Energy Cascades in Granular Matter

Eli Ben-Naim

Theory Division

Los Alamos National Laboratory

with: Jon Machta (Massachusetts)

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http://cnls.lanl.gov/~ebn

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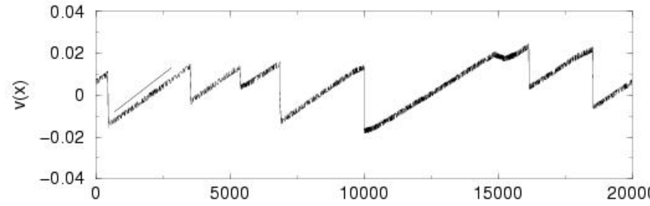
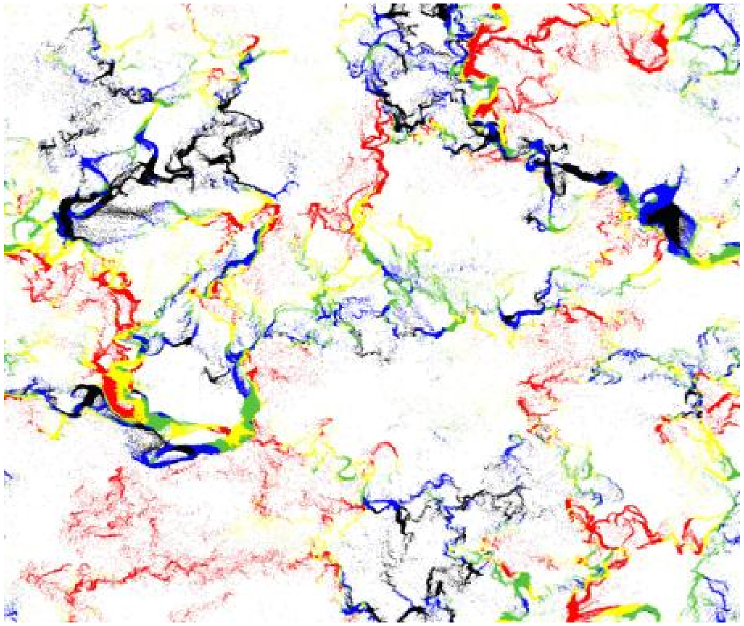
Energy dissipation in granular media

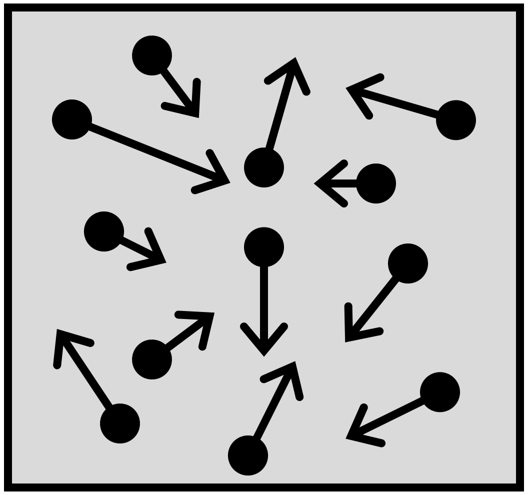
- Responsible for collective phenomena
  - » Clustering
  - » Hydrodynamic instabilities
  - » Shocks
  - » Pattern formation
- Anomalous statistical mechanics:
  - ➢No energy equipartition
  - ➢Nonequilibrium distributions





$$P(E) \neq \exp(-E/kT)$$

Inelastic gas



- Vigorous driving
- Spatially uniform system
- Particles undergo binary collisions
- Velocity changes due to
  1. Inelastic collisions (lose energy)
  2. Energy input (gain energy)
- What is the typical velocity (granular “temperature”)?

$$T = \left\langle v^2 \right\rangle$$

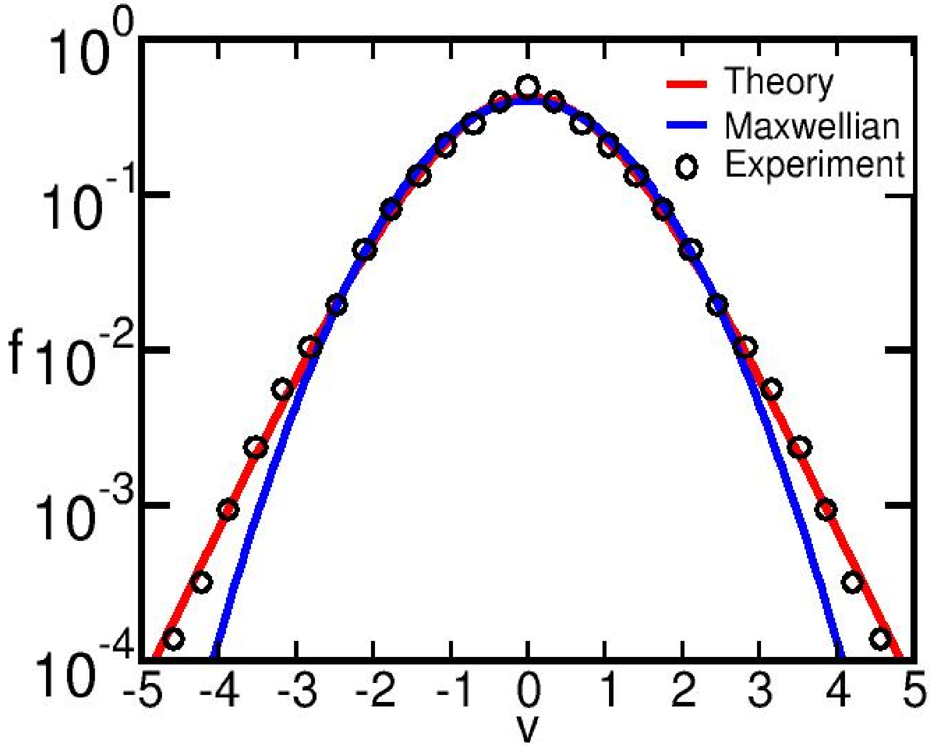
- What is the velocity distribution?

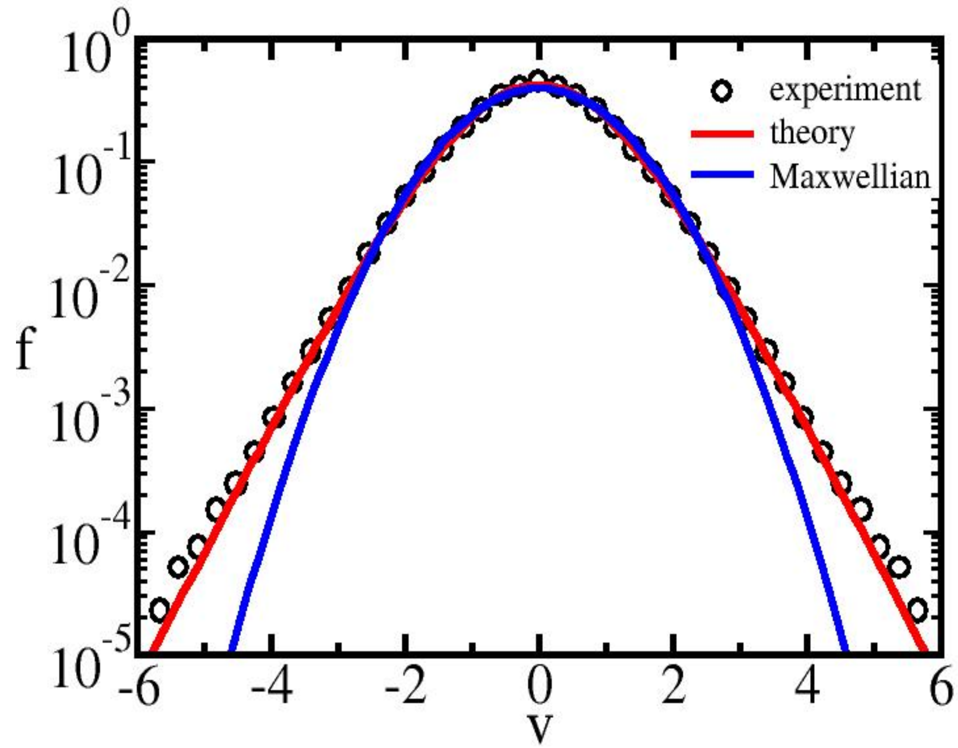
$$f(v)$$

Nonequilibrium velocity distributions

$$f(\mathrm{v}) \sim \exp\left(-|\mathrm{v}|^{\delta}\right) \quad 1 \le \delta \le 3/2$$

$$\left\langle \mathrm{v}^4 \right\rangle \Big/ \left\langle \mathrm{v}^2 \right\rangle^2 = {}^{117}\!/_{33} \cong 3.55$$





Mechanically vibrated beads

Rouyer & Menon 2000

Electrostatically driven powders

Aronson & Olafsen 2002

Theory: ebn & krapivsky 2002

Inelastic Collisions

- **Relative velocity reduced by 0<r<1**

$$\mathrm{v}_1 - \mathrm{v}_2 = -r(u_1 - u_2)$$

- **Momentum is conserved**

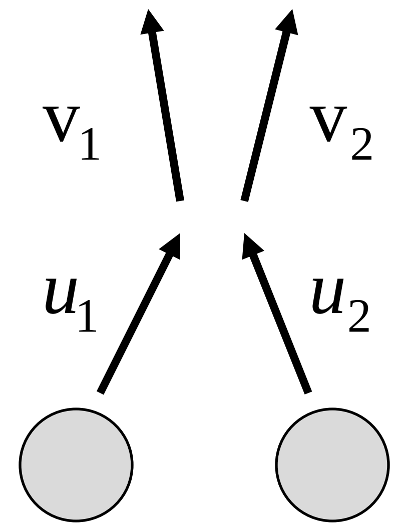
$$\mathrm{v}_1 + \mathrm{v}_2 = u_1 + u_2$$

- **Energy is dissipated**

$$\Delta E \propto -(1-r)(\Delta \mathrm{v})^2$$

- **Limiting cases**

$$\mathrm{r} = \begin{cases} 0 & \text{completely inelastic } (\Delta \mathrm{E} = \max) \\ 1 & \text{elastic } (\Delta E = 0) \end{cases}$$



Freely decaying states

- **Energy loss in a collision** $\Delta T = \langle v^2 \rangle$
- **Collision rate** $\Delta t \sim 1/(\Delta v)^\lambda$
- **Energy balance equation**

$$\frac{\Delta T}{\Delta t} \sim -(\Delta v)^{2+\lambda} \quad \Rightarrow \quad \frac{dT}{dt} \sim -T^{1+\lambda/2}$$

- **Temperature decays, system comes to rest**

$$T \sim t^{-2/\lambda} \quad \Rightarrow \quad P(v) \to \delta(v)$$

Trivial steady-state

Haff, JFM 1982

Kinetic Theory

- **Collision rule (linear)**

$$(u_1, u_2) \to (pu_1 + qu_2, qu_1 + pu_2) \qquad r = 1 - 2p$$

- **Boltzmann equation**

$$\frac{\partial P(\mathrm{v})}{\partial t} = \iint du_1 du_2 P(u_1) P(u_2) \underbrace{|u_1 - u_2|^{\lambda}}_{\text{collision rate}} [\underbrace{\delta(\mathrm{v} - pu_1 - qu_2)}_{\text{gain}} - \underbrace{\delta(\mathrm{v} - u_2)}_{\text{loss}}]$$

- **Collision rate related to interaction potential**

$$U(r) \sim r^{-\gamma} \qquad \lambda = 1 - 2\frac{d-1}{\gamma} = \begin{cases} 0 & \text{Maxwell molecules } (\alpha = 2, \text{2D}) \\ 1 & \text{Hard spheres } (\alpha = \infty) \end{cases}$$

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Are there nontrivial steady states?

An exact solution

- **One-dimensional Maxwell molecules**
- **Fourier transform obeys a closed equation**

$$F(k) = F(pk)F(k - pk) \qquad F(k) = \int d\mathrm{v}\, e^{ik\mathrm{v}} f(\mathrm{v})$$

- **Exponential solution**

$$F(k) = \exp\left(-|k|\mathrm{v}_0\right)$$

- **Lorentzian velocity distribution**

$$f(\mathrm{v}) = \frac{1}{\pi \mathrm{v}_0} \frac{1}{1 + \left(\mathrm{v}/\mathrm{v}_0\right)^2}$$

Nontrivial steady states do exist

Properties of stationary state

- Perfect balance between collisional loss and gain
- Power-law high-energy tail

$$P(v) \sim v^{-\sigma} \qquad \sigma = 2$$

- Infinite energy, infinite dissipation!

Is this stationary state physical?

Cascade Dynamics (1D)

- **Collision rule: arbitrary velocities**

$$(u_1, u_2) \to (pu_1 + qu_2, pu_2 + qu_1)$$

- **Large velocities cascade**

$$\mathrm{v} \to (p\mathrm{v}, q\mathrm{v})$$

- **High-energies: linearized equation**

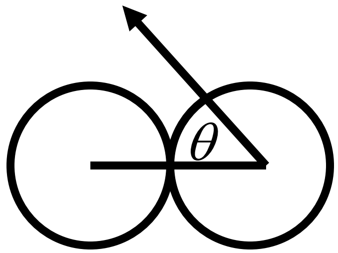
$$\frac{1}{p^{1+\lambda}} f\left(\frac{\mathrm{v}}{p}\right) + \frac{1}{q^{1+\lambda}} f\left(\frac{\mathrm{v}}{q}\right) - f(\mathrm{v}) = 0$$

- **Power-law tail**

$$f(\mathrm{v}) \sim \mathrm{v}^{-\sigma} \qquad \sigma = 2 + \lambda$$

Cascade Dynamics

- **Collision process: large velocities**



$$\mathrm{v} \to (\alpha \mathrm{v}, \beta \mathrm{v})$$

- **Stretching parameters related to impact angle**

$$\alpha = (1 - \mathrm{p})\cos\theta \qquad \beta = [1-(1-p^2)\cos^2\theta]^{1/2}$$

- **Energy decreases, velocity magnitude increases**

$$\alpha^2 + \beta^2 \le 1 \qquad \alpha + \beta \ge 1$$

- **Steady state equation**

$$\left\langle \left(\frac{1}{\alpha^{d+\lambda}} f\left(\frac{\mathrm{v}}{\alpha}\right) + \frac{1}{\beta^{d+\lambda}} f\left(\frac{\mathrm{v}}{\beta}\right) - f(\mathrm{v}) \right) \cos^{\lambda}\theta \right\rangle = 0$$

Power-laws are generic

- Velocity distributions always has power-law tail

$$f(\mathrm{v}) \sim \mathrm{v}^{-\sigma}$$

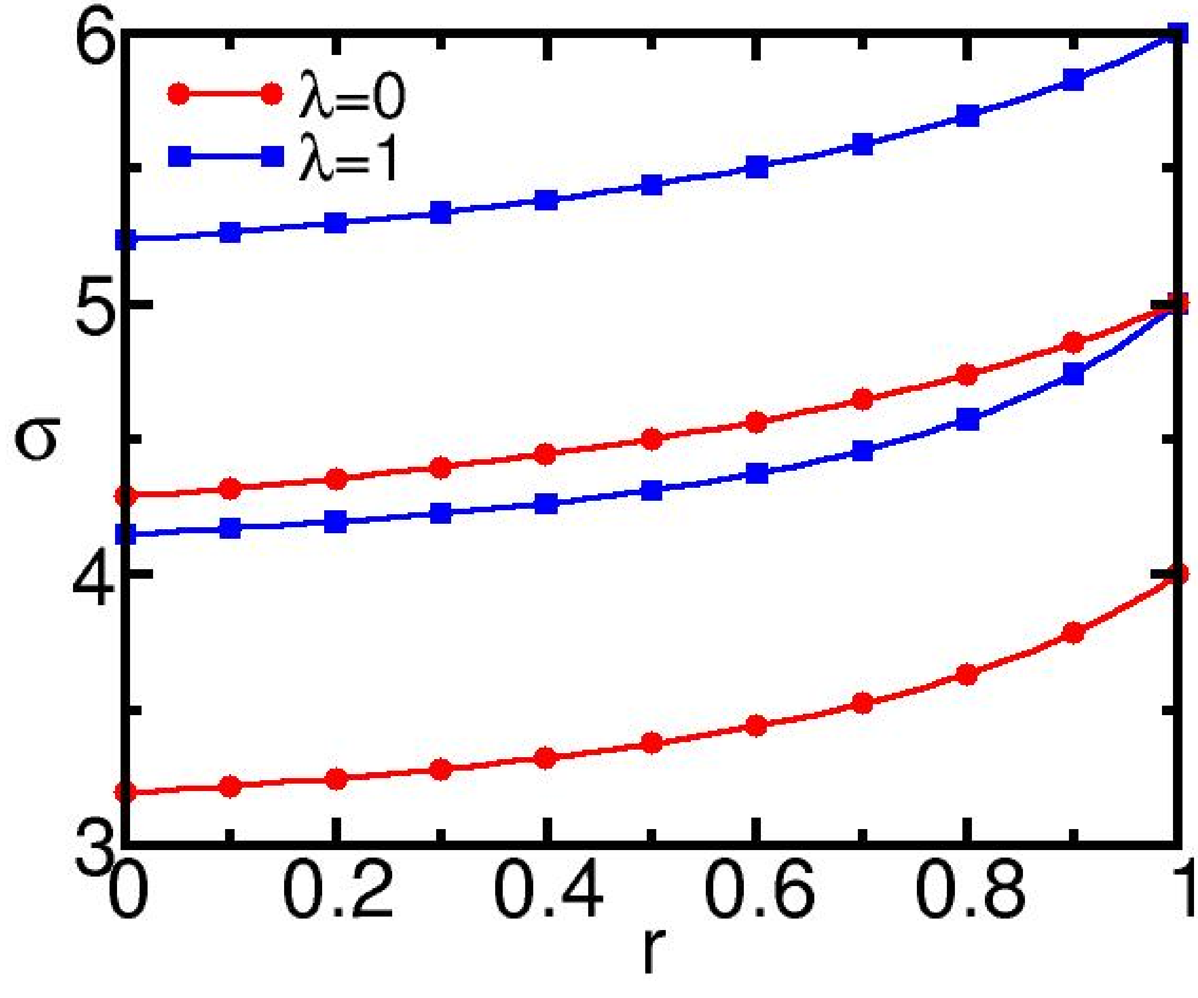
- Exponent varies with parameters

$$\frac{1-{}_1F_2\left(\frac{d+\lambda-\sigma}{2},\frac{\lambda+1}{2},\frac{d+\lambda}{2},1-p^2\right)}{(1-p)^{\sigma-d-\lambda}}=\frac{\Gamma\left(\frac{\sigma-d+1}{2}\right)\Gamma\left(\frac{d+\lambda}{2}\right)}{\Gamma\left(\frac{\sigma}{2}\right)\Gamma\left(\frac{\lambda+1}{2}\right)}$$

- Tight bounds $1 \le \sigma - d - \lambda \le 2$
- Elastic limit is singular $\sigma \to \mathrm{d} + 2 + \lambda$

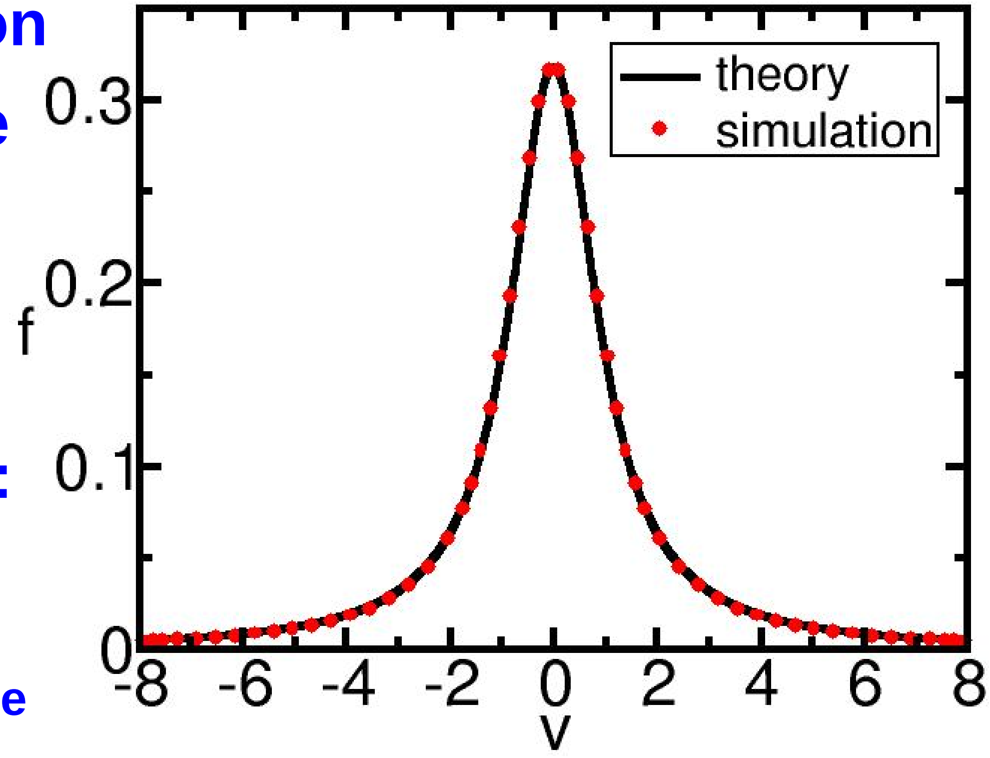
Dissipation always divergent
Energy finite or infinite

The Characteristic Exponent



Monte Carlo Simulations

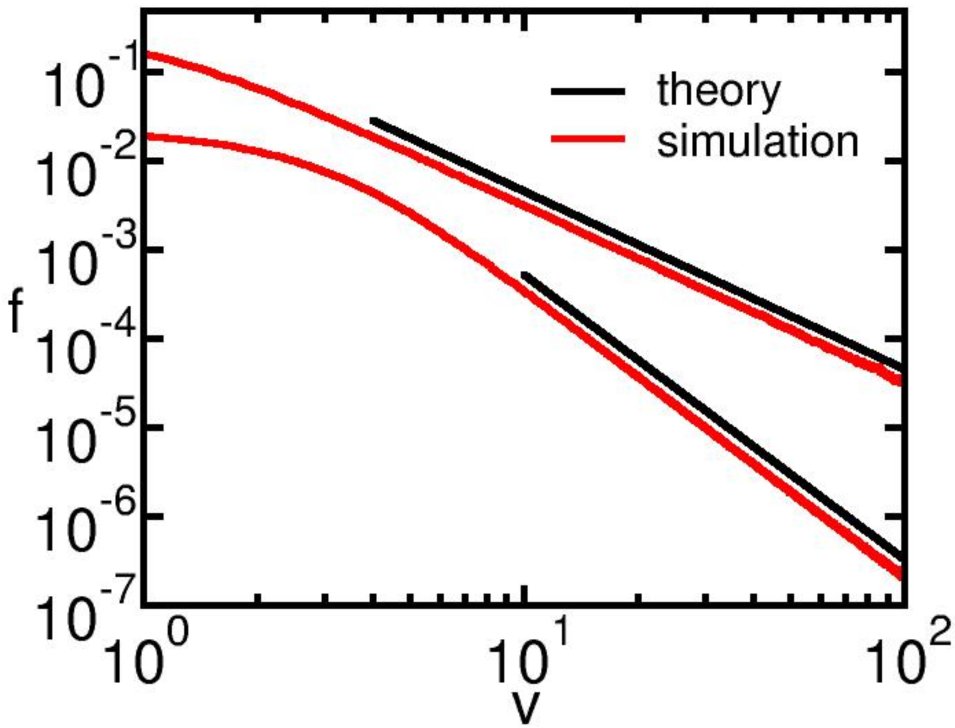
- Compact initial distribution
- Inject energy at very large velocity scales only
- Maintain constant total energy
- "Lottery" implementation:
  - Keep track of total energy dissipated, E_T
  - With small rate, boost a particle by E_T



Excellent agreement between theory and simulation

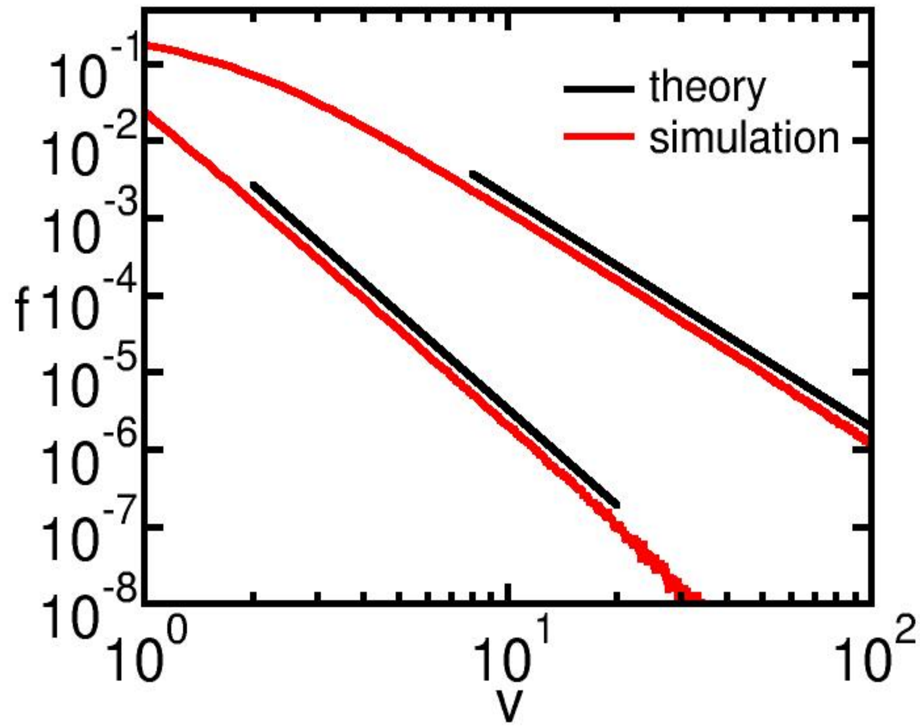
Further confirmation

Maxwell molecules (1D, 2D)



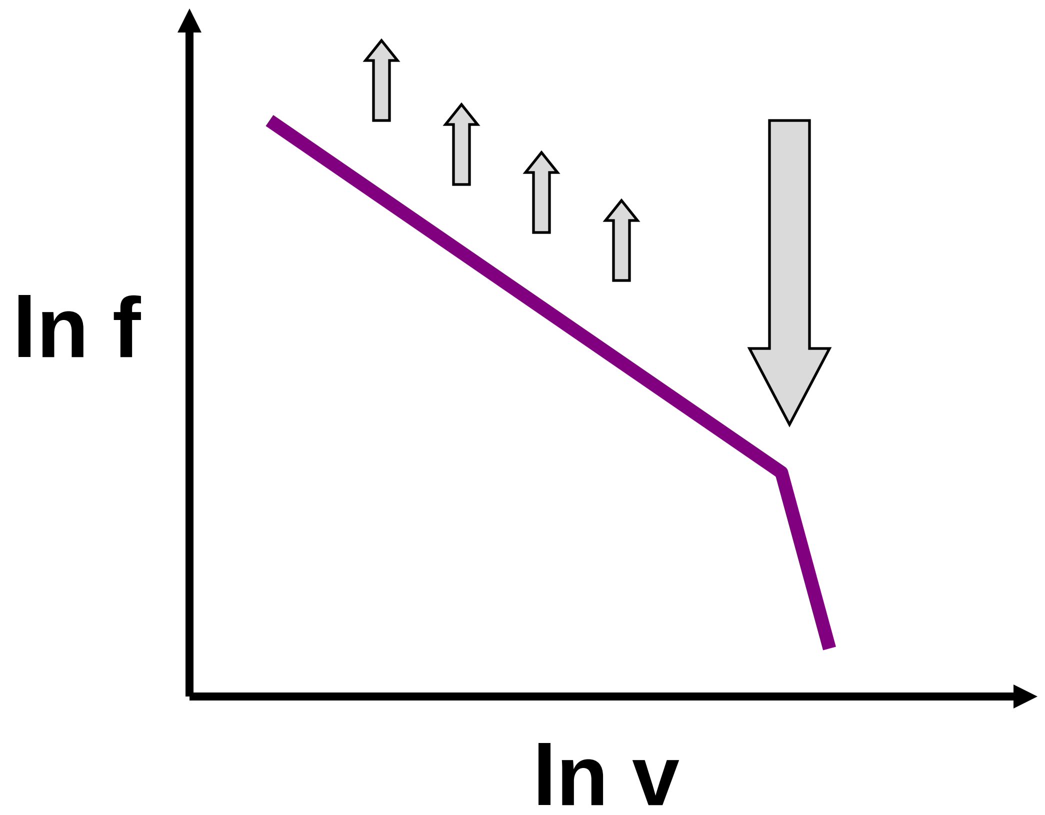
$N=10^7$

Hard spheres (1D, 2D)



$N=10^5$

Injection, cascade, dissipation



- Energy is injected at large velocity scales
- Energy cascades from large velocities to small velocities
- Energy dissipated at small velocity scales

Conclusions

- New class of nonequilibrium stationary states
- Energy cascades from large velocities to small velocities
- Power-law high-energy tail
- Energy input at large scales balances dissipation
- Temperature insufficient to characterize velocities
- Experimental realization: requires a different driving mechanism