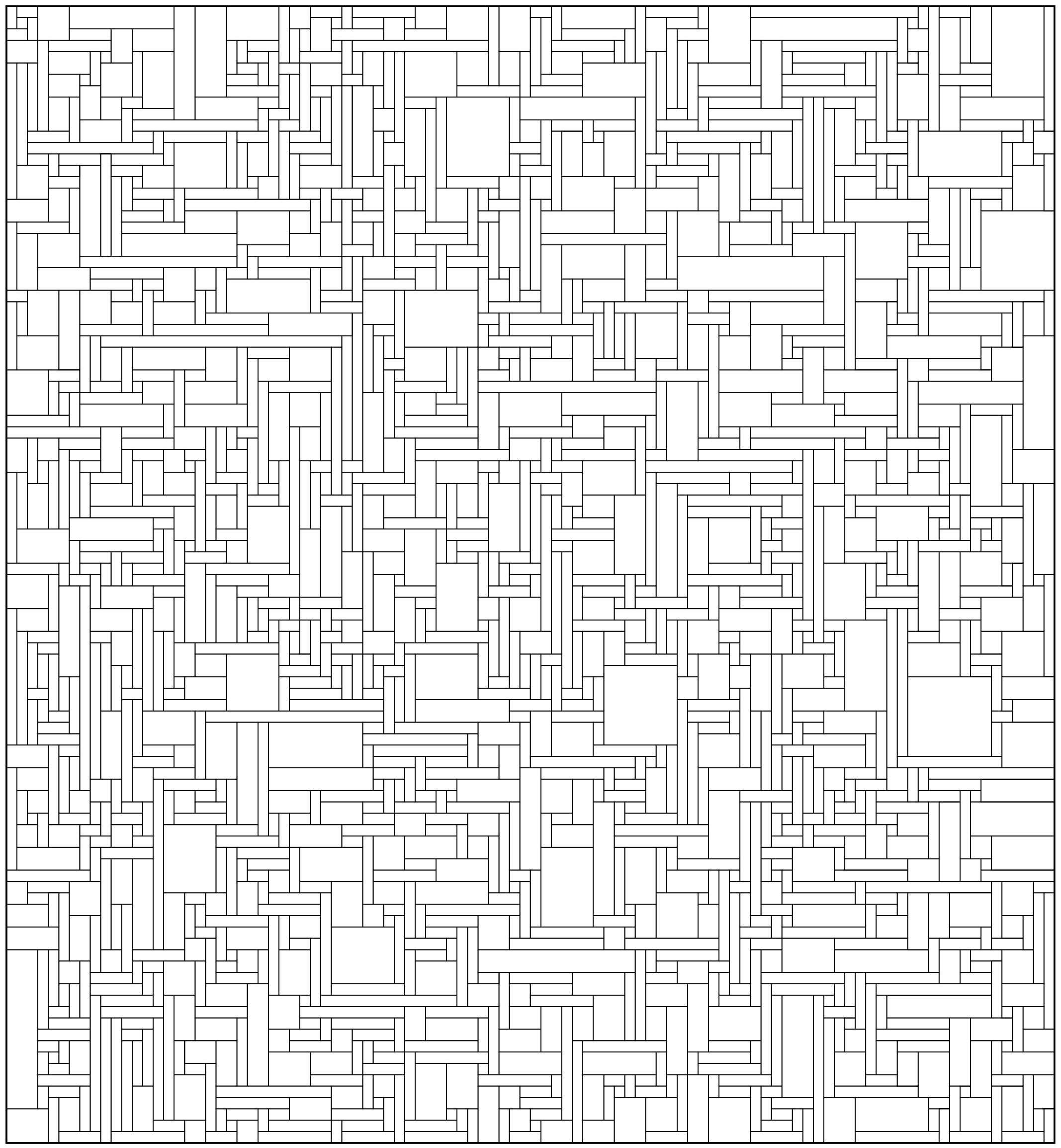
The jammed state

no two neighbors share a common side

Tiling is:
- Polydisperse
- Dynamical



Features of the jammed state



- Local alignment
- Motifs
- Finite rectangle density

$$\rho = 0.1803$$

- Finite tile density

$$T = 0.009949$$

- Finite stick density

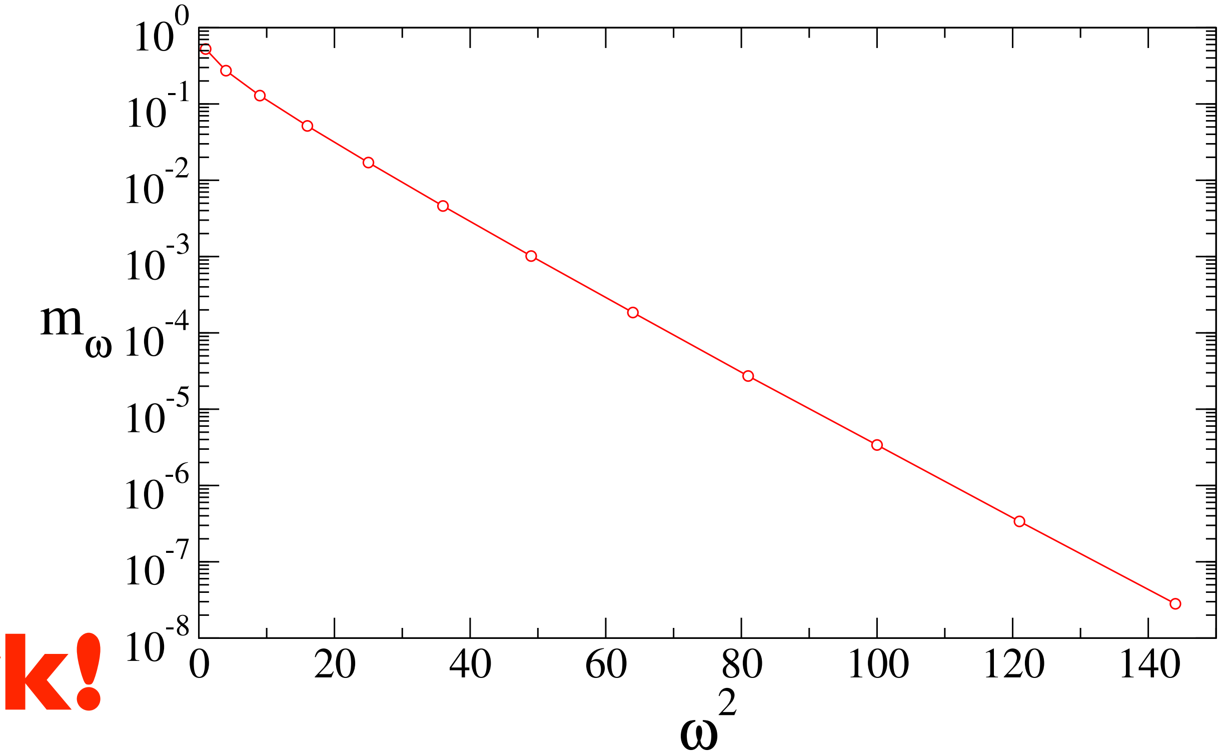
$$S = 0.1322$$

- Finite square density

$$H = 0.02306$$

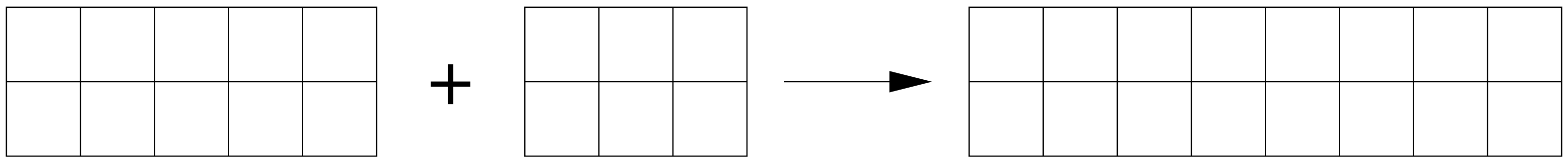
- Area distribution of rectangles with width w

$$m_\omega \sim \exp\left(-\text{const.} \times \omega^2\right)$$



No theoretical framework!

Mean-field fragmentation process



- Start with N 1x1 tiles (elementary building blocks)
- Pick two rectangles completely at random
- Pick an orientation at random (vertical or horizontal)
- Merge rectangles if they are perfectly compatible

$$(i_1, j) + (i_2, j) \to (i_1 + i_2, j)$$
$$(i, j_1) + (i, j_2) \to (i, j_1 + j_2)$$

- System is jammed when f rectangles have:

f distinct horizontal sizes and f distinct vertical sizes

System reaches a jammed state

An example of a jammed state

- Characterize rectangle by horizontal and vertical size

$$(i, j)$$

- Characterize rectangle by maximal and minimal size

$$(\omega, \ell) \qquad \omega = \min(i, j) \qquad \ell = \max(i, j)$$

- Example of a jammed state for N=10,000

$$1 \times 3144, 2 \times 498, 3 \times 113, 4 \times 45, 5 \times 6, 6 \times 14, 9 \times 12$$

$$3237 \times 1, 475 \times 2, 61 \times 3, 14 \times 4, 48 \times 5, 29 \times 7, 25 \times 10$$

- Ordered widths of f=14 rectangles

$$\{1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 7, 9, 10\}$$

Width sequence has gaps

Number of jammed rectangles

- Average number of rectangles grows algebraically with N

$$F \sim N^{\alpha}$$

- Nontrivial exponent

$$\alpha = 0.229 \pm 0.002$$

- Typical width of rectangles grows algebraically with N

$$\omega \sim N^{\alpha}$$

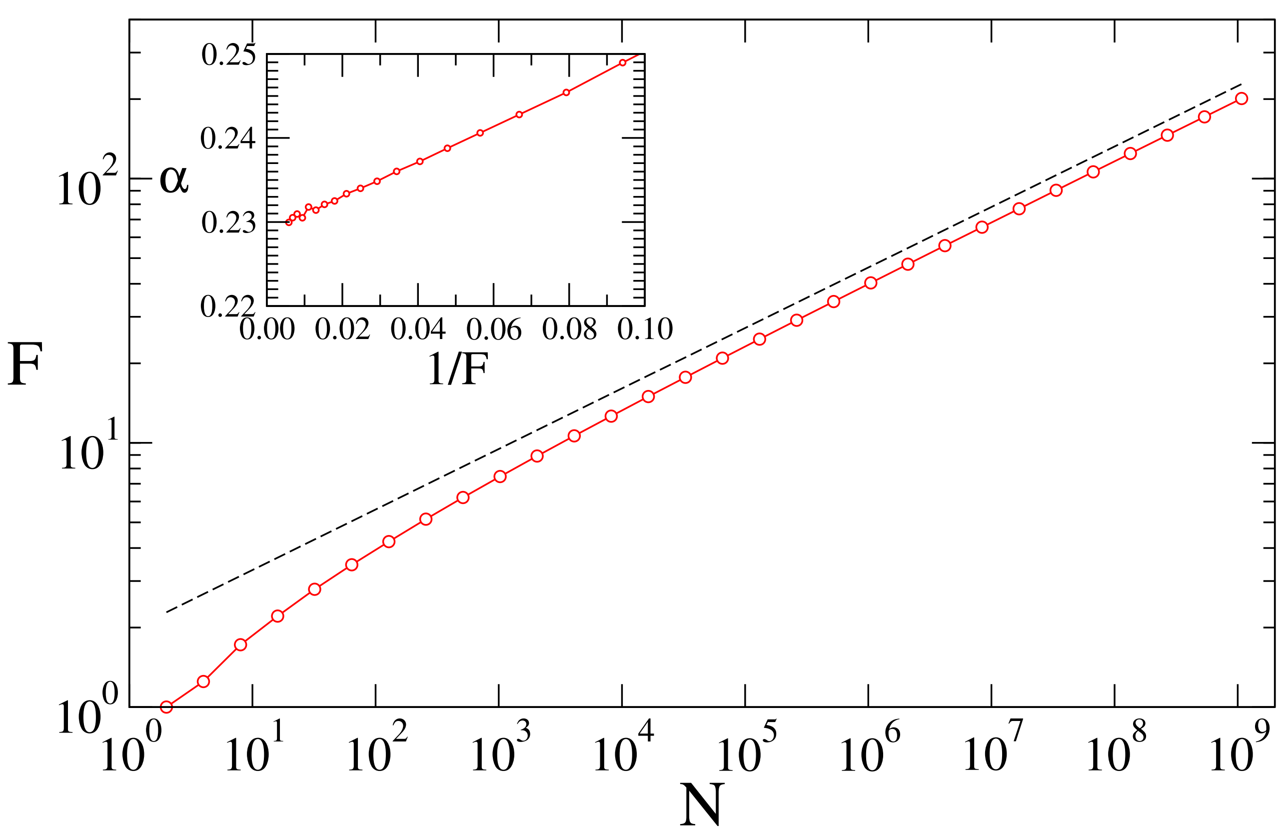
- Area density of rectangles of width w decays as a power law

$$m_{\omega} \sim \omega^{-\gamma} \quad \text{with} \quad \gamma = \alpha^{-1} - 2$$

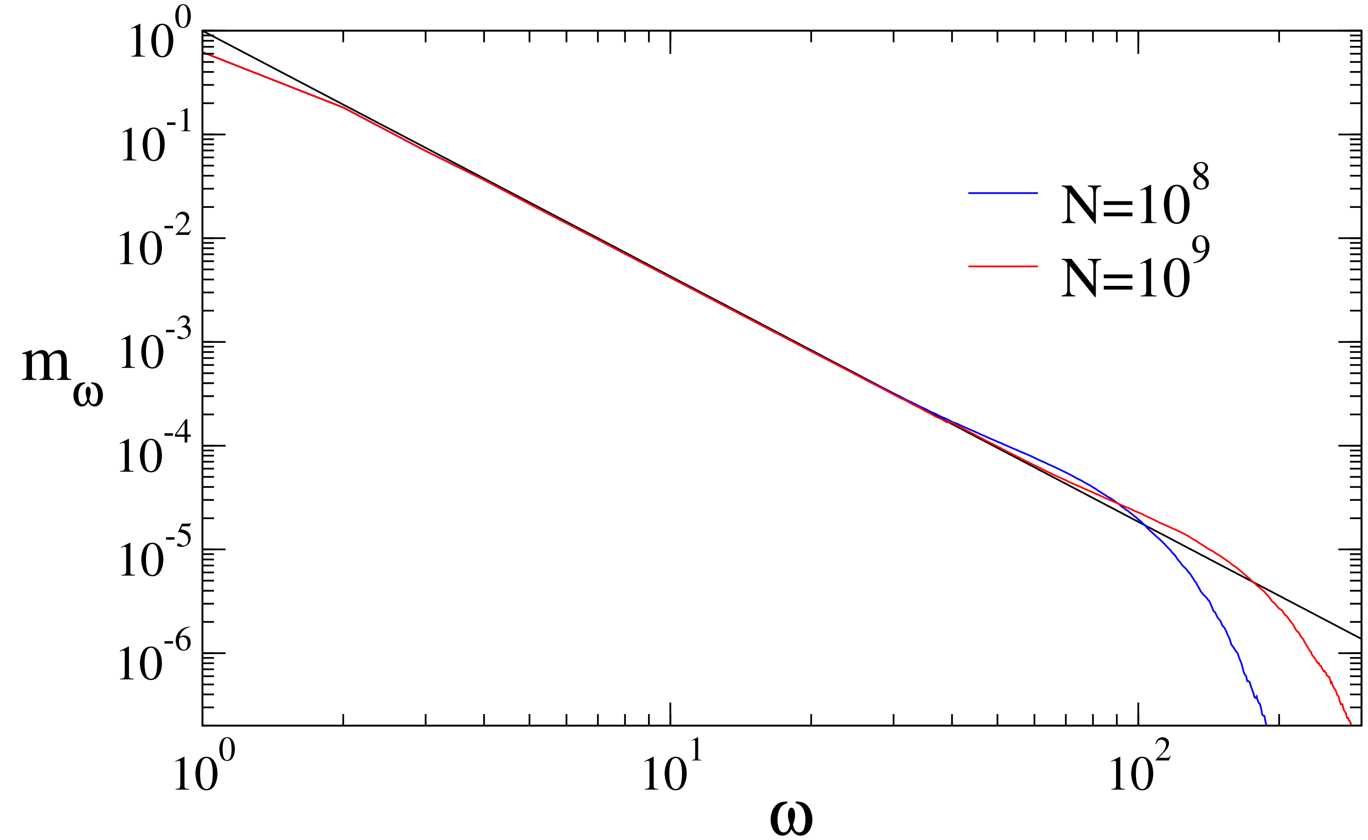
A single exponent characterizes the jammed state

Numerical simulations

$F \sim N^{\alpha}$



$m_\omega \sim \omega^{-\gamma}$



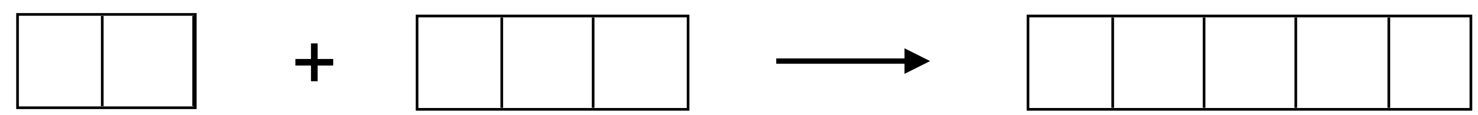
ω	1	2	3	4	5	6
m_ω	0.622	0.182	0.0694	0.0365	0.0214	0.0139
M_ω	0.622	0.804	0.873	0.910	0.931	0.945

Rectangles with finite width are macroscopic!
Rectangles of width 1,2,3,4,5 contain 95% of total area
Still, the area distribution has a broad power-law tail!

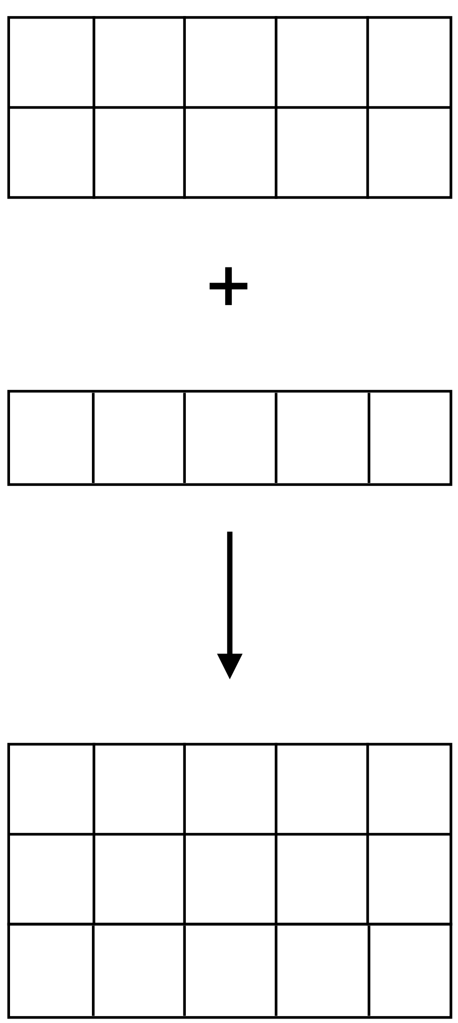
Two aggregation modes: fast and slow
two length scales

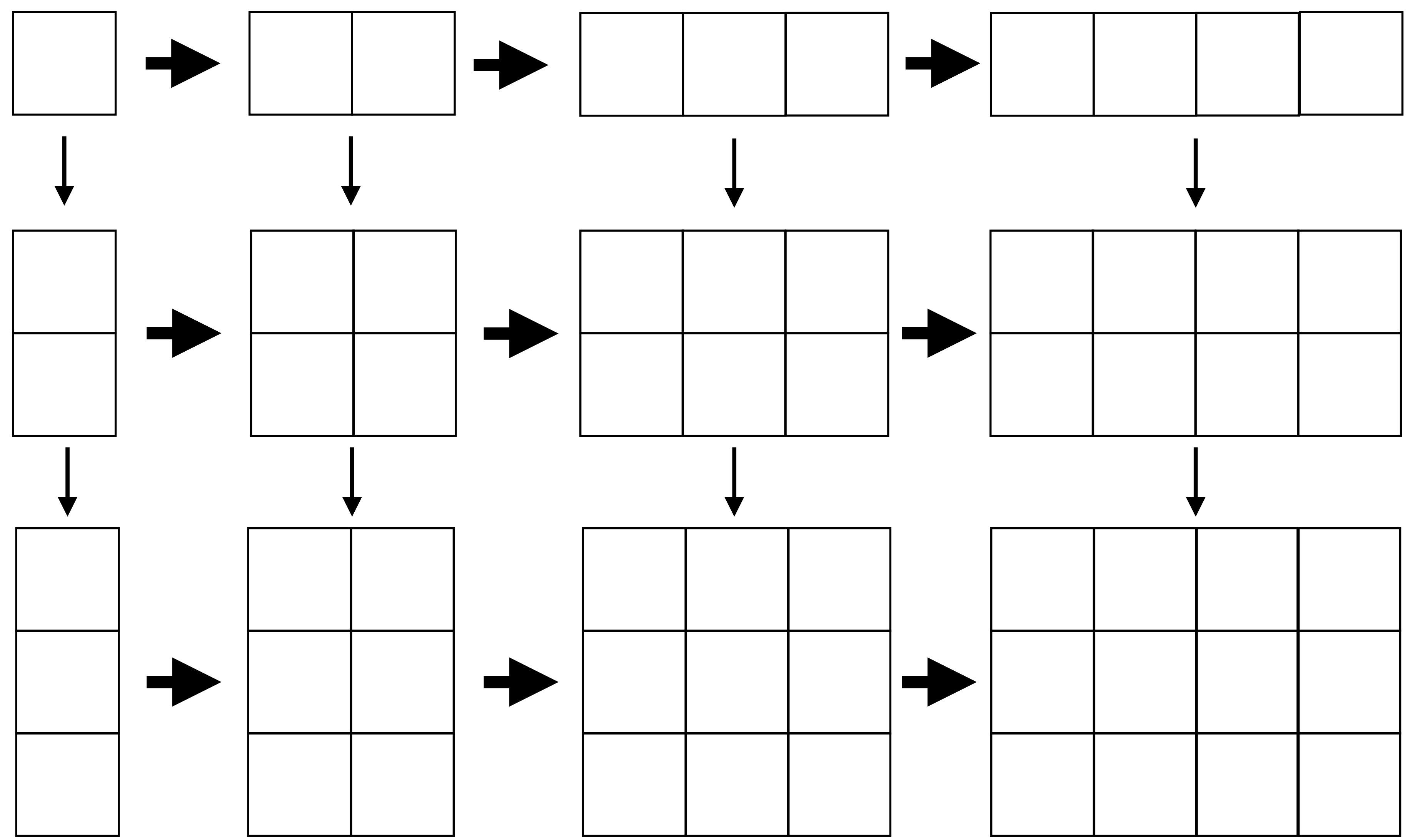
$$\ell \sim t \qquad w \sim t^{\alpha} \qquad \alpha = 0.229 \pm 0.001$$

elongating: aspect ratio increases



widening: aspect ratio decreases





Kinetic theory

1D
Smoluchowski 1917

- Straightforward generalization of ordinary aggregation

$$\frac{dR_{i,j}}{dt} = \sum_{i_1+i_2=i} R_{i_1,j}R_{i_2,j} - 2R_{i,j}\sum_{k\geq 1} R_{k,j} + \sum_{j_1+j_2=j} R_{i,j_1}R_{i,j_2} - 2R_{i,j}\sum_{k\geq 1} R_{i,k}$$

- Allows calculation of the density of sticks

$$\frac{dS}{dt} = -S^2 - 2\sum_{i,j} R_{1,j}R_{i,j}$$

- Simple decay for the stick density and jamming time

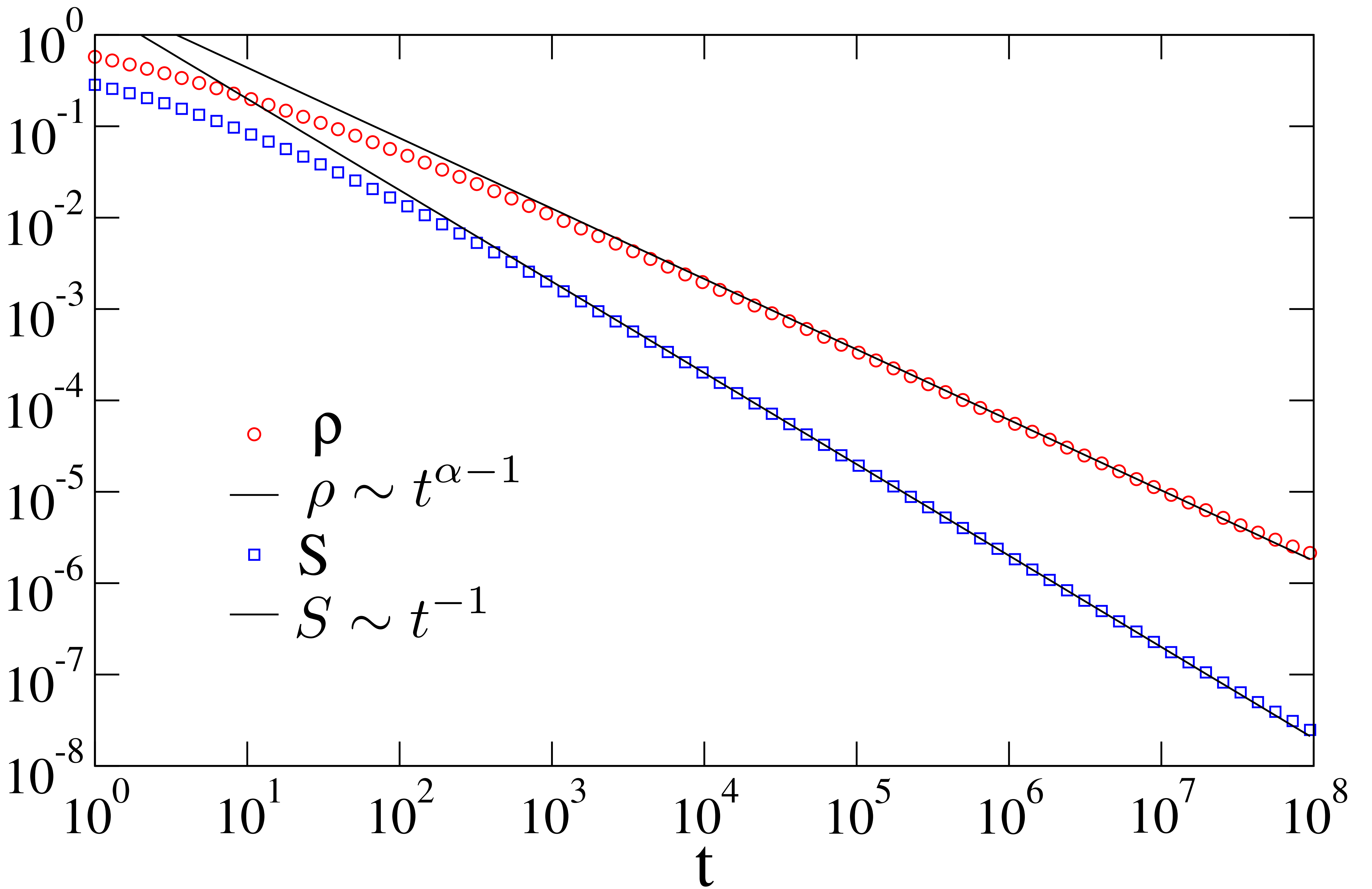
$$S \simeq t^{-1} \quad \Longrightarrow \quad \tau \sim N$$

- Jammed state properties give density decay and width growth

$$\rho \sim t^{\alpha-1} \qquad \text{and} \qquad w \sim t^{\alpha}$$

Jamming exponent characterizes the kinetics, too

Numerical validation

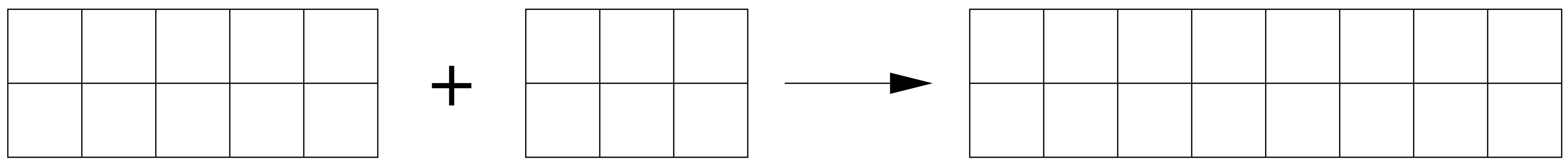


Numerics validate approximation

Suggest two aggregation modes: elongating and widening

Primary aggregation: elongation

- Aggregation between two rectangles of same **width**



aspect ratio increases

- Ordinary aggregation equation (example: sticks)

$$\frac{dR_{1,\ell}}{dt} = \sum_{i+j=\ell} R_{1,i}R_{1,j} - 2SR_{1,\ell} - 2\left(\sum_i R_{i,\ell}\right)R_{1,\ell}$$

- Length distribution as in $d=1$, length grows linearly $l \sim t$

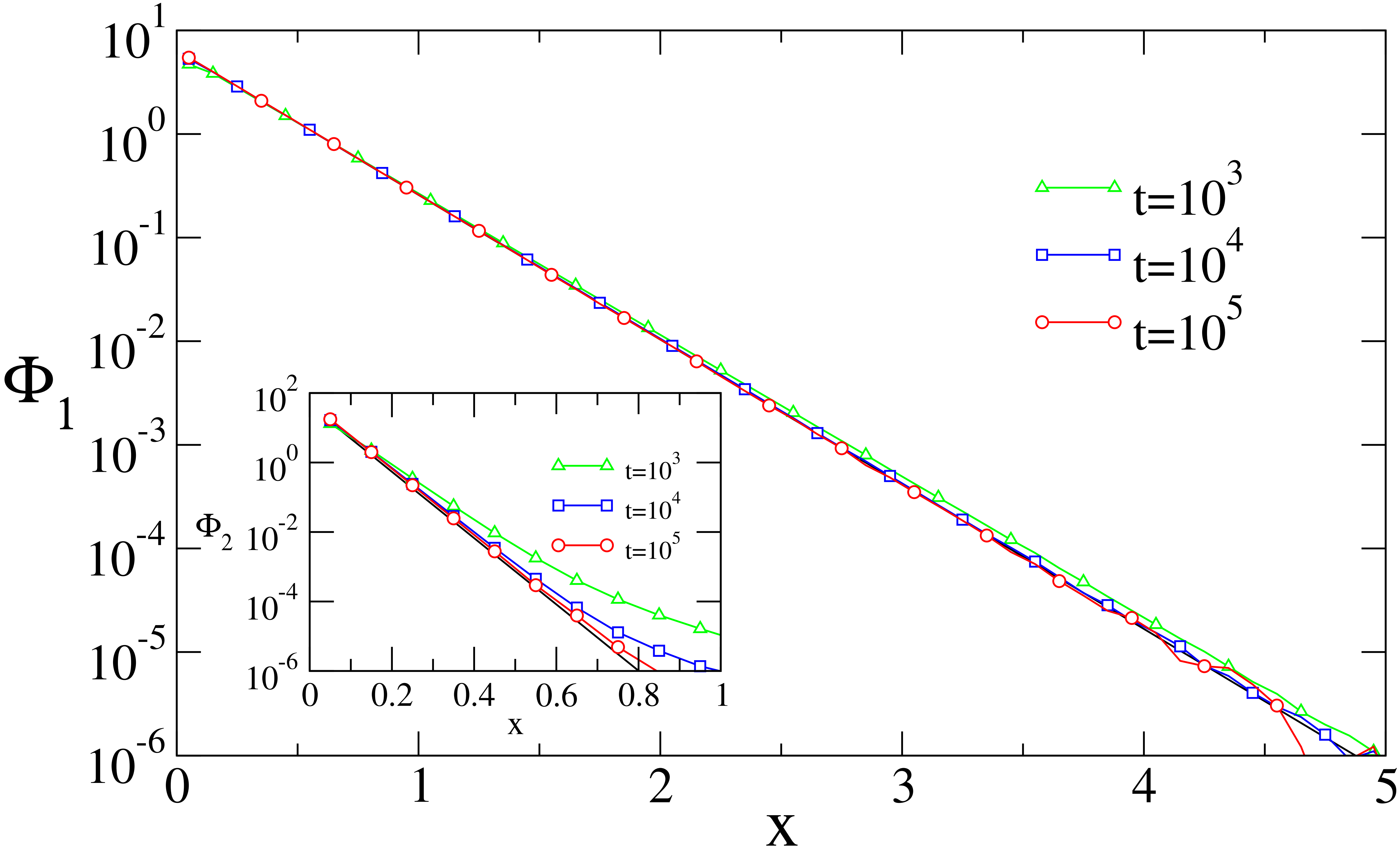
$$R_{1,\ell} \simeq (2/m_1 t^2)\exp(-2\ell/m_1\, t)$$

- Behavior extends to all rectangles with finite width

$$\mathcal{R}_{\omega,\ell}(t) \simeq t^{-2}\,\Phi_\omega\left(\ell\, t^{-1}\right) \quad \text{with} \quad \Phi_\omega(x) = (2\omega/m_\omega)\exp(-2\omega x/m_\omega)$$

Finite width: problem reduces to one-dimensional aggregation
However, total mass for each width is not known

Numerical validation

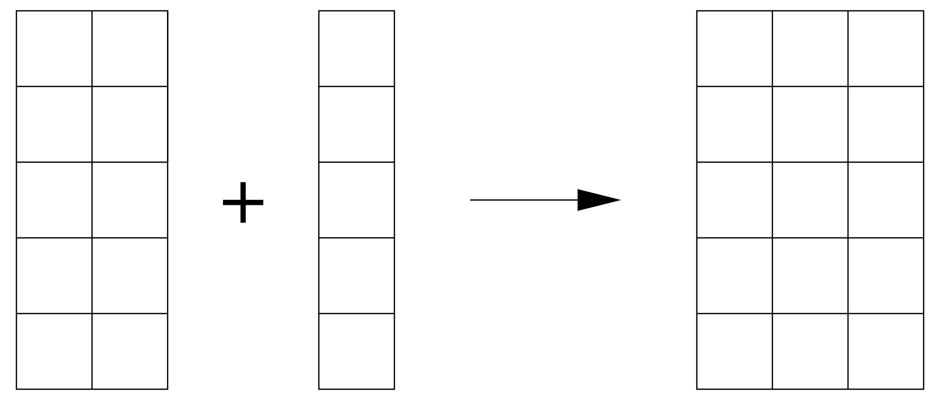


Exponential scaling function

Total mass set by the jammed state

Secondary aggregation: widening

- Aggregation between two rectangles of same **length**



aspect ratio decreases

- The area fraction is coupled to the size distribution

$$\frac{dm_\omega}{dt} = \frac{1}{2} \sum_{i+j=\omega} \sum_{\ell} \omega \ell \mathcal{R}_{i,\ell} \mathcal{R}_{j,\ell} - \sum_{j} \sum_{\ell} \omega \ell \mathcal{R}_{j,\ell} \mathcal{R}_{\omega,\ell}$$

- Insights about relaxation toward jammed state $\mu_\omega = \frac{2\omega}{m_\omega}$

$$m_\omega(t) - m_\omega(\infty) \simeq C_\omega\, t^{-1} \quad \text{with} \quad C_\omega = -2\omega \sum_{i+j=\omega} \frac{\mu_i \mu_j}{(\mu_i + \mu_j)^2} + 4\omega \sum_j \frac{\mu_\omega \mu_j}{(\mu_\omega + \mu_j)^2}$$

Closure & theoretical determination of α remains elusive

Conclusions II

- Random aggregation of compatible rectangles
- Process reaches a jammed state where all rectangles are incompatible
- Number of jammed rectangles grows as a power-law
- Area distribution decays as a power law
- A single, nontrivial, exponent characterize both the jammed state and the time-dependent behavior
- Primary aggregation: rectangles of same width
- Secondary aggregation: rectangles of same length
- Slow transfer of "mass" from thin to wide rectangles
- Kinetic theory successfully describes primary aggregation process only