Kinetic Theory of Synchronization

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Talk, paper available from: http://cnls.lanl.gov/~ebn

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Phase Synchronization & Alignment

- Biology: synchronization of fruit flies, alignment of microtubules by molecular motors
- Ecology: flocking of birds, fish movement
- Granular matter: granular chains and solid rods
- Mechanics: coupled oscillators
- Material Science: nematic phases in liquid crystals

Traditional approach is Hamiltonian/Equilibrium

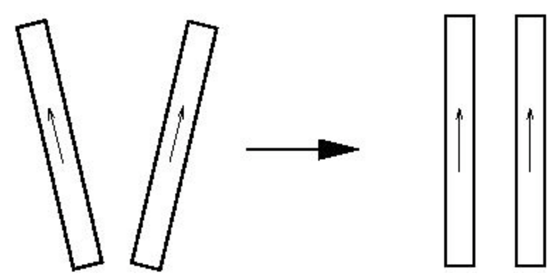
Kinetic Model

- Each rod has an orientation

Aronson & Tsimring 05

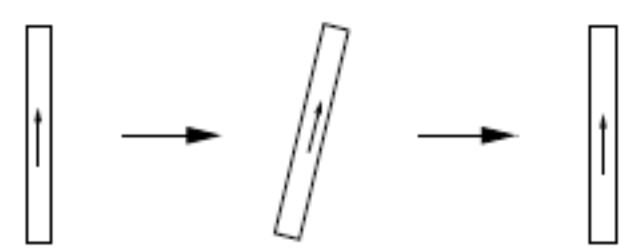
$$0 \leq \theta \leq \pi$$

- Alignment by pairwise interactions (nonlinear)



$$(\theta_1, \theta_2) \to \begin{cases} \left(\frac{\theta_1+\theta_2}{2}, \frac{\theta_1+\theta_2}{2}\right) & |\theta_1 - \theta_2| < \pi \\ \left(\frac{\theta_1+\theta_2+2\pi}{2}, \frac{\theta_1+\theta_2+2\pi}{2}\right) & |\theta_1 - \theta_2| > \pi \end{cases}$$

- Diffusive wiggling (linear)



$$\frac{d\theta_j}{dt} = \eta_j(t)$$

$$\langle \eta_j(t) \eta_j(t') \rangle = 2D\delta(t - t')$$

Kinetic Theory

- Nonlinear integro-differential equation

$$\frac{\partial P(\theta)}{\partial t} = D\frac{\partial^2 P(\theta)}{\partial \theta^2} + \int_{-\pi}^{\pi} d\phi\, P\left(\theta - \frac{\phi}{2}\right) P\left(\theta + \frac{\phi}{2}\right) - P(\theta)$$

- Fourier transform

$$P_k = \langle e^{-ik\theta}\rangle = \int_{-\pi}^{\pi} d\theta e^{-ik\theta} P(\theta) \qquad P(\theta) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} P_k e^{ik\theta}$$

- Closed nonlinear equation (as in Inelastic Maxwell Model)

$$(1 + Dk^2)P_k = \sum_{i+j=k} A_{i-j} P_i P_j$$

- Coupling constants

$$A_q = \frac{\sin\frac{\pi q}{2}}{\frac{\pi q}{2}} = \begin{cases} 1 & q = 0 \\ 0 & q = 2, 4, \cdots \\ (-1)^{\frac{q-1}{2}}\frac{2}{\pi|q|} & \end{cases}$$

Linear Stability Analysis

- Small perturbation to uniform state

$$P(\theta, t) = \frac{1}{2\pi} + p(\theta, t)$$

- Linear evolution equation for small perturbation

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial \theta^2} + \int_{-\pi}^{\pi} d\phi \, \frac{p(\theta - \phi/2) + p(\theta + \phi/2)}{2\pi} - p$$

- Growth rate of perturbation

$$p(\theta, t) \propto e^{ik\theta + \lambda t} \qquad \Longrightarrow \qquad \lambda_k = 2A_k - 1 - Dk^2$$

- Uniform state stable only when diffusion large

$$D > D_c = 2A_1 - 1 = \frac{4}{\pi} - 1$$

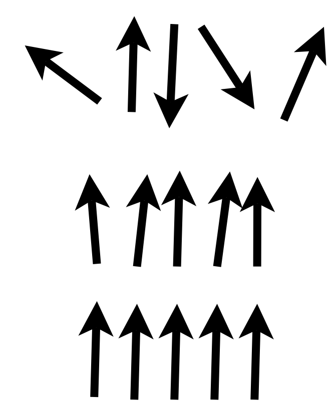
The Order Parameter

- Lowest order Fourier mode

$$R = |\langle e^{i\theta} \rangle| = |P_{-1}|$$

- Probes state of system

$$R = \begin{cases} 0 & \text{disordered state} \\ 0.4 & \text{partially ordered} \\ 1 & \text{perfectly ordered state} \end{cases}$$



The Fourier Equation

- Compact Form

$$P_k = \sum_{i+j=k} G_{i,j} P_i P_j$$

- Transformed coupling constants

$$G_{i,j} = \frac{A_{i-j}}{1+D(i+j)^2-2A_{i+j}}$$

- Properties

$$\begin{aligned} G_{i,j} &= G_{j,i} \\ G_{i,j} &= G_{-i,-j} \\ G_{i,j} &= 0, \qquad \text{for} \qquad |i-j| = 2, 4, \ldots . \end{aligned}$$

Solution

- Repeated iterations (product of three modes)

$$P_k = \sum_{\substack{i+j=k \\ i\neq 0,\, j\neq 0}} \sum_{\substack{l+m=j \\ l\neq 0,\, m\neq 0}} G_{i,j}\, G_{l,m}\, P_i\, P_l\, P_m .$$

- When k=2,4,8,...

$$\begin{aligned} P_2 &= G_{1,1} P_1^2 \\ P_4 &= G_{2,2} P_2^2 = G_{2,2} G_{1,1}^2 P_1^4 . \end{aligned}$$

- Generally

$$\begin{aligned} P_3 &= 2G_{1,2} P_1 P_2 + 2G_{-1,4} P_{-1} P_4 + \cdots \\ &= 2G_{1,2} G_{1,1} P_1^3 + 2G_{-1,4} G_{2,2} G_{1,1}^2 P_1^4 P_{-1} \cdots \end{aligned}$$

Partition of Integers

- Diagramatic solution

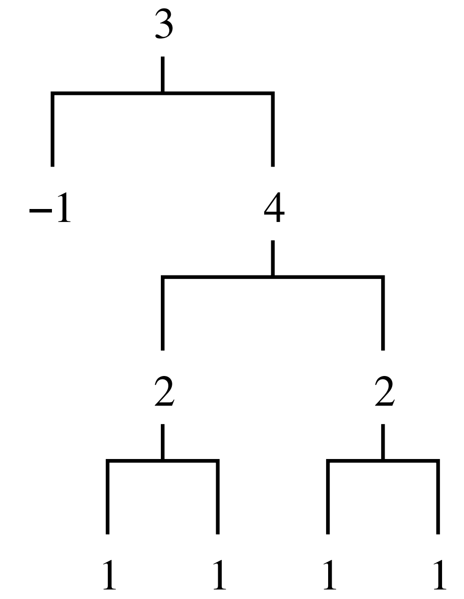
$$P_k = R^k \sum_{n=0}^{\infty} p_{k,n} R^{2n}$$

- Partition

$$k = \underbrace{1 + 1 + \cdots + 1 + 1}_{k+n} \underbrace{-1 - \cdots - 1}_{n} .$$

- Partition rules

$$\begin{aligned} k &= i + j \\ i &\neq 0 \\ j &\neq 0 \\ G_{i,j} &\neq 0 \end{aligned}$$



$$p_{3,1} = 2G_{-1,4}G_{2,2}G_{1,1}^2$$

All modes expressed in terms of order parameter

The Order Parameter

- Diagramatic solution

$$R = R^k \sum_{n=0}^{\infty} p_{1,n} \, R^{2n} \qquad R = r_3 R^3 + r_5 R^5 + \cdots$$

- Landau theory (critical diffusivity as in linear stability analysis)

$$R = \frac{C}{D_c - D} R^3 + \cdots \qquad D_c = \frac{4}{\pi} - 1$$

- Strong Diffusion: trivial solution

$$R = 0 \qquad \text{when} \qquad D > D_c$$

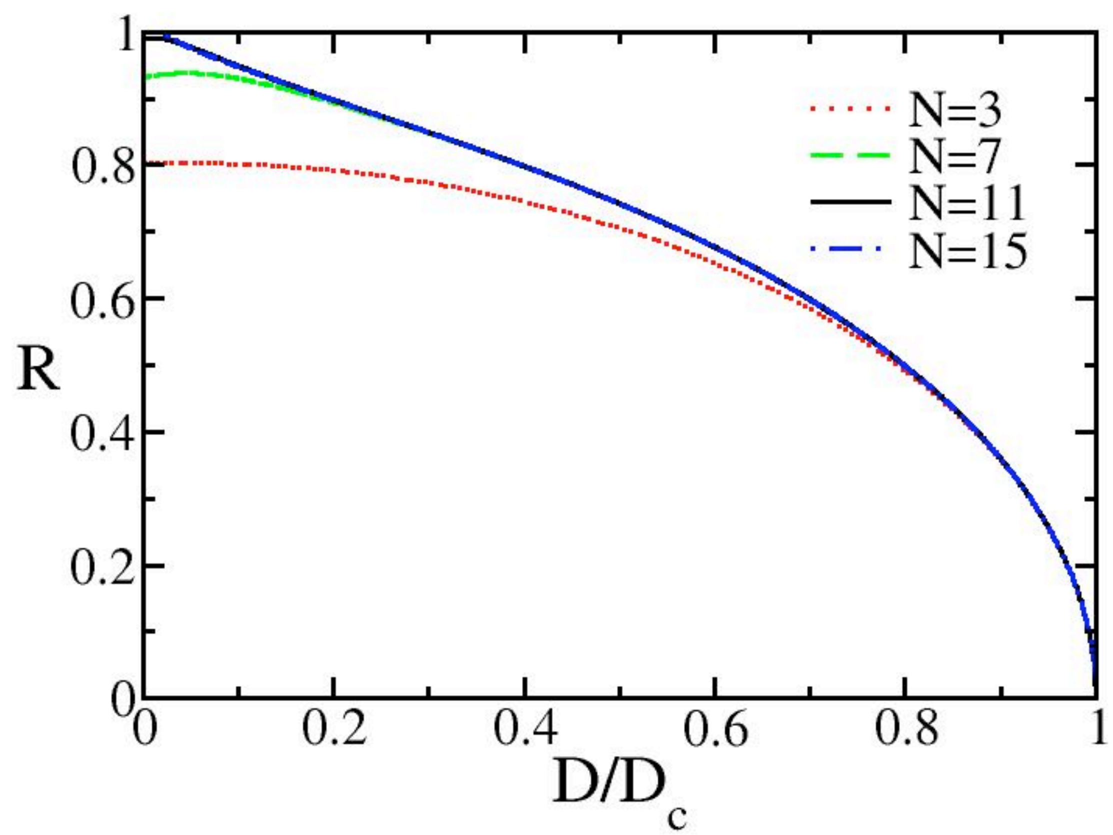
- Weak Diffusion: nontrivial solution

$$R \sim \sqrt{D_c - D} \qquad \text{when} \qquad D < D_c$$

Closed equation for order parameter

Nonequilibrium Phase Transition

- Critical diffusion constant $D_c = \frac{4}{\pi} - 1$
- Weak diffusion: ordered phase $R > 0$
- Strong diffusion: disordered phase $R = 0$
- Critical behavior $R \sim (D_c - D)^{1/2}$

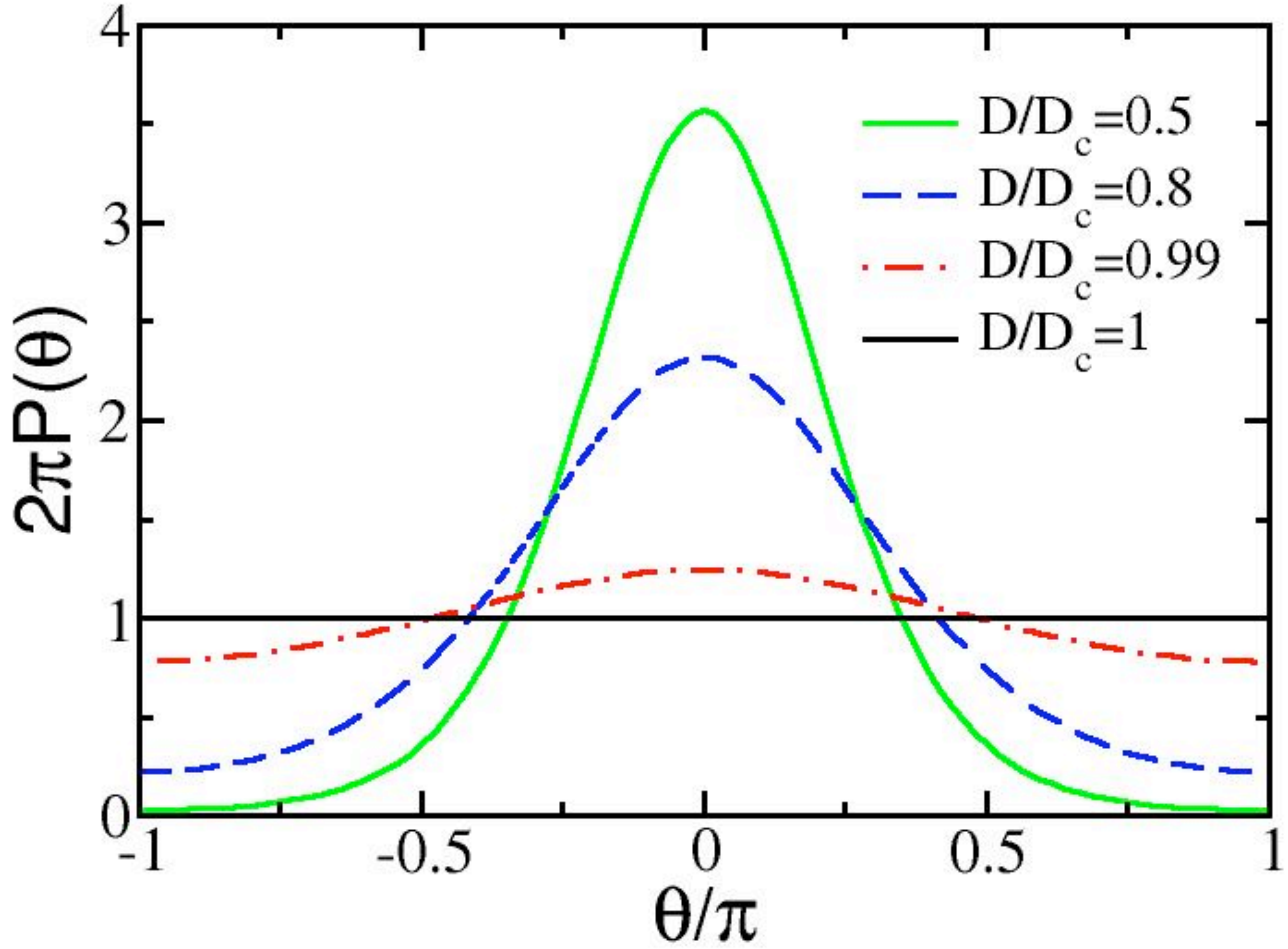


Distribution of Orientation

- Fourier modes decay exponentially with R

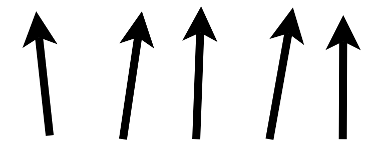
$$P_k \sim R^k$$

- Small number of modes sufficient



$$P(\theta) = \frac{1}{2\pi} + \frac{1}{\pi} R\cos\theta + \frac{1}{\pi} G_{1,1} R^2 \cos(2\theta) + \frac{2}{\pi} G_{1,2} G_{1,1} R^3 \cos(3\theta) + \cdots$$

Infinitesimal Diffusion

- Near perfect alignment 
- Scaling transformation reduce to inelastic collisions

$$P(\theta) = \frac{1}{\sqrt{D}} f\left(\frac{\theta}{\sqrt{D}}\right) \qquad \Longrightarrow \qquad f(x) = \frac{d^2 f(x)}{dx^2} + \int_{-\infty}^{\infty} dy\, f\left(x - \frac{y}{2}\right) f\left(x + \frac{y}{2}\right)$$

- Closed form solution in Fourier space

$$(1+q^2)F(q) = F^2(q/2) \qquad \Longrightarrow \qquad F(q) = \prod_{n=0}^{\infty}\left[1 + \frac{q^2}{4^n}\right]^{-2^n}$$

- Sharply localized distribution

$$f(x) \simeq C e^{-x} \qquad x \gg 1$$

- Order parameter: close to one, taylor series

$$R = 1 - 2D + \frac{18}{7} D^2 + \cdots$$

General Alignment Rates

- Orientation-dependent alignment rate

$$K \equiv K(|\theta_1 - \theta_2|) \qquad \Longrightarrow \qquad D_c = \frac{1}{\pi}\int_{-\pi}^{\pi} d\phi\, e^{i\phi/2} K(\phi) - 1$$

- Diagramatic solution holds

- Hard rods

$$K(\phi) \propto |\sin\phi| \qquad D_c = \frac{1}{3}$$

- Hard spheres: system always disordered

$$K(\phi) \propto |\phi|$$

Boltzmann equation can be solved!

Phase transition may or may not exist

Arbitrary Alignment Rates

- Kinetic theory: arbitrary alignment rates

$$0 = D\frac{d^2P}{d\theta^2} + \int_{-\pi}^{\pi} d\phi\, \underline{K(\phi)}P\left(\theta - \frac{\phi}{2}\right) P\left(\theta + \frac{\phi}{2}\right) - P(\theta)\int_{-\pi}^{\pi} d\phi\, \underline{K(\phi)}P(\theta + \phi)$$

- Fourier transform of alignment rate

$$A_q = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi\, e^{iq\phi/2}K(\phi)$$

- Recover same Fourier equation using

$$G_{i,j} = \frac{1}{2}\,\frac{A_{i-j} + A_{j-i} - A_{2i} - A_{2j}}{1 + D(i+j)^2 - 2A_{i+j}}$$

When Fourier spectrum is discrete: exact solution is possible for arbitrary alignment rates

Conclusions

- Nonequilirbium description through kinetic model based on binary interactions
- Nonequilibrium phase transition
- Weak noise: ordered phase (nematic)
- Strong noise: disordered phase (isotropic)
- Solution relates to iterated partition of integers
- Only when Fourier spectrum is discrete: exact solution possible for arbitrary averaging rates

Publication

E. Ben-Naim and P.L. Krapivsky,
Phys. Rev. E 73, 031109 (2006).