# Dynamics of Random Graphs with Bounded Degrees


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thanks: Wolfgang Losert (Maryland)

E. Ben-Naim and P.L. Krapivsky, J. Stat. Mech. P11008 (2011) & EPL **97**, 48003 (2012)

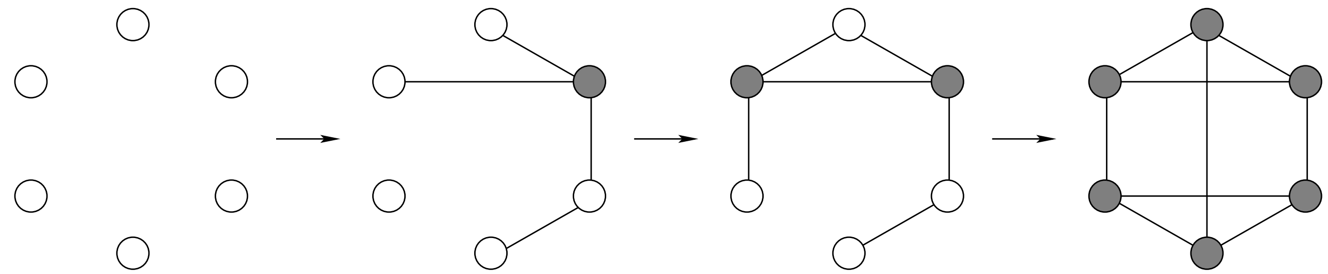
Talk, paper available from: http://cnls.lanl.gov/~ebn

APS March Meeting, February 27, 2012

# Plan

- Evolving random graphs with bounded degrees
- Degree distribution
- Hamilton-Jacobi theory of evolving random graphs with unbounded degrees
- Hamilton-Jacobi theory of evolving random graphs with bounded degrees
- Finite-size scaling laws

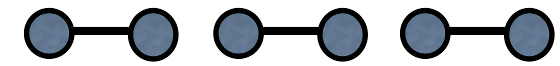
# Evolving Random Graph

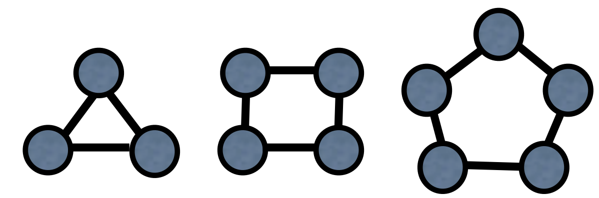


- Initial state: regular random graph (degree = $0$ )
- Define two classes of nodes
  - Active nodes: degree $< d$
  - Inactive nodes: degree $= d$
- Sequential linking
  - Pick two active nodes
  - Draw a link
- Final state: regular random graph (degree = $d$ )

Erdos 60, Wormald 84

# Percolation Transition

✓ d=1 <u>microscopic</u> graphs, dimers 

✓ d=2 <u>mesoscopic</u> graphs, rings  $N_k = k^{-1}$

? d≥2 one <u>macroscopic</u> graph = "giant component"

- Nonpercolating phase: microscopic graphs only
- Percolating phase: one giant component coexists with many microscopic graphs

## Question

How many links (per node) are needed for the giant component to emerge?

## Answer

0.577200 (when d=3)

# Degree Distribution

- Distribution of nodes with degree *j* is $n_j$
- Density of active nodes $\nu = n_0 + n_1 + \cdots + n_{d-1}$ $\nu = 1 - n_d$
- Linking Process

$$(i, j) \to (i+1, j+1) \qquad i, j < d$$

- Active nodes control linking process, effectively linear evolution equation $\tau = \int_0^t dt'\, \nu(t')$

$$\frac{dn_j}{dt} = \nu\,(n_{j-1} - n_j)$$

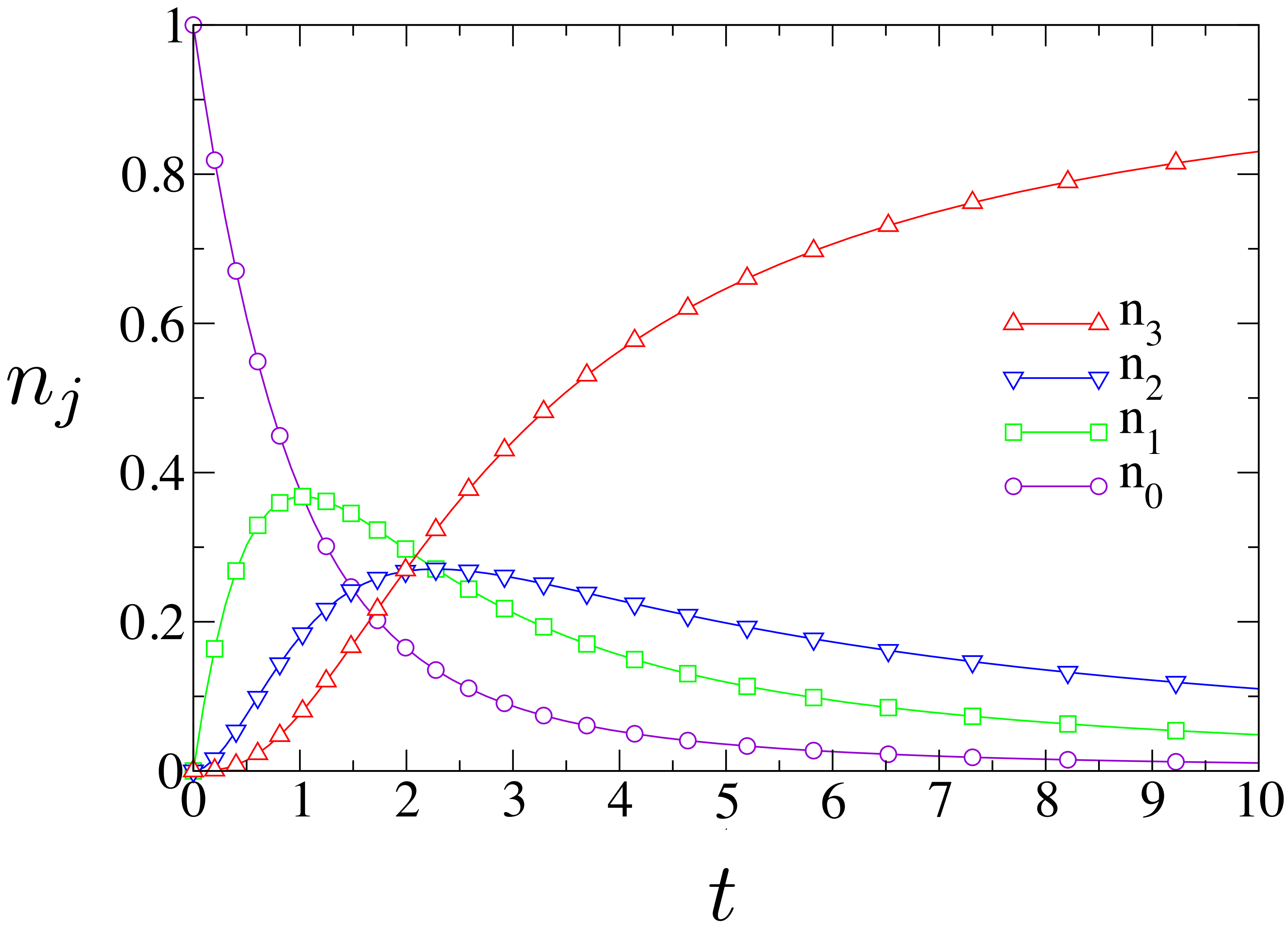
- Solve using an effective time variable

$$n_j = \frac{\tau^j}{j!} e^{-\tau} \qquad\qquad j < d$$

**Truncated Poisson Distribution**

Wormald 99

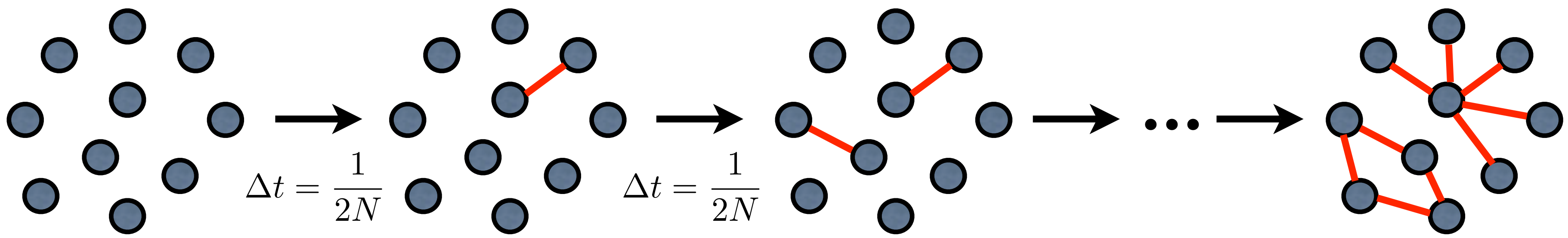
# Degree Distribution



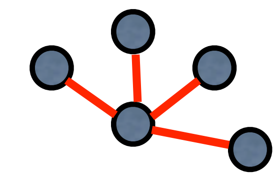

Isolated nodes dominate initially
All nodes become inactive eventually

# Unbounded Random Graphs

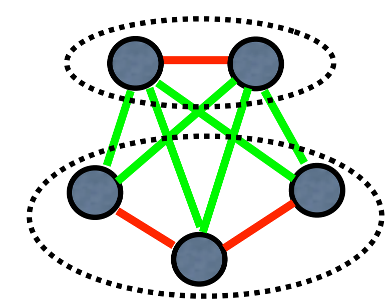
Erdos-Renyi




- Cluster = a connected graph component



- Links involving two separate components lead to merger
- Aggregation rate = product of cluster sizes



$$K_{ij} = ij$$

- Master equation for size distribution

$$\frac{dc_k}{dt} = \frac{1}{2} \sum_{i+j=k} ij c_i c_j - k c_k \qquad c_k(t=0) = \delta_{k,1}$$

- Master equation for generating function

$$\frac{\partial \mathcal{C}}{\partial t} + x \frac{\partial \mathcal{C}}{\partial x} = \frac{1}{2}\left(x \frac{\partial \mathcal{C}}{\partial x}\right)^2 \qquad \mathcal{C}(x,t) = \sum_k c_k(t) x^k$$

# Hamilton-Jacobi Theory I

- Master equation is a first-order PDE

$$\frac{\partial \mathcal{C}}{\partial t} + x\,\frac{\partial \mathcal{C}}{\partial x} = \frac{1}{2}\left(x\,\frac{\partial \mathcal{C}}{\partial x}\right)^2 \qquad \mathcal{C}(x,0) = x$$

- Recognize as a Hamilton-Jacobi equation

$$\frac{\partial \mathcal{C}(x,t)}{\partial t} + H(x,p) = 0$$

- By identifying "momentum" and "Hamiltonian"

$$p = \frac{\partial \mathcal{C}}{\partial x} \qquad \text{and} \qquad H = xp - \frac{1}{2}(xp)^2$$

- Hamilton-Jacobi equations generate two coupled ODEs

$$\frac{dx}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial x} \quad \Longrightarrow \quad \frac{dx}{dt} = x(1-xp), \quad \frac{dp}{dt} = -p(1-xp)$$

$$x(0) = 1-g \qquad p(0) = 1$$

Initial coordinate unknown, final coordinate known!

Hamiltonian is a conserved quantity

# Solution I

- Coordinate and momentum are immediate

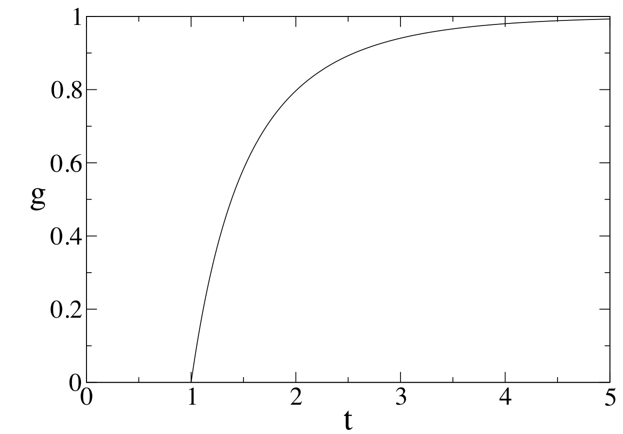
$$x = (1-g)e^{gt} \qquad p = e^{-gt}$$

- Size of giant component found immediately

$$g = 1 - \sum_k k\, c_k = 1 - p(0)$$

- Satisfies a closes equation

$$1 - g = e^{-gt}$$




- Nontrivial solution beyond the percolation threshold

$$t_g = 1$$

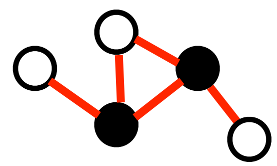
The giant component emerges when the average degree equals one

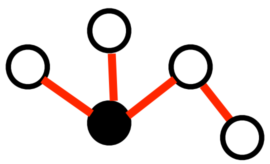
# Bounded Random Graphs

- Total size of components provides insufficient description
- Describe components by a d+1 dimensional vector whose components specify number of nodes with given degree

$$(k_0, k_1, \cdots, k_d) \qquad k = k_0 + k_1 + \cdots + k_d$$

$(0, 2, 1, 2)$





$(0, 3, 1, 1)$

- Multivariate aggregation process
- Aggregation rate is product of the number of active nodes

$$K(\mathbf{l}, \mathbf{m}) = (l - l_d)(m - m_d)$$

- Why can't we get away with two variables only?
- Node degrees are coupled!

$$0 \to 1 \to 2 \to 3$$

# Hamilton-Jacobi Theory II

- Master equation is a first-order PDE

$$\frac{\partial C}{\partial \tau} = \frac{1}{2\nu}\left(\sum_{j=0}^{d-1} x_{j+1}\frac{\partial C}{\partial x_j}\right)^2 - \sum_{j=0}^{d-1} x_j \frac{\partial C}{\partial x_j} \qquad C(\mathbf{x},0) = x_0$$

- Recognize as a Hamilton-Jacobi equation

$$\frac{\partial C(\mathbf{x},\tau)}{\partial \tau} + H(\mathbf{x},\nabla C,\tau) = 0$$

- By identifying "momentum" and "Hamiltonian"

$$H(\mathbf{x},\mathbf{p},\tau) = \sum_{j=0}^{d-1} x_j p_j - \frac{\Pi_1^2}{2\nu(\tau)} \qquad \Pi_j = \sum_{i=j}^{d} x_i\, p_{i-j}$$

- Hamilton-Jacobi equation give $2(\mathrm{d}+1)$ coupled ODEs

$$\frac{dx_j}{dt} = \frac{\partial H}{\partial p_j}, \quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial x_j} \quad \Longrightarrow \quad \frac{dx_j}{dt} = x_j - \frac{\Pi_1}{\nu}x_{j+1}, \quad \frac{dp_j}{dt} = \frac{\Pi_1}{\nu}p_{j-1} - p_j$$

Initial coordinates unknown, final coordinates known!

Equations are now in $\mathrm{d}+1$ dimensions!

Hamiltonian no longer conserved!

# Solution II

- Find hidden conservation laws and explicit backward equations
- reduce 2(d+1) first order ODE to 1 second order ODE

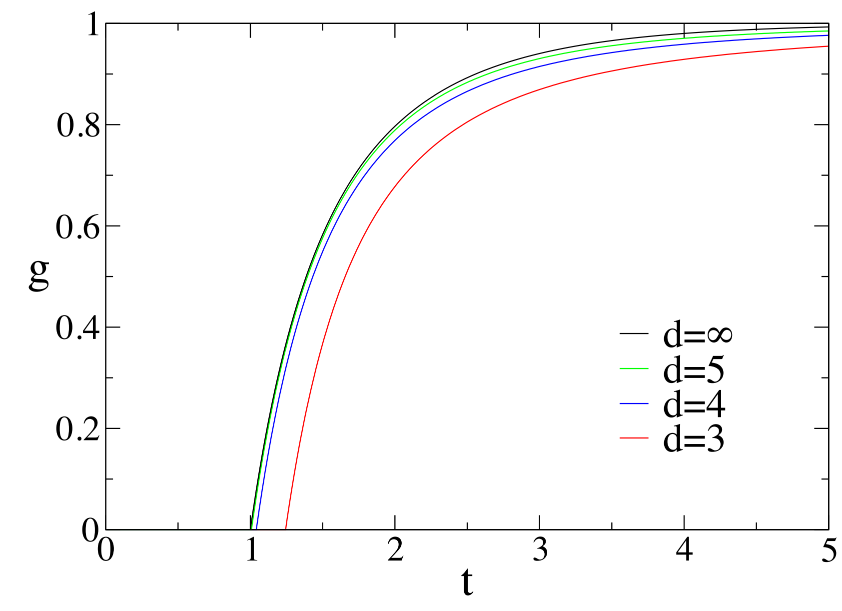
$$\frac{d^2u}{d\tau^2} + \frac{n_{d-1}}{\nu}\frac{du}{d\tau} - x_d\frac{p_{d-1}}{\nu} = 0$$

- Nontrivial solution when d>2
- Numerical solution gives percolation threshold (d=3)

$$t_g = 1.243785, \quad L_g = 0.577200$$

- The size distribution of components at the critical point

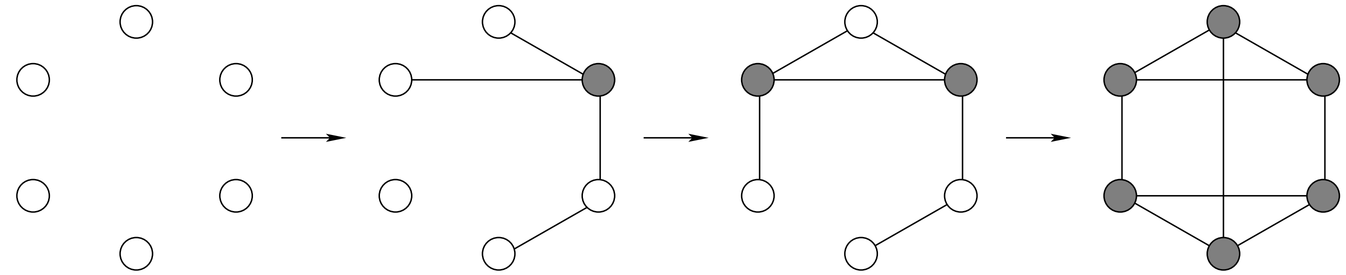
$$c_k \simeq A\, k^{-5/2}$$




- Mean-field percolation

Hamilton-Jacobi theory gives all percolation parameters

# Finite-size scaling



Degree distribution

$$n_j \simeq \frac{(d-1)!}{j!}\, t^{-1} (\ln t)^{-(d-1-j)}$$

Regular random graph emerges in several steps

1. Giant component emerges at finite time

$$t_1 = 1.243785$$ deterministic

2. Graph becomes fully connected emerges at time

$$N n_0 \sim 1 \Longrightarrow \qquad t_2 \sim N (\ln N)^{-(d-1)}$$ stochastic

3. Regular random graph emerges at time

$$N n_{d-1} \sim 1 \Longrightarrow \qquad t_3 \sim N$$ stochastic

Giant fluctuations in completion time

# Summary

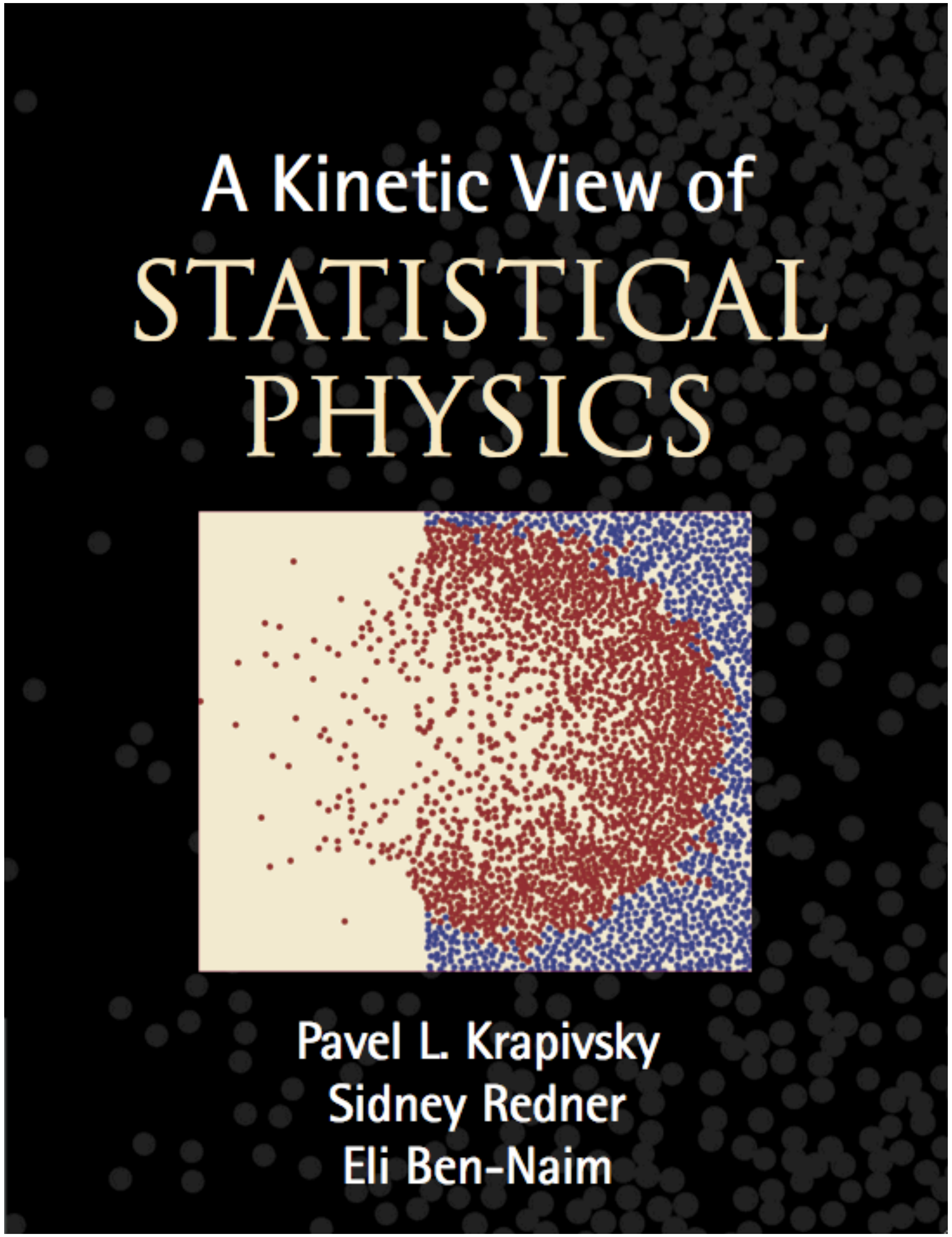
- Dynamic formation of regular random graphs
- Degree distribution is truncated Poissonian
- Hamilton-Jacobi formalism powerful
- Percolation parameters with essentially arbitrary precision
- Mean-field percolation universality class
- A multitude of finite-size scaling properties
- Giant fluctuations in completion time

Theory applicable to broader set of evolving graphs

chapter 5
aggregation

chapter 12
population
dynamics

chapter 13
complex
networks




Cambridge University Press 2010