Nontrivial Exponents in Records Statistics

Eli Ben-Naim
Los Alamos National Laboratory

with: Pearson Miller (Yale)

P. W. Miller and E. Ben-Naim, J. Stat. Mech. P10025 (2013)

Talk, paper available from: http://cnls.lanl.gov/~ebn

APS March Meeting, Denver (CO), March 3, 2014

Marathon World Record

Year	Athlete	Country	Record	Improvement
2002	Khalid Khannuchi	USA	2:05:38	
2003	Paul Tergat	Kenya	2:04:55	0:43
2007	Haile Gebrsellasie	Ethiopia	2:04:26	0:29
2008	Haile Gebrsellasie	Ethiopia	2:03:59	0:27
2011	Patrick Mackau	Kenya	2:03:38	0:21
2013	Wilson Kipsang	Kenya	2:03:23	0:15

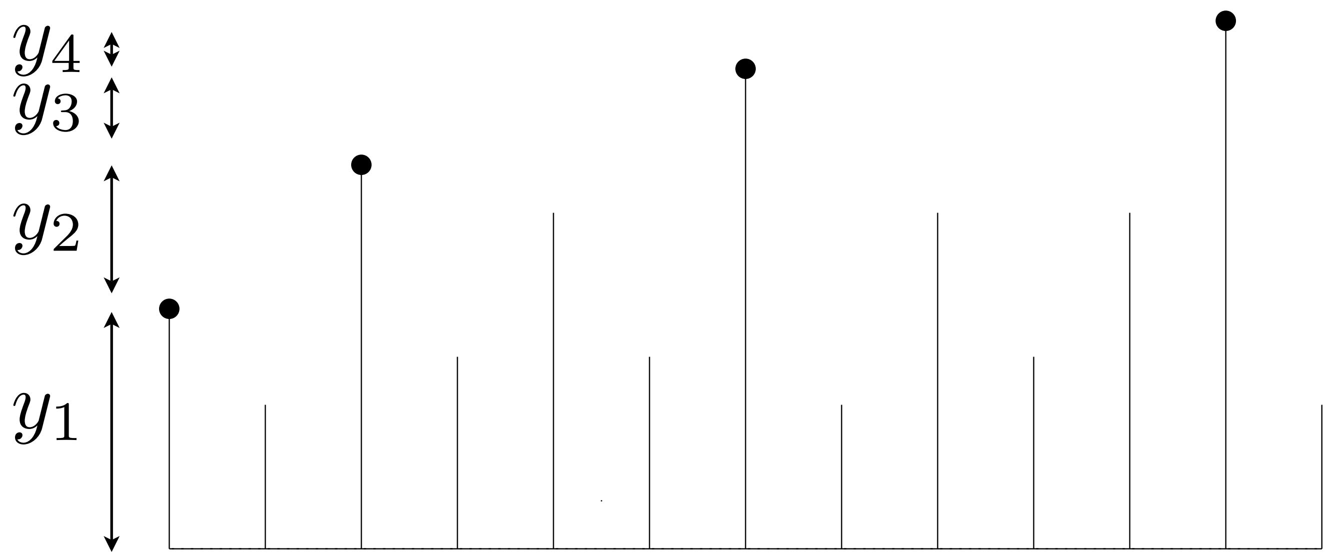
Incremental sequence of records

every record improves upon previous record by yet smaller amount

Are incremental sequences of records common?

source: wikipedia

Incremental Records



Incremental sequence of records

every record improves upon previous record by yet smaller amount

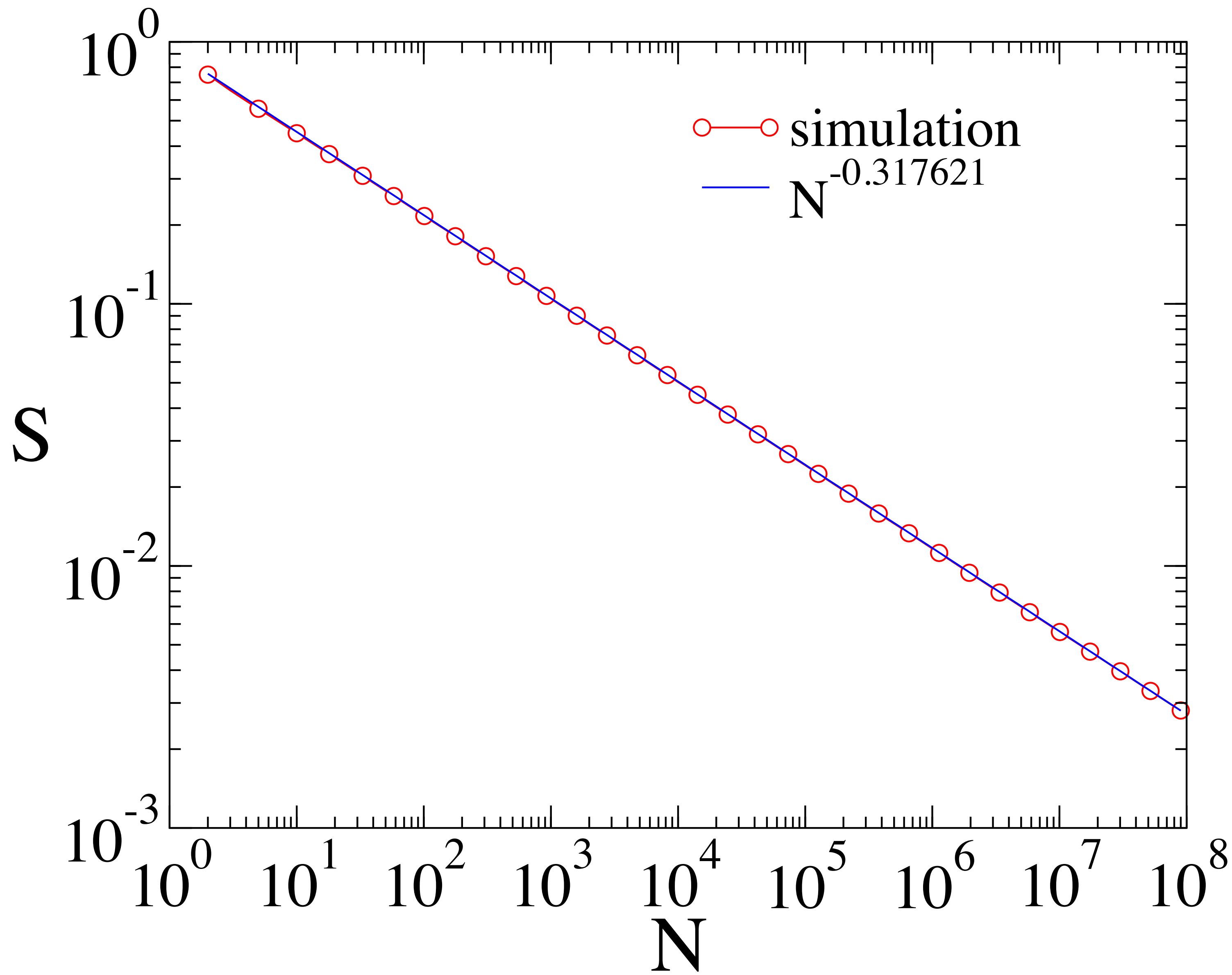
$$\text{random variable} = \{0.4, 0.4, 0.6, 0.7, 0.5, 0.1\}$$

$$\text{latest record} = \{0.4, 0.4, 0.6, 0.7, 0.7, 0.7\} \quad \uparrow$$

$$\text{latest increment} = \{0.4, 0.4, 0.2, 0.1, 0.1, 0.1\} \quad \downarrow$$

What is the probability all records are incremental?

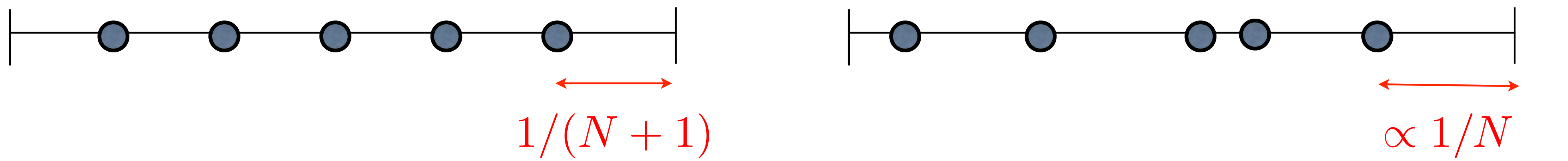
Probability all records are incremental



$S_N \sim N^{-\nu}$ $\qquad \nu = 0.31762101$

Power law decay with nontrivial exponent
Question is free of parameters!

Uniform distribution



- The variable x is randomly distributed in $[0{:}1]$

$$\rho(x) = 1 \quad \text{for} \quad 0 \le x \le 1$$

- Probability record is smaller than x

$$R_N(x) = x^N$$

- Average record

$$A_N = \frac{N}{N+1} \qquad \Longrightarrow \qquad 1 - A_N \simeq N^{-1}$$

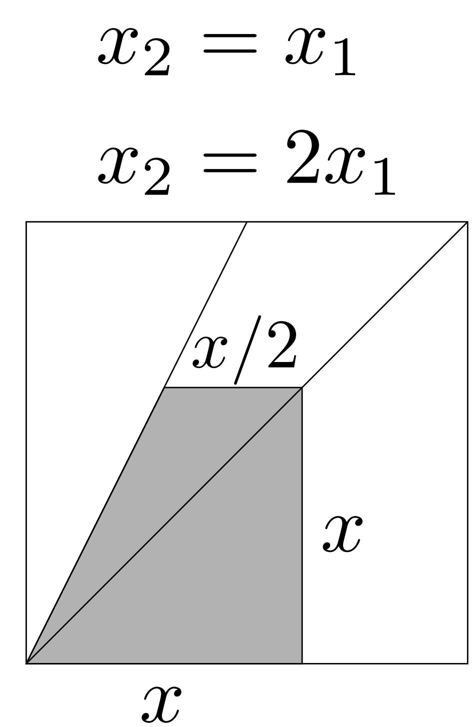
- Number of records

$$M_N = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N}$$

Distribution of records

- Probability a sequence is incremental and record $< x$

$$G_N(x) \quad \Longrightarrow \quad S_N = G_N(1)$$



- One variable

$$G_1(x) = x \quad \Longrightarrow \quad S_1 = 1$$

- Two variables

$x_2 - x_1 > x_1$

$$G_2(x) = \frac{3}{4}\,x^2 \quad \Longrightarrow \quad S_2 = \frac{3}{4}$$

- In general, conditions are scale invariant $x \to a\,x$
- Distribution of records for incremental sequences

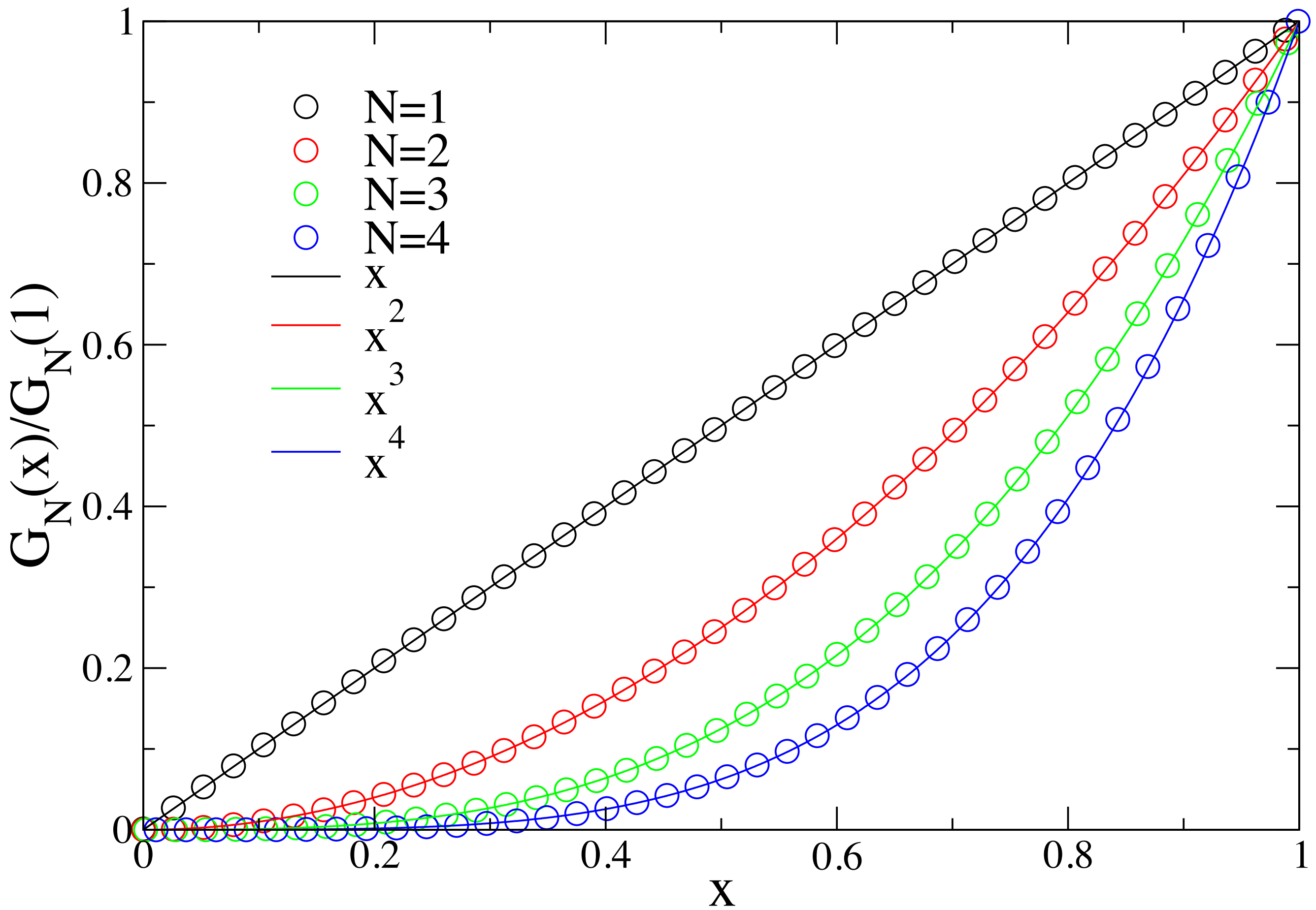
$$G_N(x) = S_N\, x^N$$

- Distribution of records for all sequences equals x^N

Statistics of records are standard

Fisher-Tippett 28
Gumbel 35

Scaling behavior



- Distribution of records for incremental sequences

$$G_N(x)/S_N = x^N = [1-(1-x)]^N \to e^{-s}$$

- Scaling variable

$$s = (1-x)N$$

Exponential scaling function

Distribution of increment+records

- Probability density $S_N(x,y)dxdy$ that:
  1. Sequence is incremental
  2. Current record is in range $(x,x+dx)$
  3. Latest increment is in range $(y,y+dy)$ *with* $0<y<x$
- Gives the probability a sequence is incremental

$$S_N = \int_0^1 dx \int_0^x dy\, S_N(x,y)$$

- Recursion equation incorporates memory

$$S_{N+1}(x,y) = \underbrace{x\, S_N(x,y)}_{\text{old record holds}} + \underbrace{\int_y^{x-y} dy'\, S_N(x-y,y')}_{\text{a new record is set}}$$

- Evolution equation includes integral, has memory

$$\frac{\partial S_N(x,y)}{\partial N} = -(1-x)S_N(x,y) + \int_y^{x-y} dy'\, S_N(x-y,y')$$

Scaling transformation

- Assume record and increment scale similarly

$$y \sim 1 - x \sim N^{-1}$$

- Introduce a scaling variable for the increment

$$s = (1-x)N \quad \text{and} \quad z = yN$$

- Seek a scaling solution

$$S_N(x,y) = N^2 S_N \, \Psi(s,z)$$

- Eliminate time out of the master equation

$$\left(2 - \nu + s + s\frac{\partial}{\partial s} + z\frac{\partial}{\partial z}\right)\Psi(s,z) = \int_z^\infty dz' \, \Psi(s+z, z')$$

Reduce problem from <u>three</u> variables to <u>two</u>

Factorizing solution

- Assume record and increment decouple

$$\Psi(s,z) = e^{-s}\, f(z)$$

- Substitute into equation for similarity solution

$$\left(2-\nu+s+s\frac{\partial}{\partial s}+z\frac{\partial}{\partial z}\right)\Psi(s,z) = \int_z^\infty dz'\,\Psi(s+z,z')$$

- First order integro-differential equation

$$zf'(z)+(2-\nu)f(z) = e^{-z}\int_z^\infty f(z')dz'$$

- Cumulative distribution of scaled increment $g(z)=\int_z^\infty f(z')dz'$
- Convert into a second order differential equation

$$zg''(z)+(2-\nu)g'(z)+e^{-z}g(z)=0 \qquad \begin{aligned} g(0)&=1\\ g'(0)&=-1/(2-\nu)\end{aligned}$$

Reduce problem from <u>two</u> variable to <u>one</u>

Distribution of increment

- Assume record and increment decouple

$$zg''(z) + (2-\nu)g'(z) + e^{-z}g(z) = 0 \qquad \begin{aligned} g(0) &= 1 \\ g'(0) &= -1/(2-\nu) \end{aligned}$$

- Two independent solutions

$$g(z) = z^{\nu-1} \quad \text{and} \quad g(z) = \text{const.} \quad \text{as} \quad z \to \infty$$

- The exponent is determined by the tail behavior

$$\beta = 0.317621014462...$$

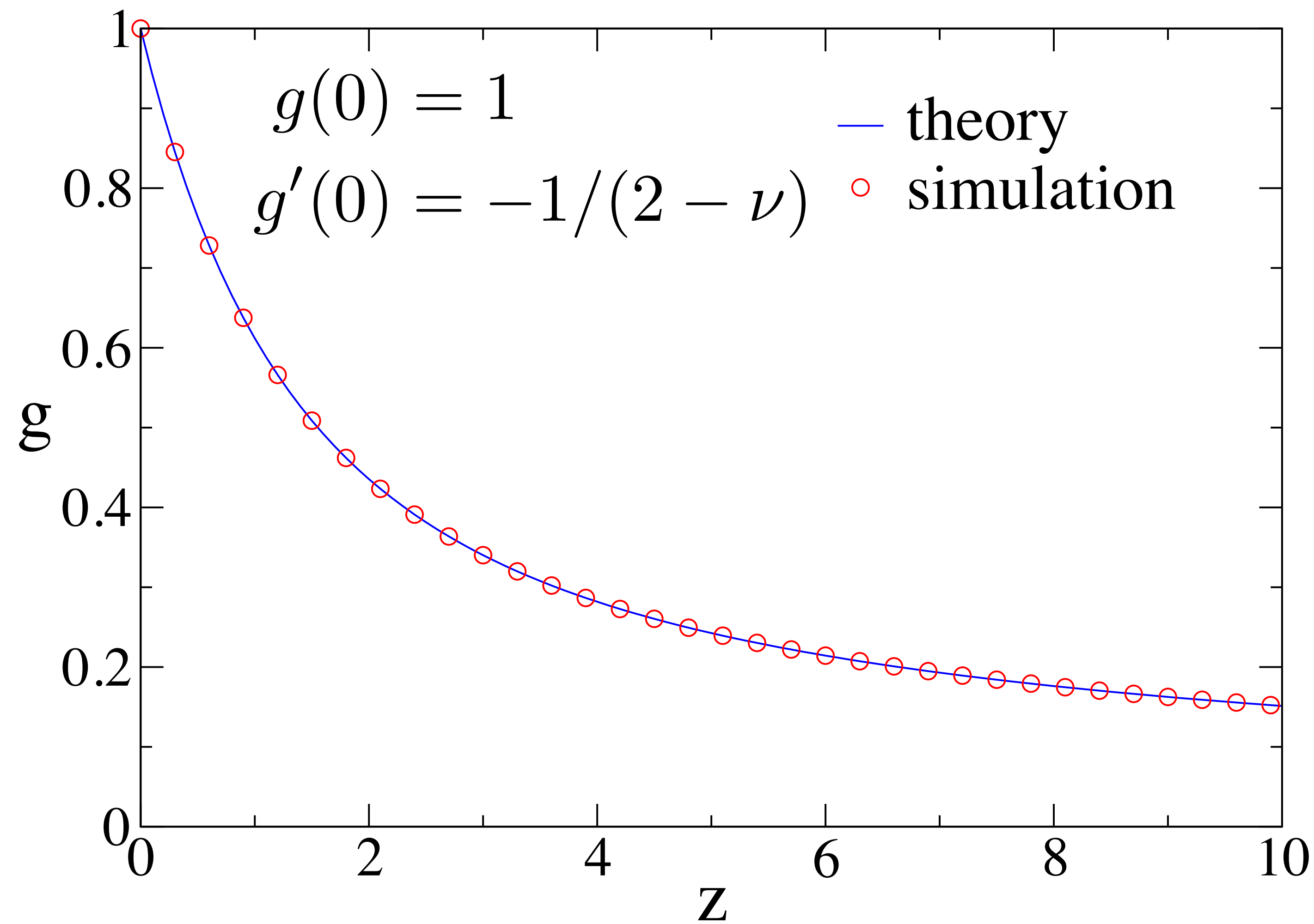
- The distribution of increment has a broad tail

$$P_N(y) \sim N^{-1} y^{\nu-2}$$

Increments can be relatively large
problem reduced to second order ODE

Numerical confirmation

Monte Carlo simulation versus integration of ODE



$$\frac{\langle sz \rangle}{\langle s \rangle \langle z \rangle} \to 1$$

Increment and record become uncorrelated

Conclusions

- Studied persistent configuration of record sequences
- Linear evolution equations (but with memory)
- Dynamic formulation: treat sequence length as time
- Similarity solutions for distribution of records
- Probability a sequence of records is incremental decays as power-law with sequence length
- Power-law exponent is nontrivial, obtained analytically
- Distribution of record increments is broad

First-passage properties of extreme values are interesting

EB, PL Krapivsky, Phys. Rev. E 88, 022145 (2013)