Opinion Dynamics: Rise and Fall of Political Parties

Eli Ben-Naim
Los Alamos National Laboratory

with: Paul Krapivsky, Sidney Redner (Boston University), Arnd Scheel (Minnesota)
thanks: Michael Cross (Caltech), Lev Tsimring (San Diego)

Talk, papers available from: http://cnls.lanl.gov/~ebn

Kinetic description of Social Dynamics, College Park, Maryland, November 6, 2012
DC, Election Day

Plan

I. Pure compromise dynamics

   A. Continuous opinions

   B. Discrete opinions

II. Noisy compromise dynamics

   A. Single-party dynamics

   B. Two-party dynamics

   C. Multi-party dynamics

Themes

1. Bifurcations
2. Pattern Formation
3. Scaling
4. Coarsening

1. Pure compromise dynamics

The compromise process

- Opinion measured by a continuum variable

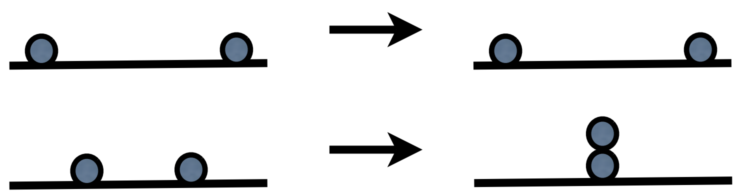
$$-\Delta < x < \Delta$$

1. **Compromise**: reached by pairwise interactions

$$(x_1, x_2) \to \left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}\right)$$

2. **Conviction**: restricted interaction range

$$|x_1 - x_2| < 1$$



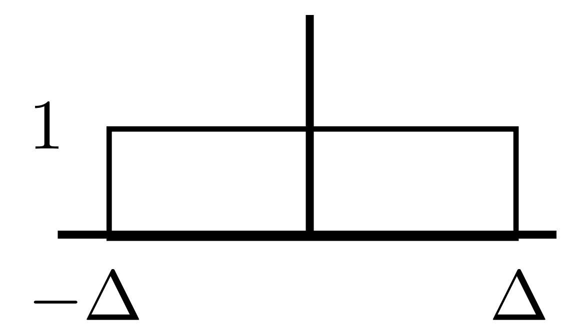
- Minimal, one parameter model
- Mimics competition between compromise and conviction

"bounded confidence"
R Axelrod, J Conf. Res. 41, 203 (1997)
G. Deffuant, G Weisbuch et al, Adv. Comp. Sys 3, 87 (2000)

Problem set-up

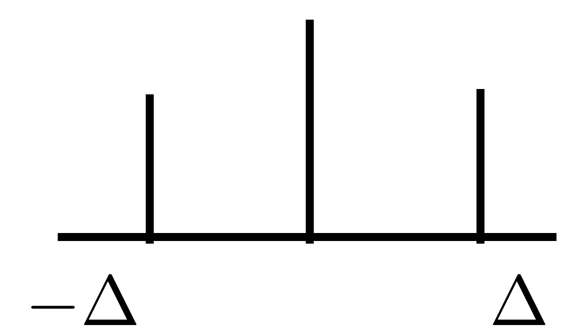
- Given uniform initial (un-normalized) distribution

$$P_0(x) = \begin{cases} 1 & |x| < \Delta \\ 0 & |x| > \Delta \end{cases}$$



- Find final distribution

$$P_\infty(x) = ?$$



- Multitude of final steady-states

$$P_0(x) = \sum_{i=1}^{N} m_i\,\delta(x - x_i)$$

$$|x_i - x_j| > 1$$

- Dynamics selects one (deterministically!)

Multiple localized clusters

Further details

- Dynamic treatment

  Each individual interacts once per unit time

- Random interactions

  Two interacting individuals are chosen randomly

- Infinite particle limit is implicitly assumed

$$N \to \infty$$

- Process is galilean invariant $x \to x + x_0$

  Set average opinion to zero $\langle x \rangle = 0$

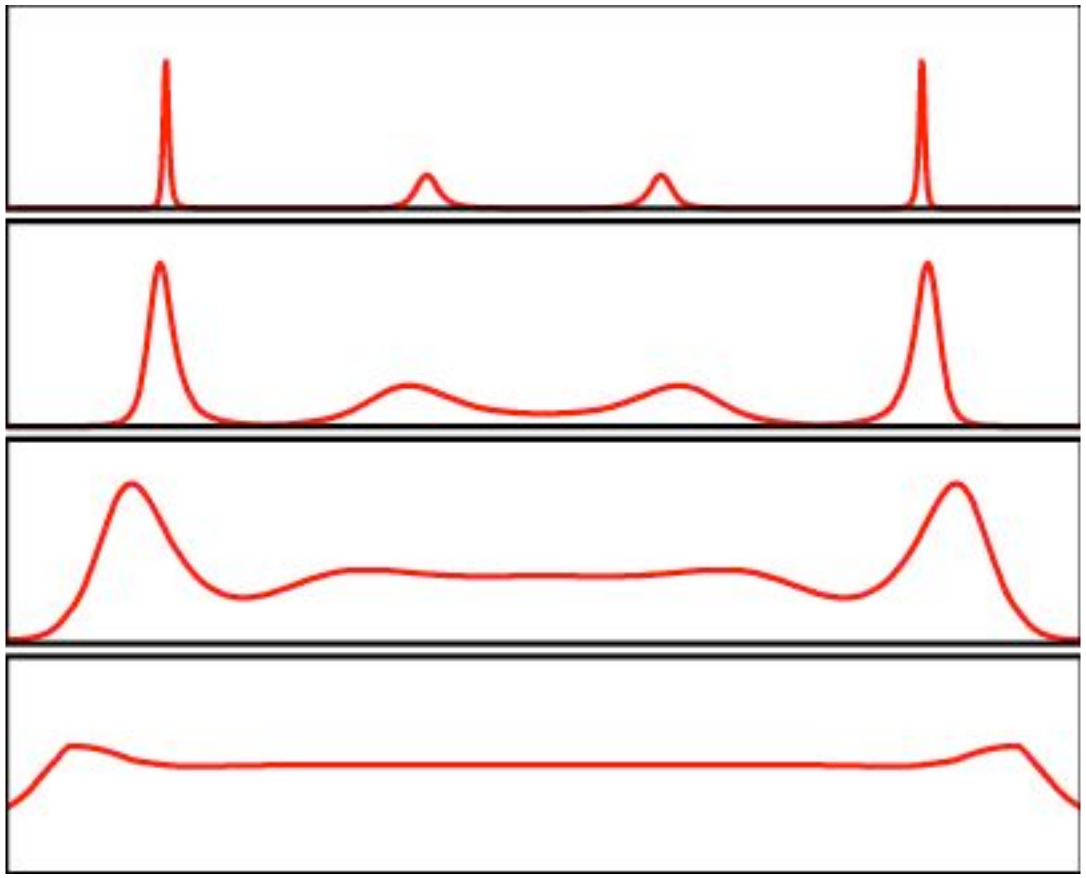
Numerical methods, kinetic theory

- Same master equation, restricted integration

$$\frac{\partial P(x,t)}{\partial t} = \iint\limits_{|x_1 - x_2| < 1} dx_1 dx_2 P(x_1,t)P(x_2,t)\left[\delta\left(x - \frac{x_1 + x_2}{2}\right) - \delta(x - x_1)\right]$$

☐ Direct Monte Carlo simulation of stochastic process

☑ Numerical integration of rate equations



Two Conservation Laws

- Total population is conserved

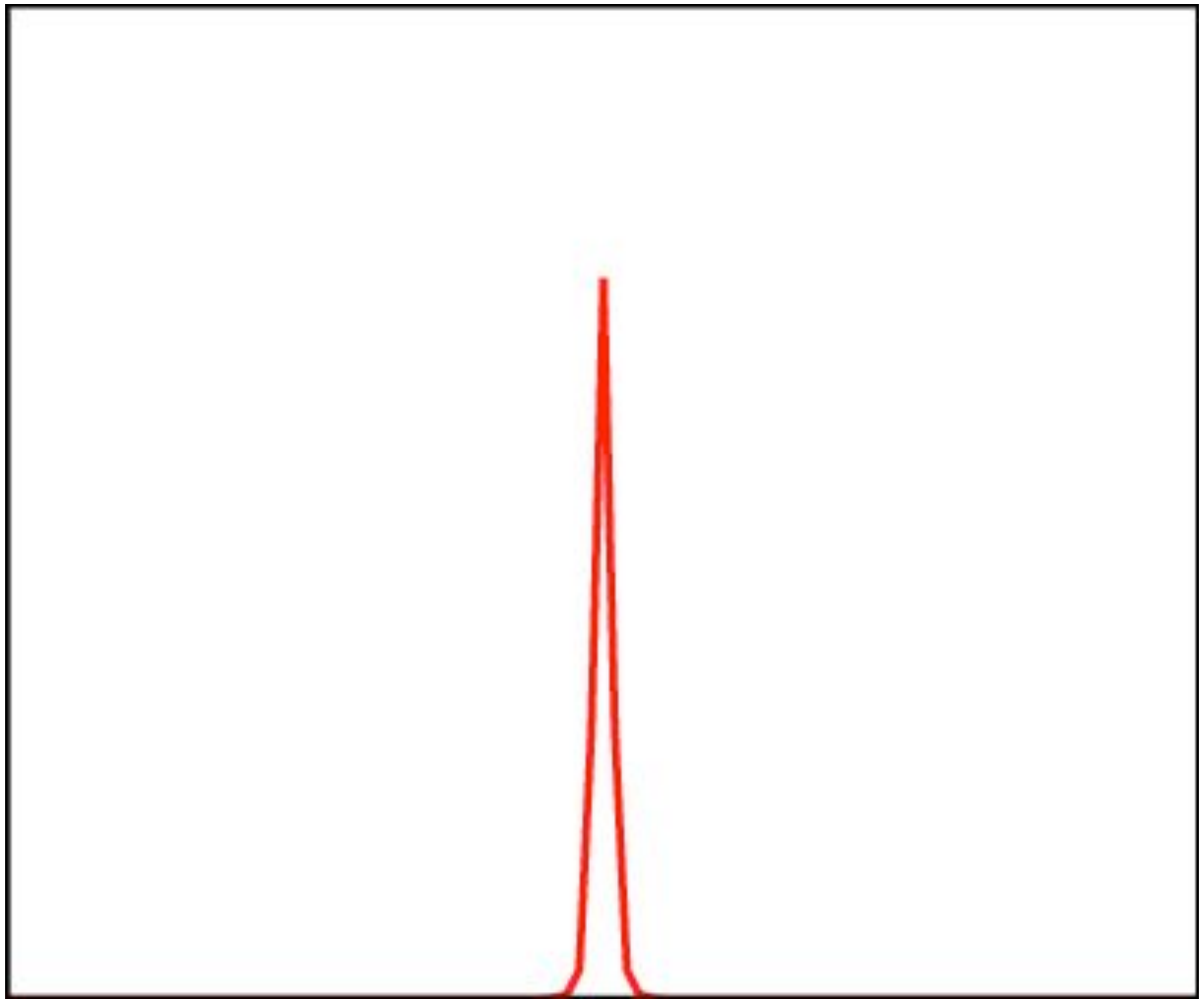
$$\int_{-\Delta}^{\Delta} dx\, P(x) = 2\Delta$$

- Average opinion is conserved

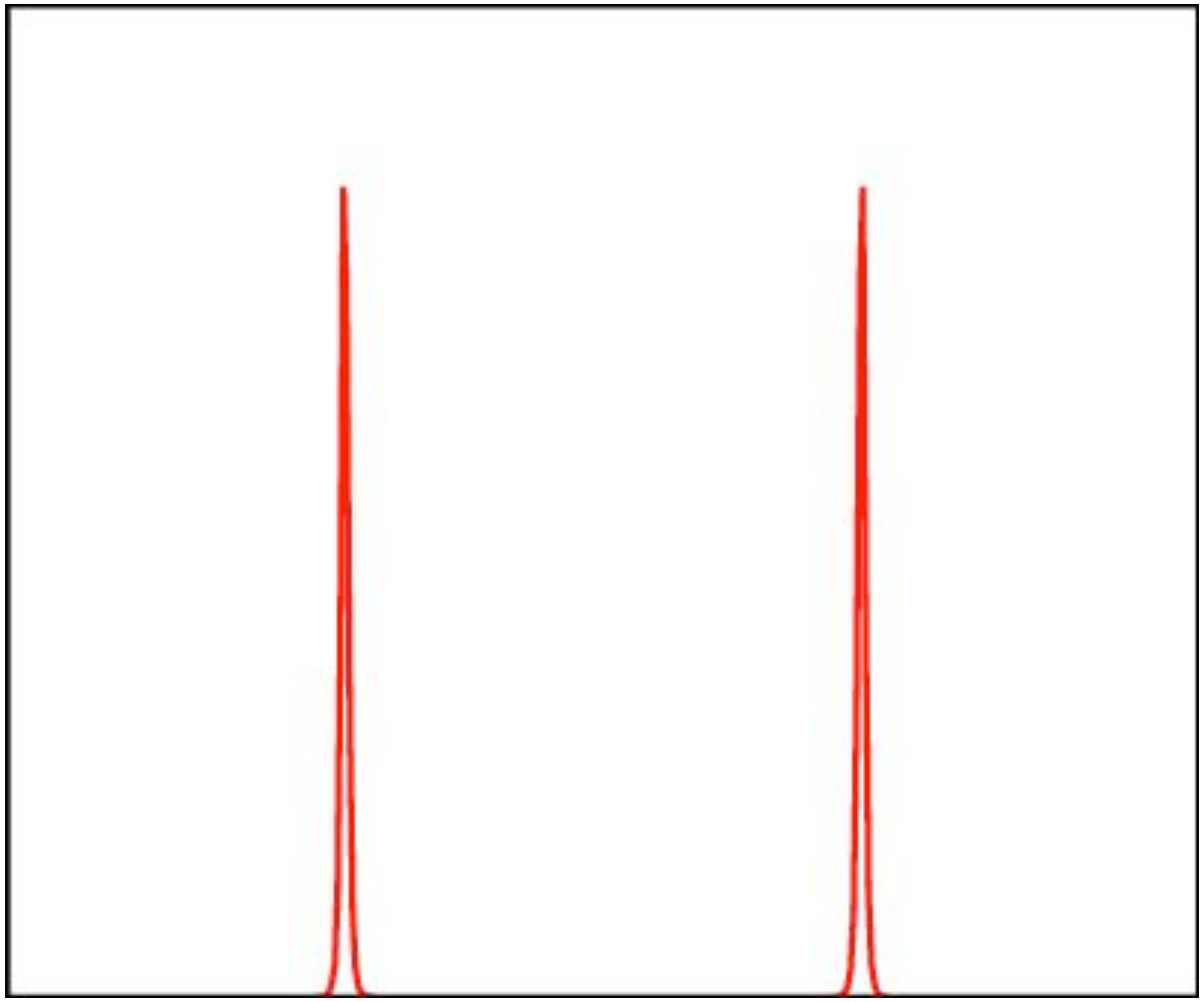
$$\int_{-\Delta}^{\Delta} dx\, x\, P(x) = 0$$

Rise and fall of central party

$0 < \Delta < 1.871$



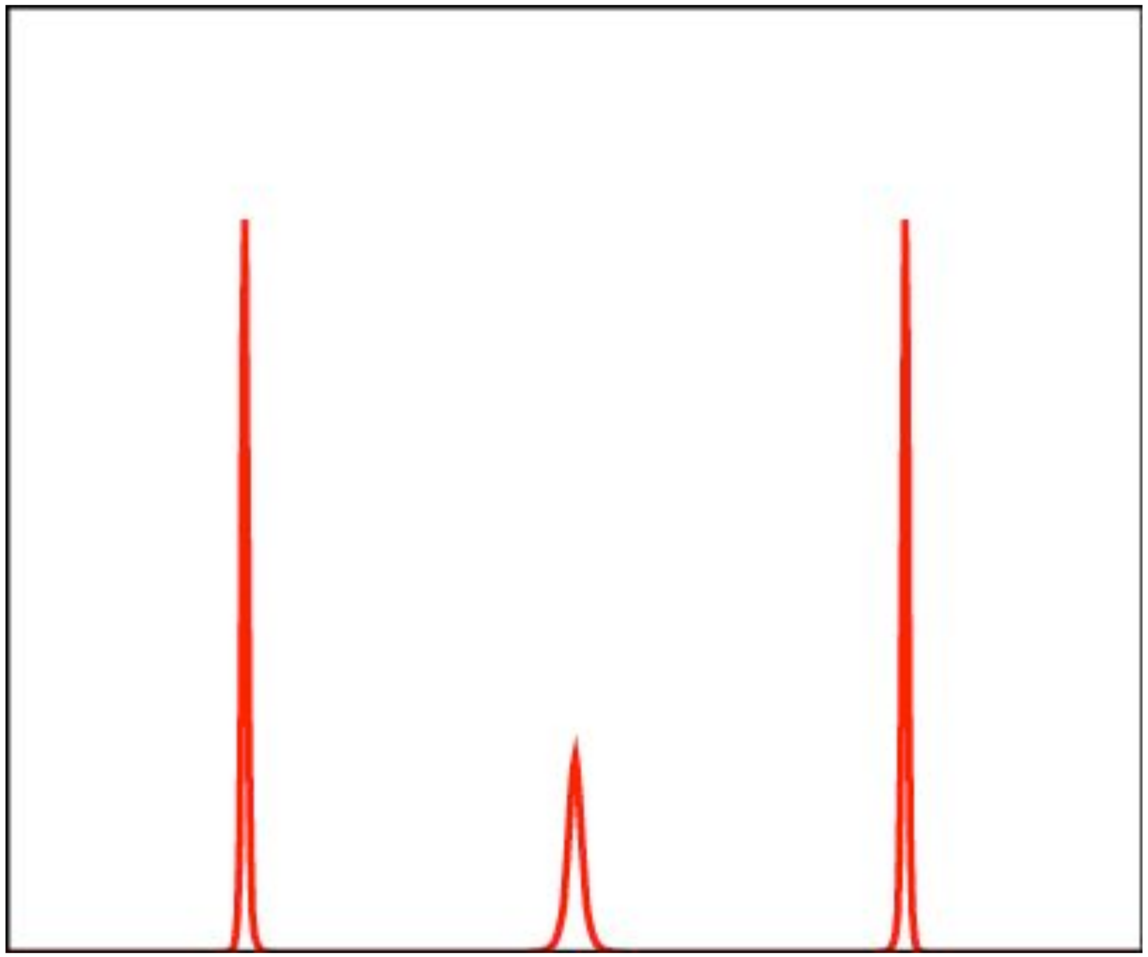
$1.871 < \Delta < 2.724$



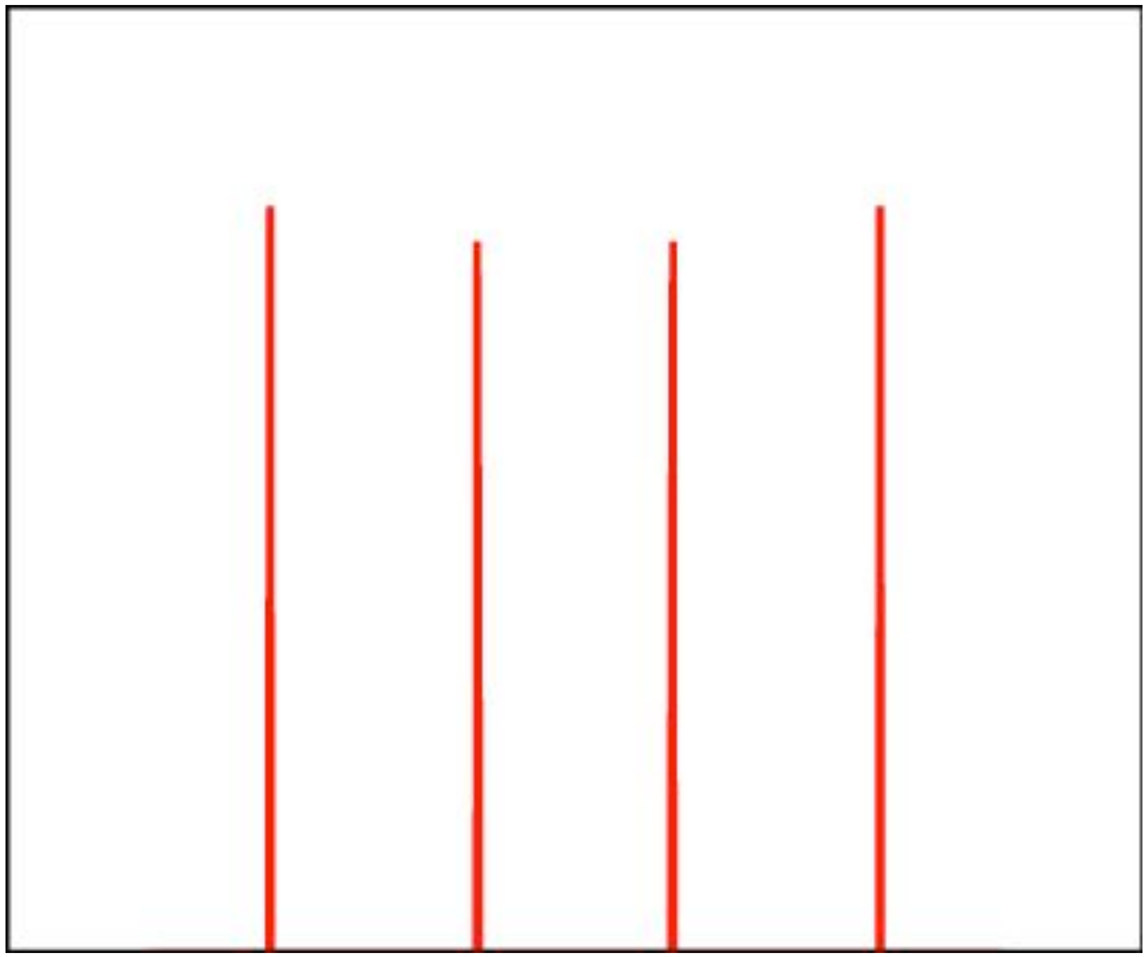
Central party may or may not exist!

Resurrection of central party

$2.724 < \Delta < 4.079$

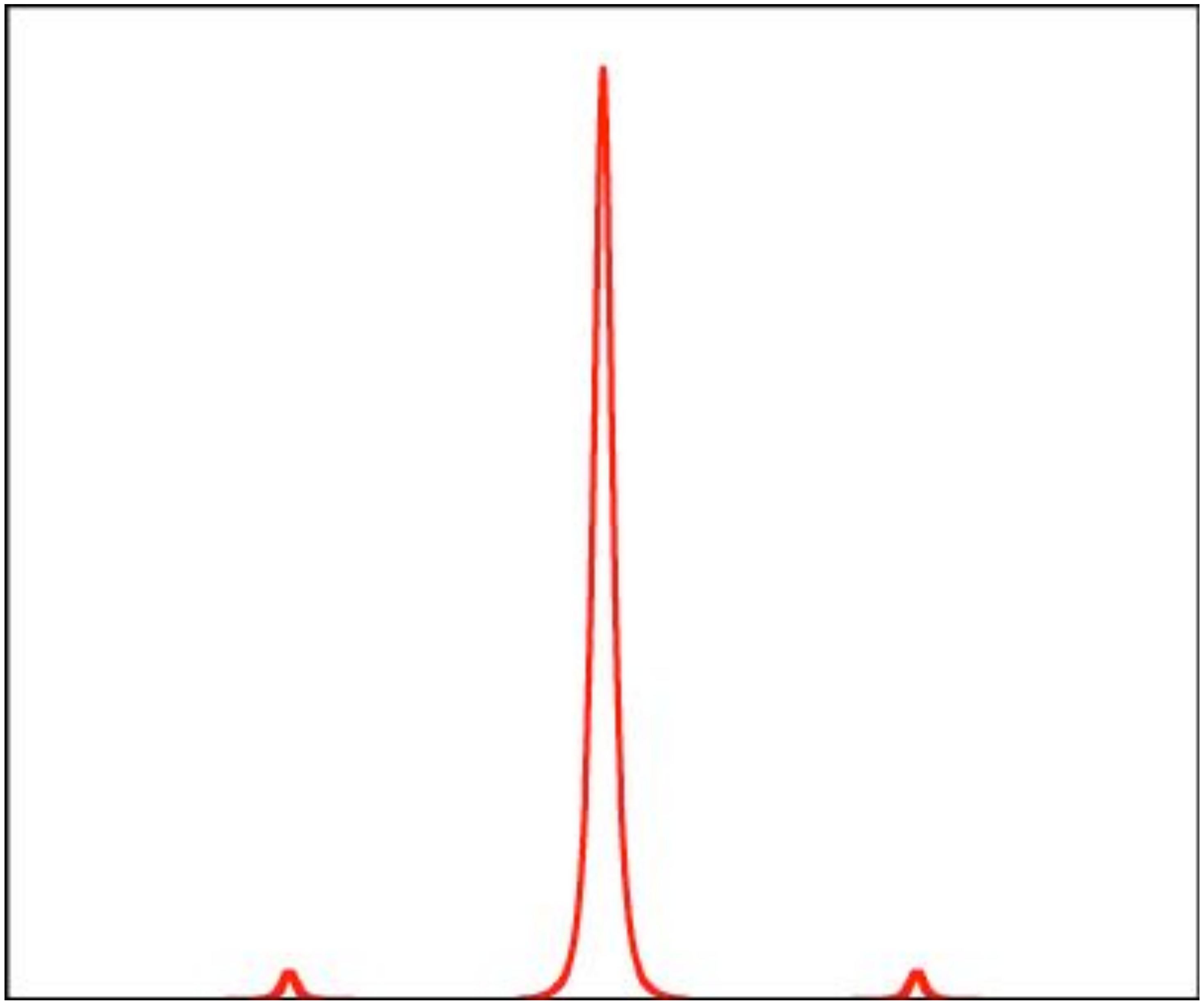


$4.079 < \Delta < 4.956$



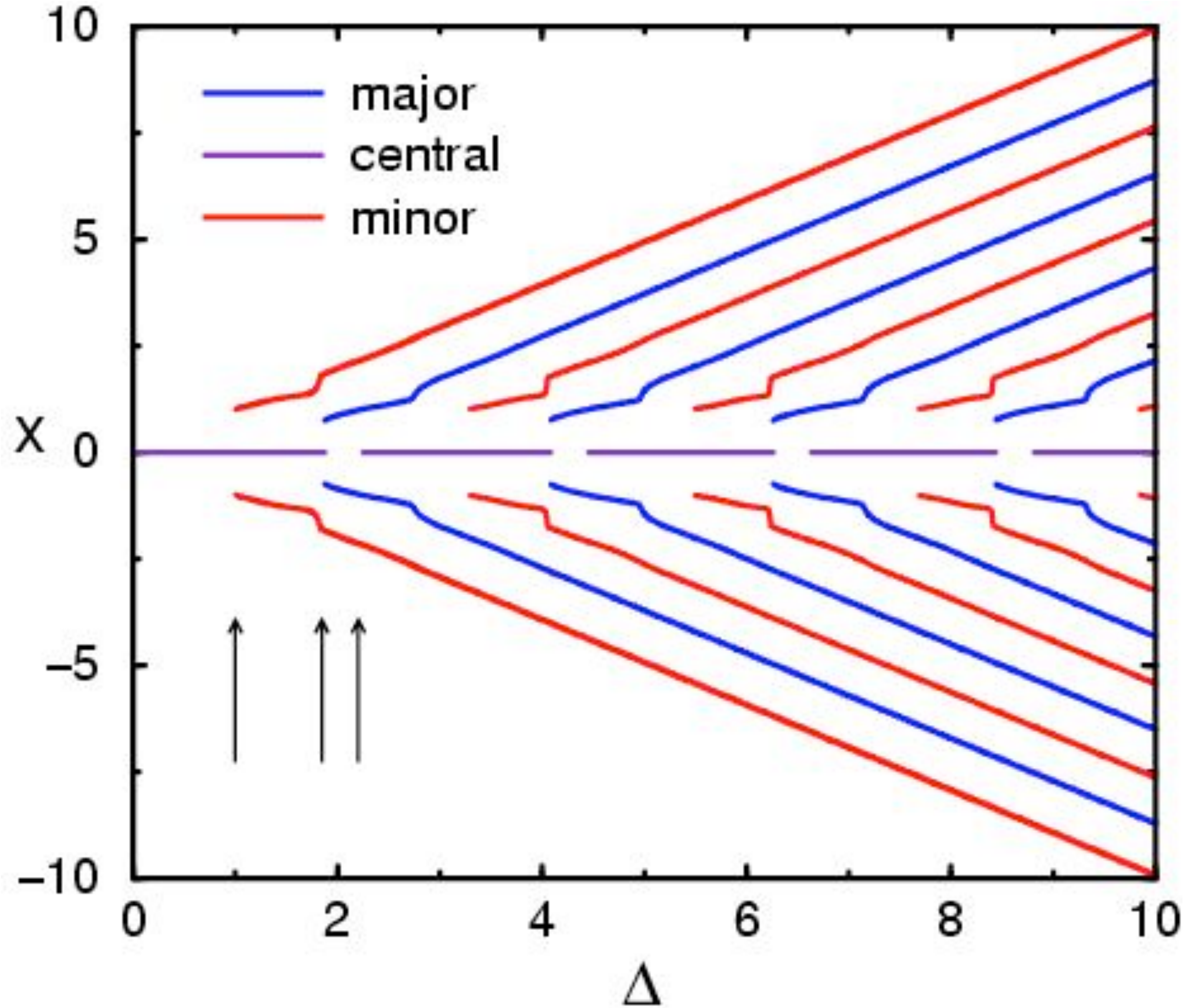
Parties may or may not be equal in size

Emergence of extremists



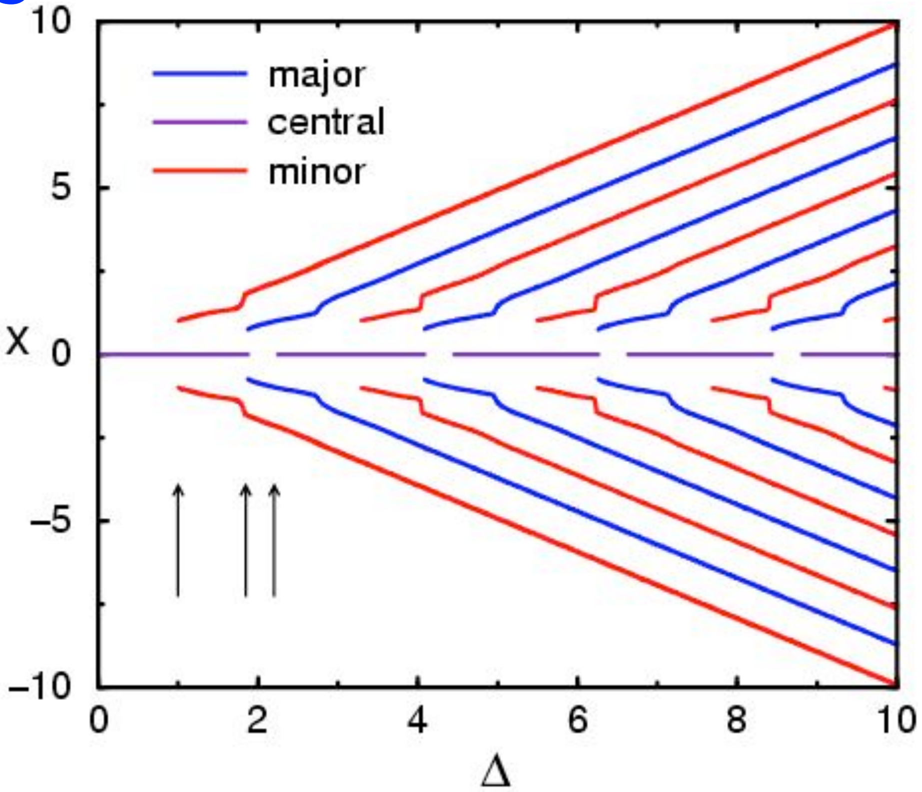
Tiny fringe parties ($m \sim 10^{-3}$)

Bifurcations and Patterns



Self-similar structure, universality

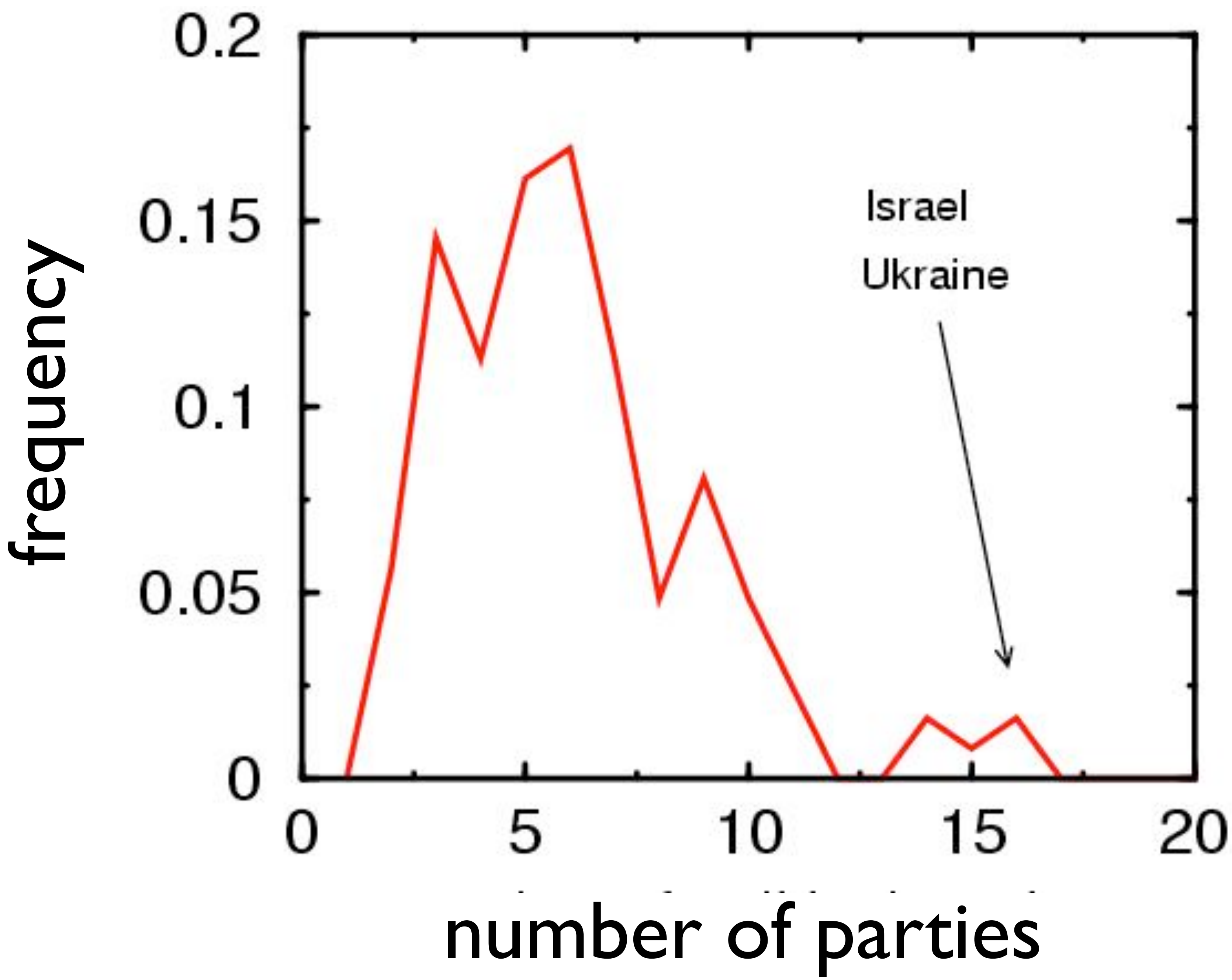
- Periodic sequence of bifurcations
    1. Nucleation of minor cluster branch
    2. Nucleation of major cluster brunch
    3. Nucleation of central cluster



- Alternating major-minor pattern
- Clusters are equally spaced
- Period L gives major cluster mass, separation

$$x(\Delta) = x(\Delta) + L \qquad L = 2.155$$

How many political parties?



- Data: CIA world factbook
- 120 countries with multi-party parliaments
- Average=5.8; Standard deviation=2.9

Cluster mass

- Masses are periodic

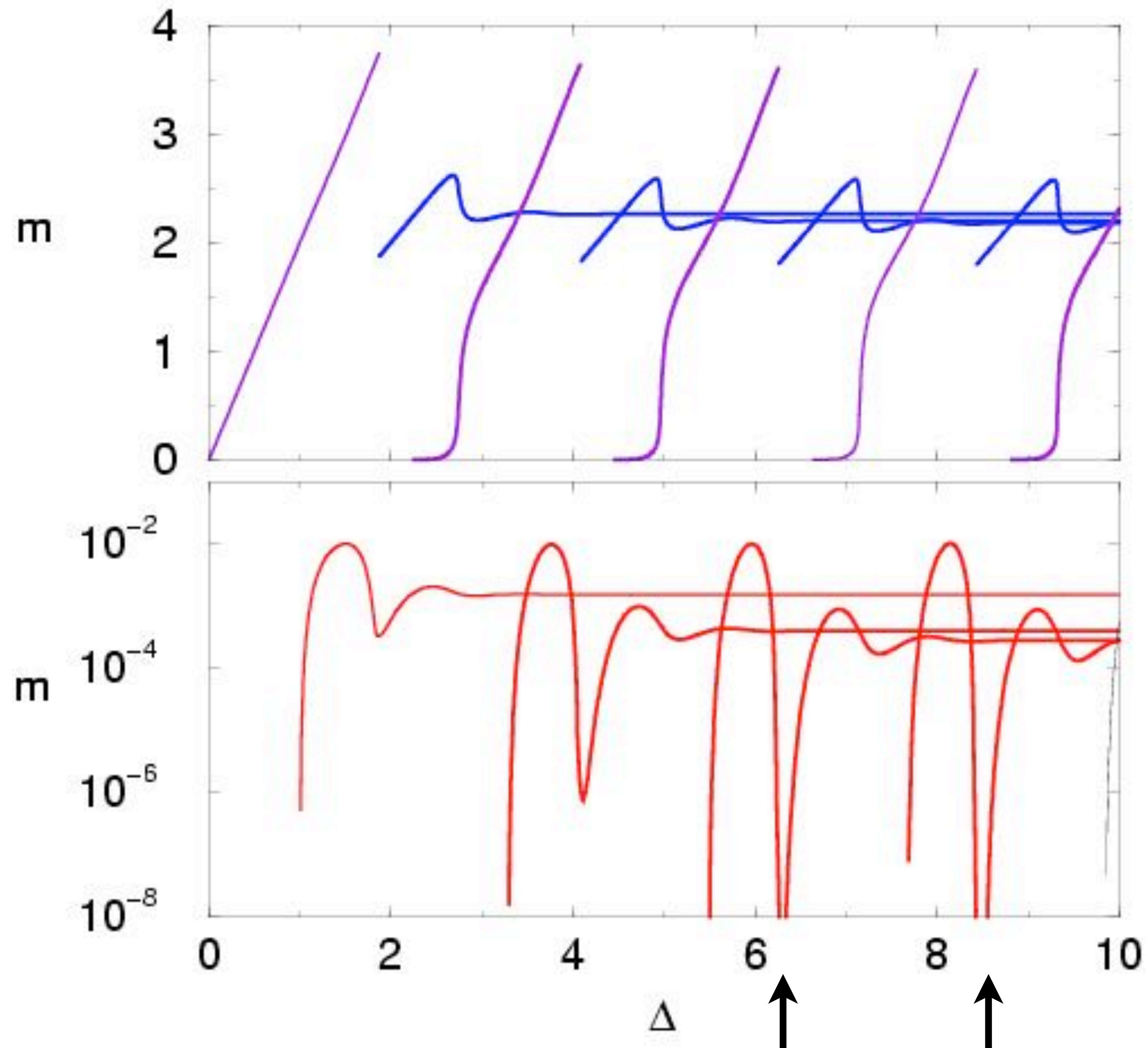
$$m(\Delta) = m(\Delta + L)$$

- Major mass

$$M \to L = 2.155$$

- Minor mass

$$m \to 3 \times 10^{-4}$$



Why are the minor clusters so small?

gaps?

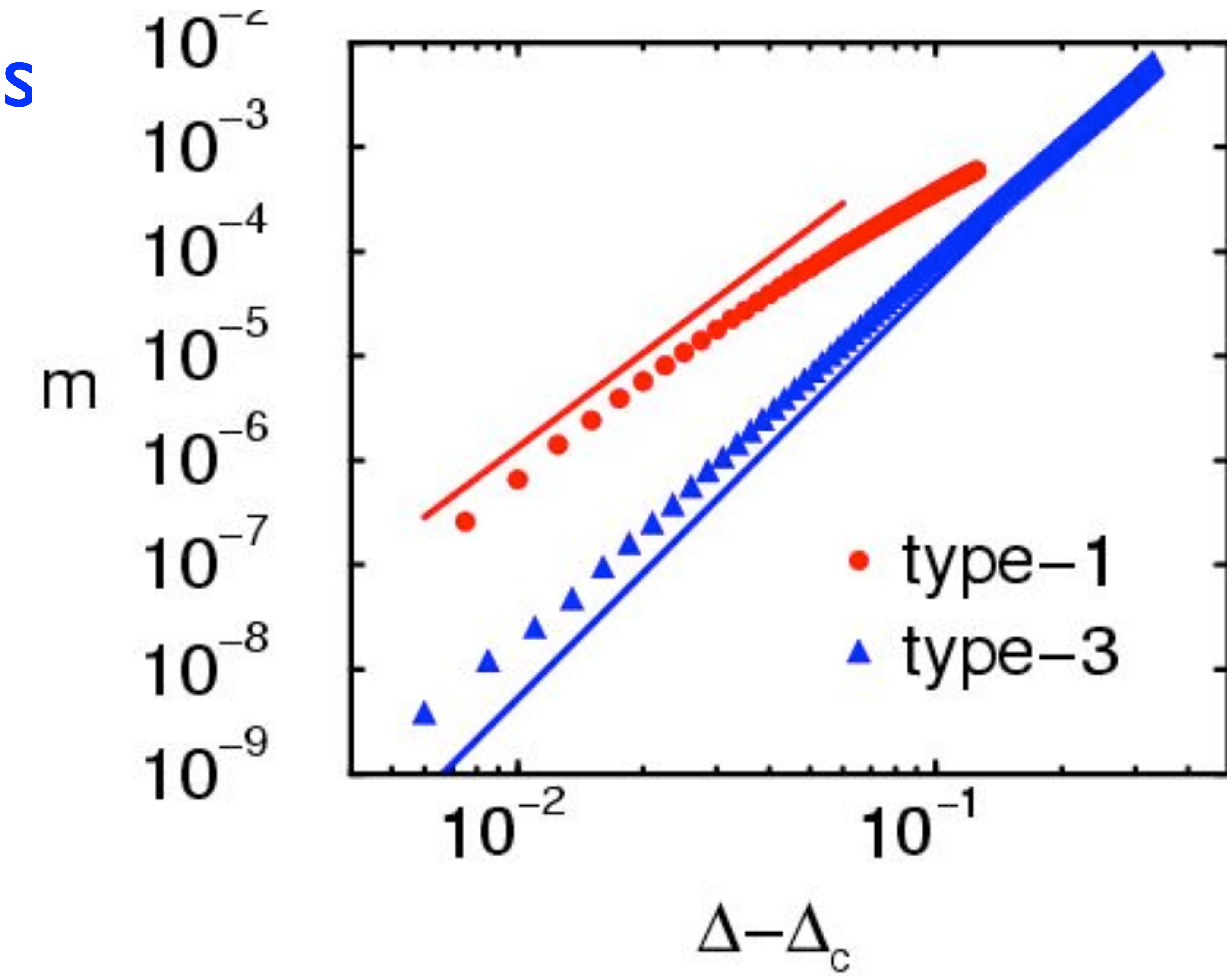
Scaling near bifurcation points

- Minor mass vanishes

$$m \sim (\Delta - \Delta_c)^{\alpha}$$

- Universal exponent

$$\alpha = \begin{cases} 3 & \text{type } 1 \\ 4 & \text{type } 3 \end{cases}$$



L-2 is the small parameter
explains small saturation mass

Consensus dynamics

- Integrable for $\Delta < 1/2$

$$\langle x^2(t)\rangle = \langle x^2(0)\rangle\, e^{-\Delta t}$$

- Final state: localized

$$P_\infty(x) = 2\Delta\,\delta(x)$$

- Rate equations in Fourier space

$$P_t(k) + P(k) = P^2(k/2)$$

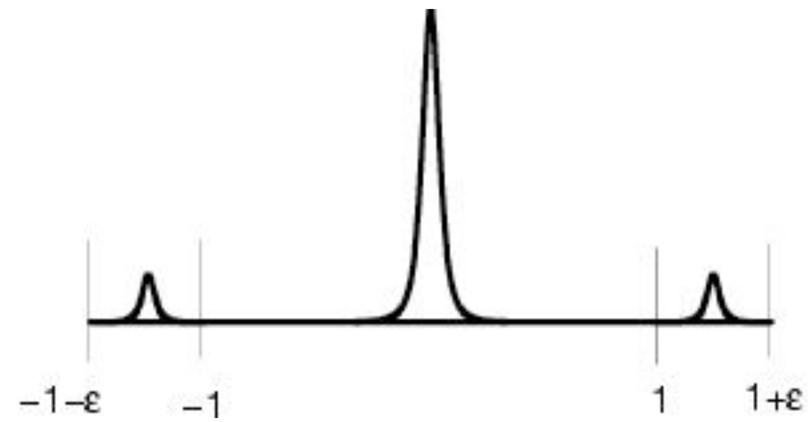
- Self-similar collapse dynamics

$$\Phi(z) \propto \left(1 + z^2\right)^{-2} \qquad z = x/\sqrt{\langle x^2\rangle}$$

The Inelastic Maxwell Model, EB & PL Krapivsky, Lecture Notes in Physics **624,** 65 (2003)

Heuristic derivation of exponent

- Perturbation theory $\Delta = 1 + \epsilon$
- Major cluster $x(\infty) = 0$
- Minor cluster $x(\infty) = \pm(1 + \epsilon/2)$



- Rate of transfer from minor cluster to major cluster

$$\frac{dm}{dt} = -m\,M \longrightarrow m \sim \epsilon\, e^{-t}$$

- Process stops when

$$x \sim e^{-t_f/2} \sim \epsilon \qquad \langle x^2 \rangle \sim e^{-t}$$

- Final mass of minor cluster

$$m(\infty) \sim m(t_f) \sim \epsilon^3 \qquad \alpha = 3$$

van Saarloos 03
scheel 10

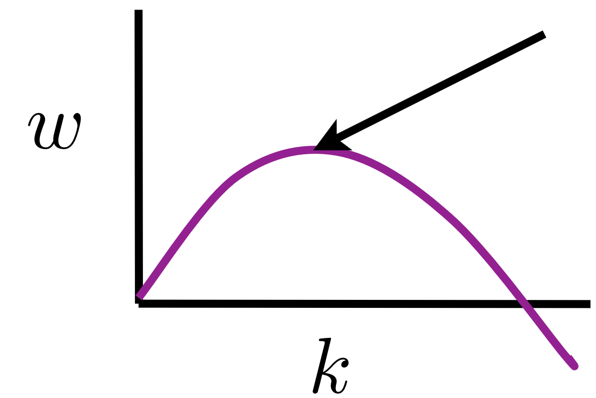
Pattern selection

- Linear stability analysis

$$P - 1 \propto e^{i(kx+wt)} \quad \Longrightarrow \quad w(k) = \frac{8}{k}\sin\frac{k}{2} - \frac{2}{k}\sin k - 2$$

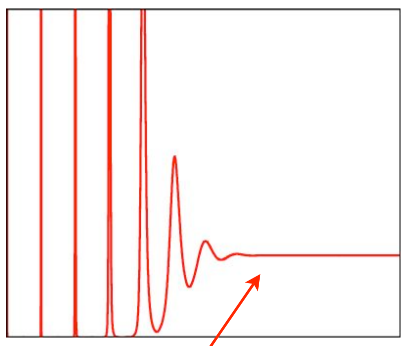
- Fastest growing mode

$$\frac{dw}{dk} \quad \Longrightarrow \quad L = \frac{2\pi}{k} = 2.2515$$



- Traveling wave (FKPP saddle point analysis)

$$v = \frac{dw}{dk} = \frac{\mathrm{Im}[w]}{\mathrm{Im}[k]} \quad \Longrightarrow \quad k_{\mathrm{select}} = k_* - \frac{w_*}{v}$$



Patterns induced by wave propagation from boundary
However, emerging period is different

$L_* = 2.037514$

$L_{\mathrm{num}} = 2.155$

$$L_{\mathrm{select}} = \frac{2\pi}{k_{\mathrm{select}}} = 2.148644$$

Wavelength obtained analytically!

Discrete opinions

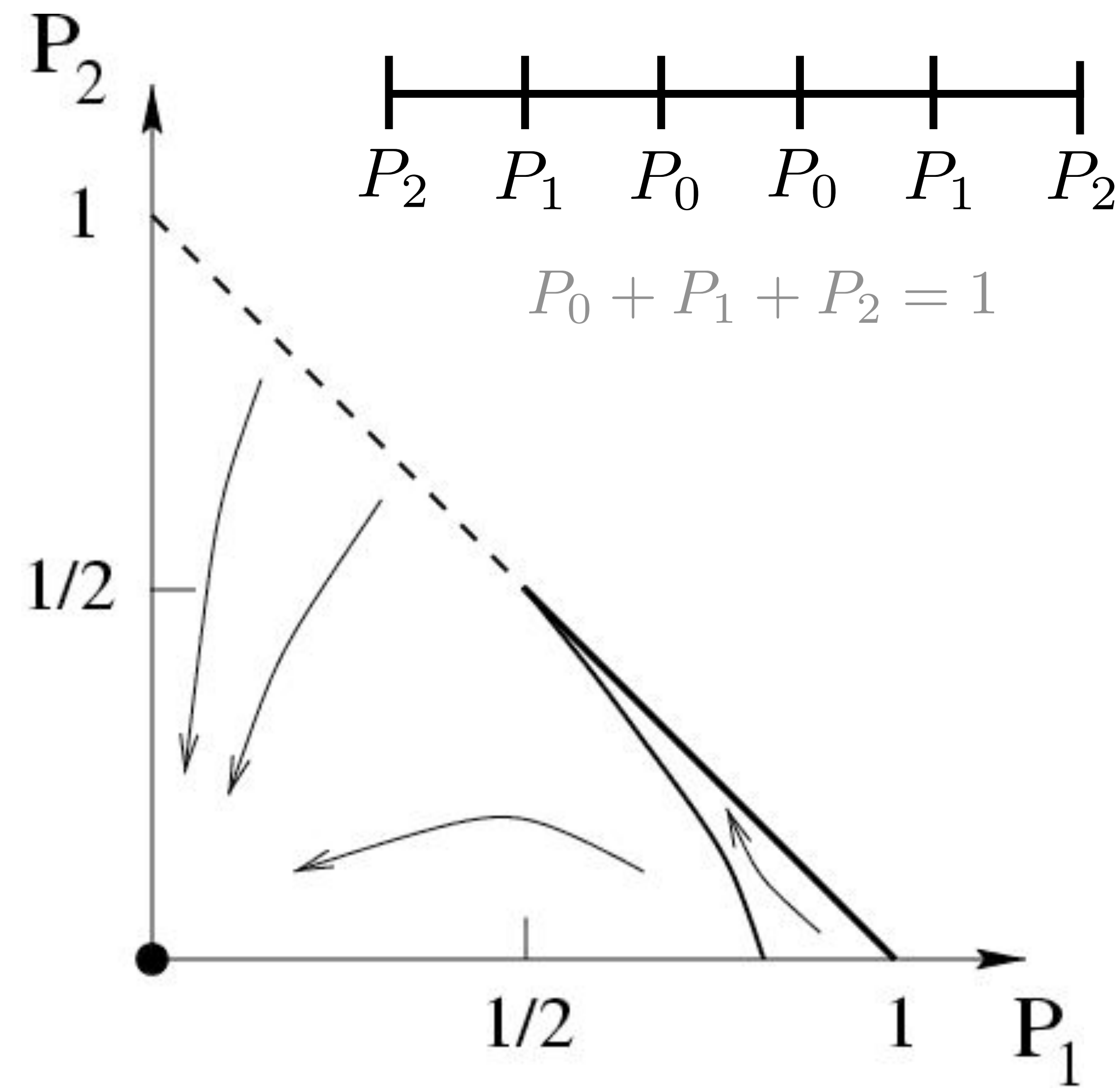
- Compromise process

$$(n-1, n+1) \to (n, n)$$

- Master equation

$$\frac{dP_n}{dt} = 2P_{n-1}P_{n+1} - P_n(P_{n-2} + P_{n+2})$$

- Simplest example: 6 states
- Symmetry + normalization:
- Two-dimensional problem

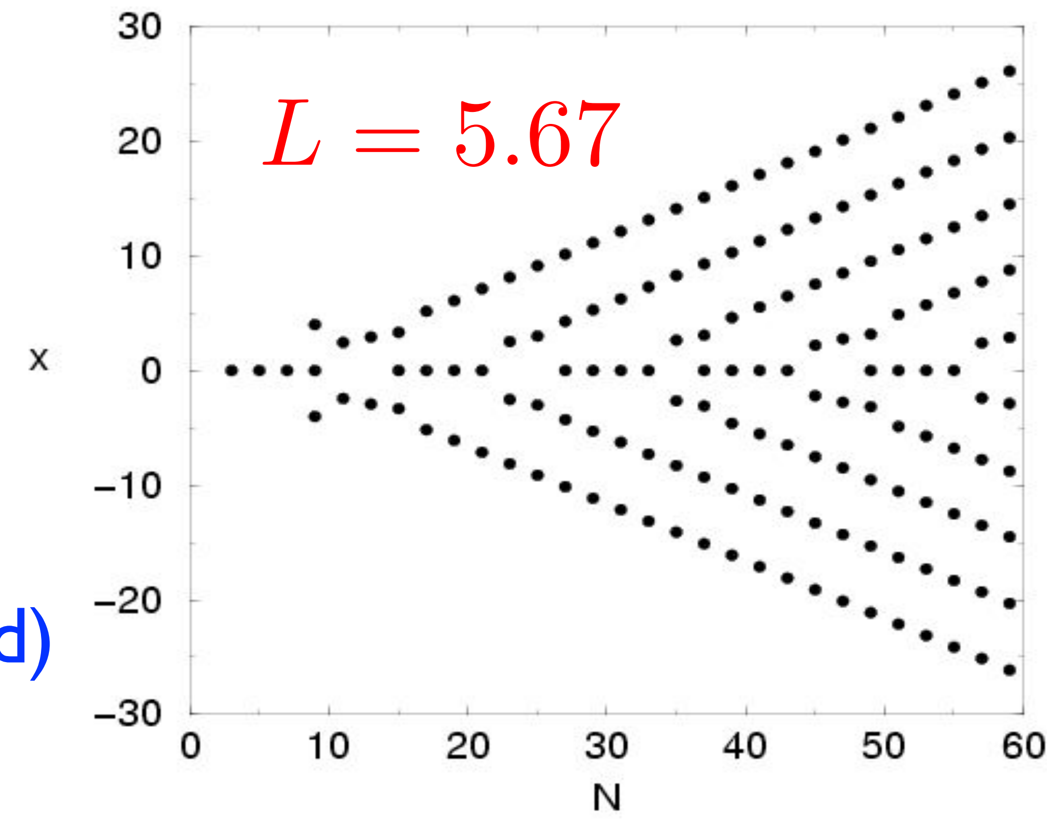


Initial condition determines final state

Isolated fixed points, lines of fixed points

Discrete opinions

- Dissipative system, volume contracts
- Energy (Lyapunov) function exists
- No cycles or strange attractors
- Uniform state is unstable (Cahn-Hilliard)



$$P_i = 1 + \phi_i \qquad \phi_t + (\phi + a\,\phi_{xx} + b\,\phi^2)_{xx}$$

Discrete case yields useful insights

Pattern selection

- Linear stability analysis

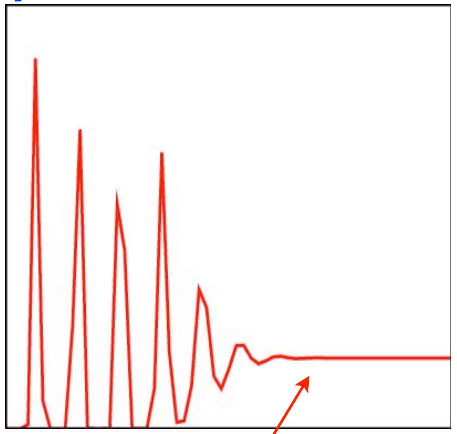
$$P - 1 \propto e^{i(kx+wt)} \longrightarrow w(k) = 4\cos k - 4\cos 2k - 2$$

- Fastest growing mode

$$\frac{dw}{dk} \quad \Longrightarrow \quad L = \frac{2\pi}{k} = 6$$

- Traveling wave (FKPP saddle point analysis)

$$v = \frac{dw}{dk} = \frac{\mathrm{Im}[w]}{\mathrm{Im}[k]} \quad \Longrightarrow \quad k_{\text{select}} = k_* - \frac{w_*}{v}$$



$$L_* = 5.671820$$

$L_* = 5.311086$

Again, wavelength can be obtained analytically

I. Conclusions

- Clusters form via bifurcations
- Periodic structure
- Alternating major-minor pattern
- Central party does not always exist
- Power-law behavior near transitions
- Pattern selection understood

I. Outlook

- Gaps
- Role of initial conditions, classification
- Role of spatial dimension, correlations
- Disorder, inhomogeneities
- Tiling/Packing in 2D
- Discord dynamics (seceder model, Halpin-Heally 03)

Many open questions

II. Noisy compromise dynamics

Diffusion (noise)

- **Diffusion:** Individuals change opinion spontaneously

$$n \xrightarrow{D} n \pm 1$$

or

- Adds noise (“temperature”)
- Linear process: no interaction
- Mimics unstable, varying opinion
- Influence of environment, news, editorials, events

Rate equations

- Compromise: reached through pairwise interactions

$$(n-1, n+1) \to (n, n)$$

- Conserved quantities: total population, average opinion
- Probability distribution $P_n(t)$
- Kinetic theory: nonlinear rate equations

$$\frac{dP_n}{dt} = 2P_{n-1}P_{n+1} - P_n(P_{n-2} + P_{n+2}) + D(P_{n-1} + P_{n+1} - 2P_n)$$

- [ ] Direct Monte Carlo simulations of stochastic process
- [x] Numerical integration of rate equations

Single-party dynamics

- Initial condition: large isolated party

$$P_n(0) = m(\delta_{n,0} + \delta_{n,-1})$$

- Steady-state: compromise and diffusion balance

$$DP_n = P_{n-1}P_{n+1}$$

- Core of party: localized to a few opinion states

$$P_0 = m \qquad P_1 = D \qquad P_2 = D^2 m^{-1}$$

- Compromise negligible for n>2

Party has a well defined core

The tail

- Diffusion dominates outside the core

$$\frac{dP_n}{dt} = D(P_{n-1} + P_{n+1} - 2P_n) \qquad P \ll D$$

- Standard problem of diffusion with source

$$P_n \sim m^{-1}\Psi(n\,t^{-1/2})$$

- Tail mass

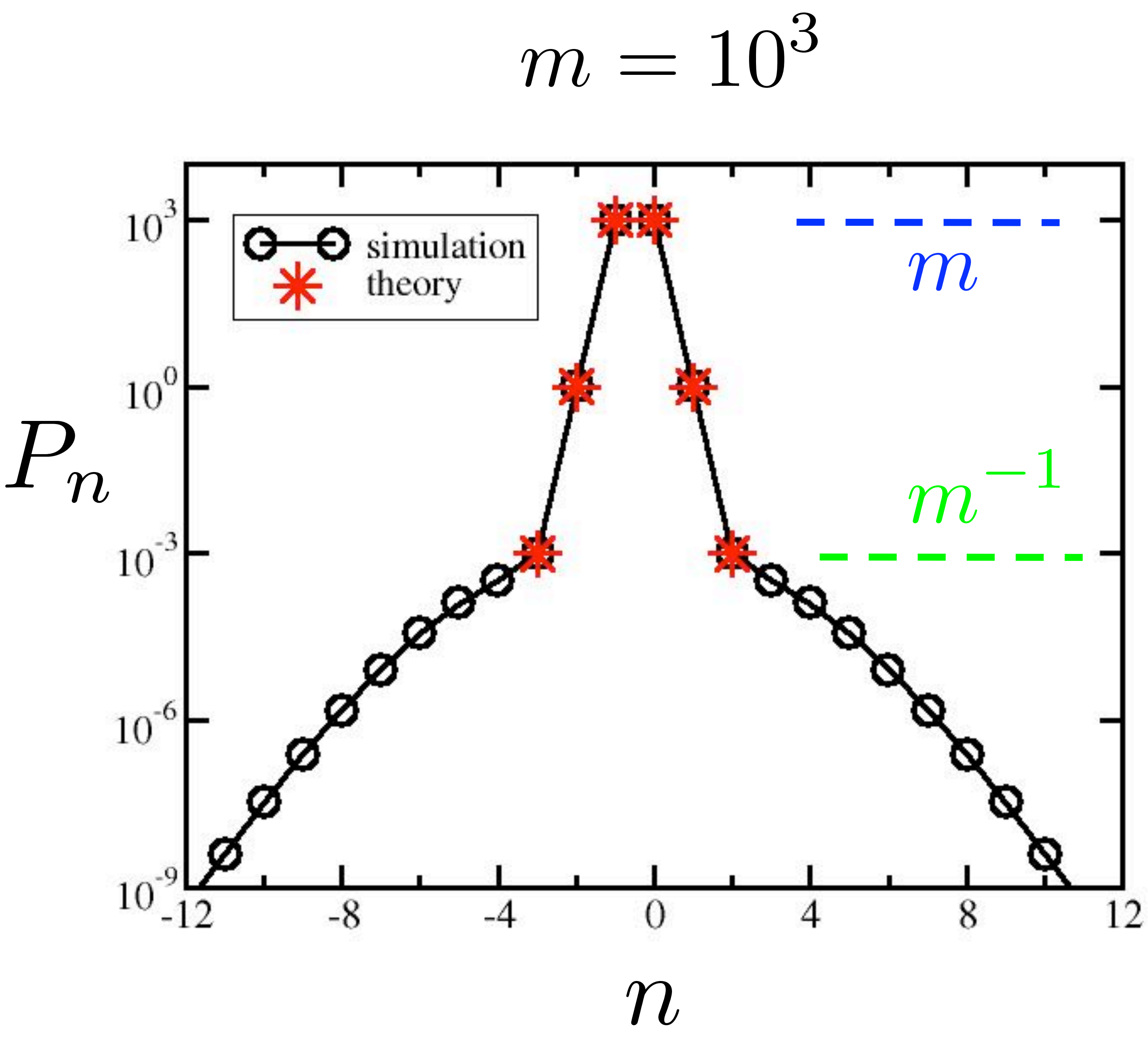
$$M_{\text{tail}} \sim m^{-1}t^{1/2}$$

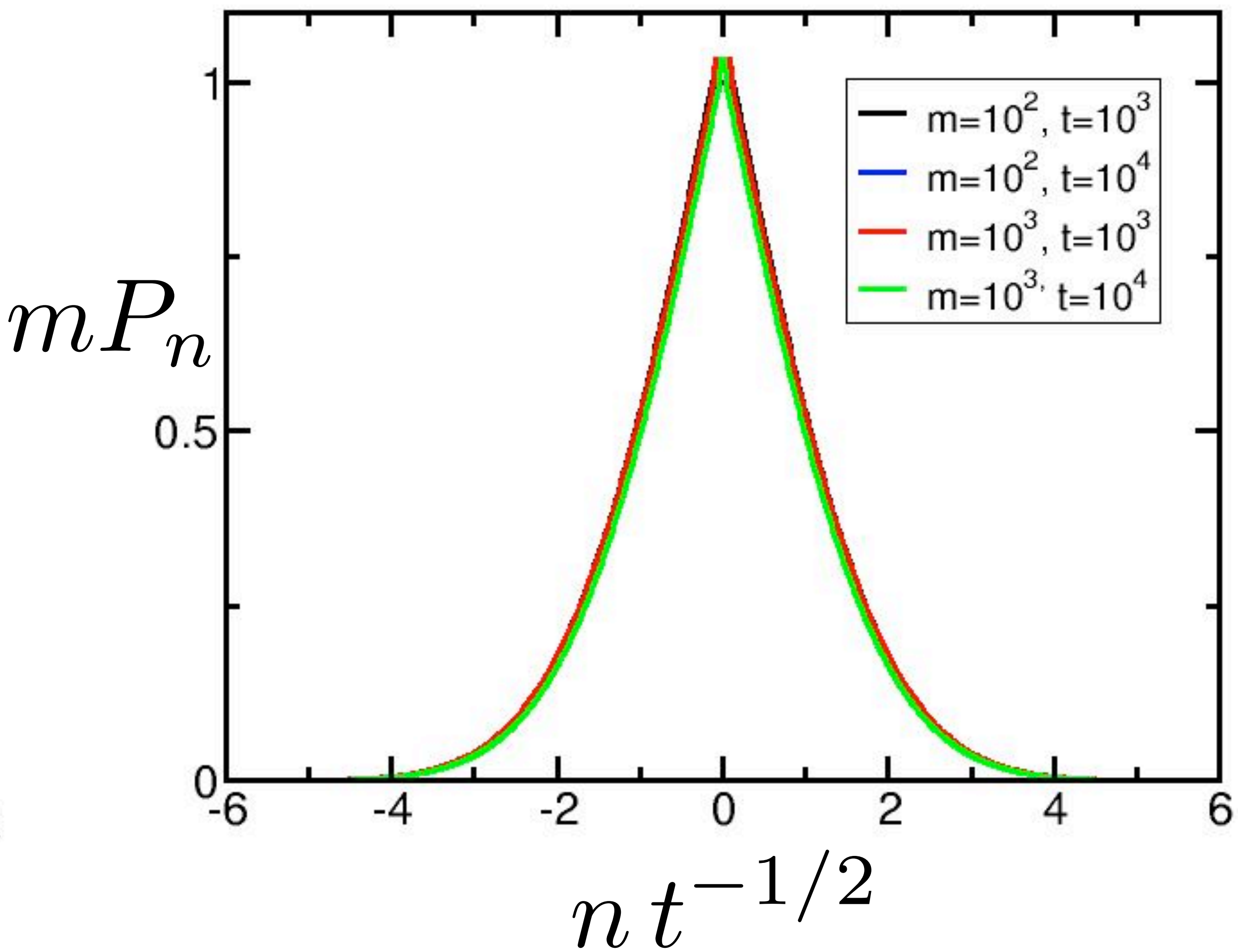
- Party dissolves when

$$M_{\text{tail}} \sim m \qquad \Longrightarrow \qquad \tau \sim m^4$$

Party lifetime grows dramatically with its size

Core versus tail





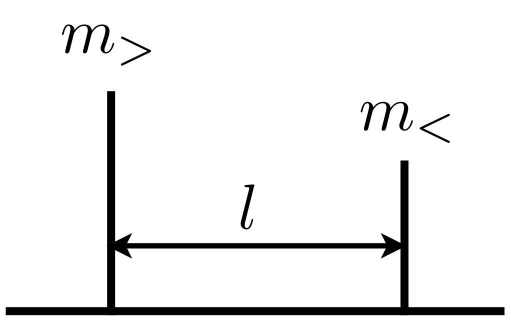
Party height=m
Party depth~m^{-1}

Self-similar shape
Gaussian tail

Qualitative features

- Exists in a quasi-steady state
- Tight core localized to a few sites
- Random opinion changes of members do not affect party position
- Party lifetime grows very fast with size
- Ultimate fate of a party: demise
- Its remnant: a diffusive cloud
- Depth inversely proportional to size, the larger the party the more stable

Two party dynamics



- Initial condition: two large isolated parties

$$P_n(0) = m_> \left(\delta_{n,0} + \delta_{n,-1}\right) + m_< \left(\delta_{n,l} + \delta_{n,l+1}\right)$$

- Interaction between parties mediated by diffusion

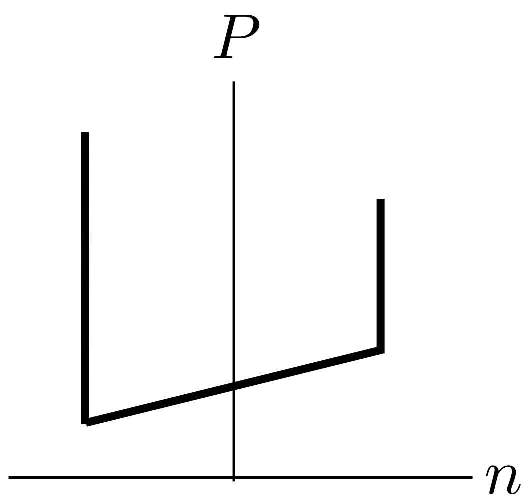
$$0 = P_{n-1} + P_{n+1} - 2P_n$$

- Boundary conditions set by parties depths

$$P_0 = \frac{1}{m_>} \qquad P_l = \frac{1}{m_<}$$

- Steady state: linear profile

$$P_n = \frac{1}{m_<} + \left(\frac{1}{m_<} - \frac{1}{m_>}\right)\frac{n}{l}$$



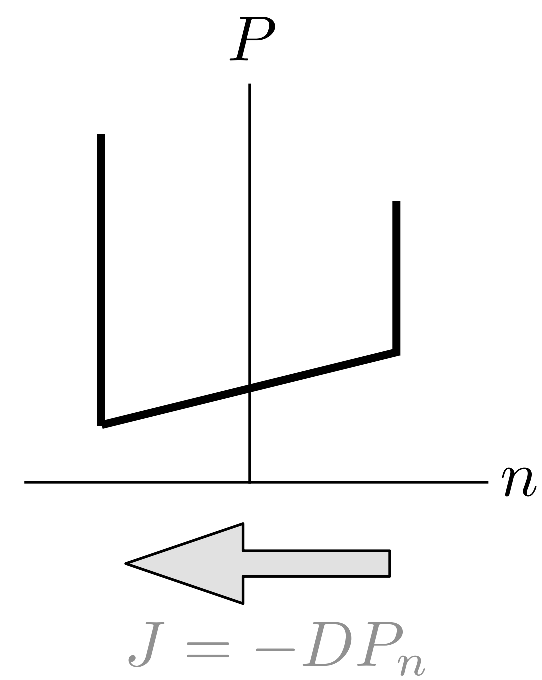
Merger

- Steady flux from small party to larger one

$$J \sim \frac{1}{l}\left(\frac{1}{m_<} - \frac{1}{m_>}\right) \sim \frac{1}{l m_<}$$

- Merger time

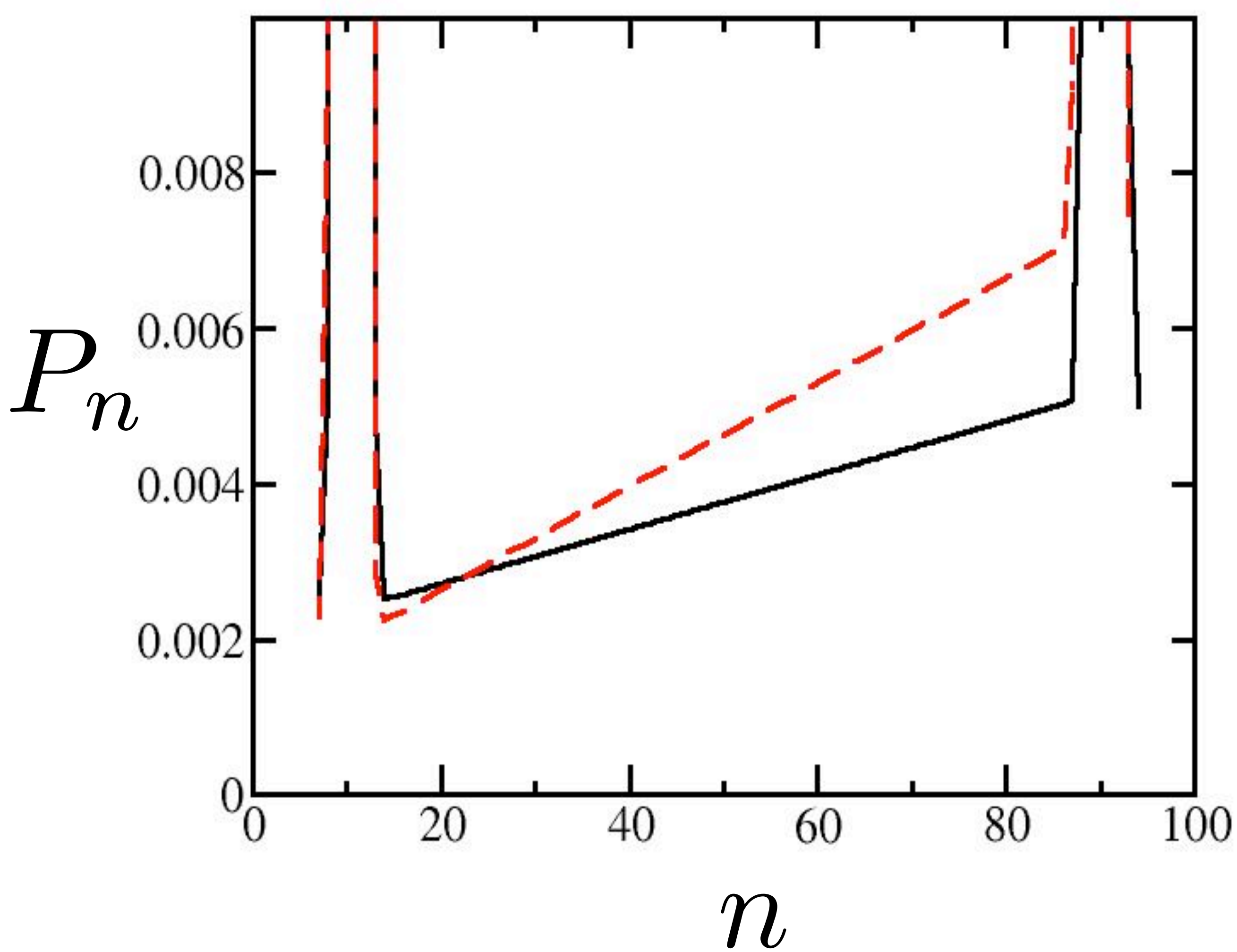
$$T \sim \frac{m_<}{J} \sim l m_<^2$$

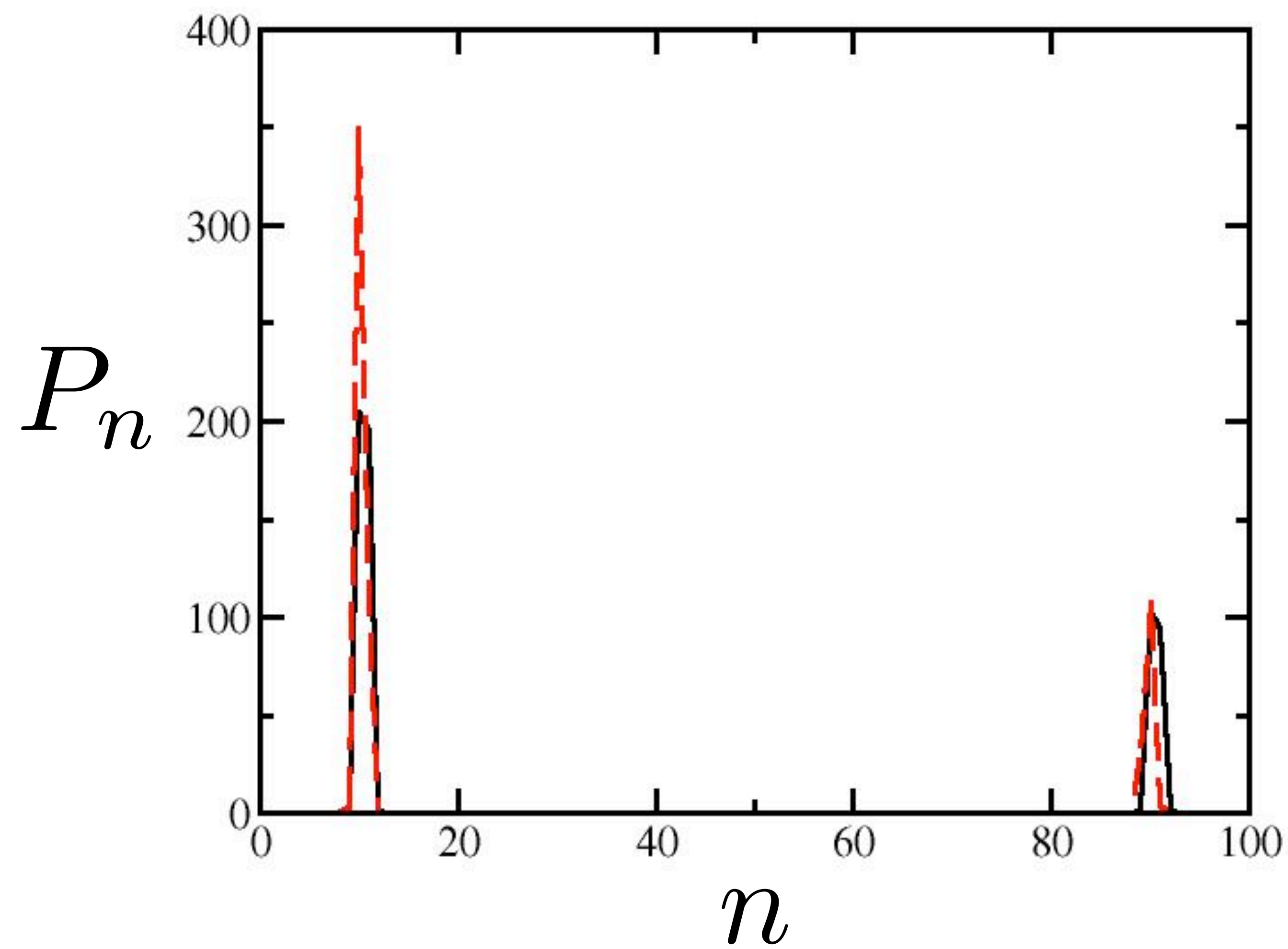


- Lifetime grows with separation (“niche”)
- Outcome of interaction is deterministic
- Larger party position remains fixed throughout merger process

Small party absorbed by larger one

Merger: numerical results





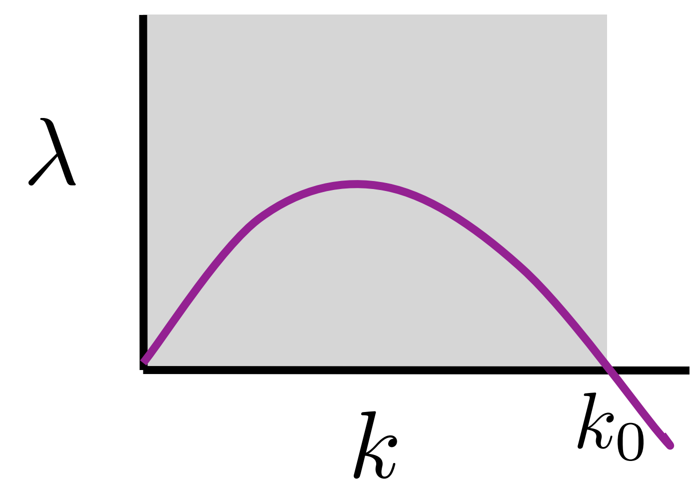
Multiple party dynamics

- Initial condition: large isolated party

$$P_n(0) = \text{randomly chosen number in } [1-\epsilon : 1+\epsilon]$$

- Linear stability analysis

$$P_n - 1 \sim e^{ikn+\lambda t}$$



- Growth rate of perturbations

$$\lambda(k) = (4\cos k - 4\cos 2k - 2) - 2D(1-\cos 2k)$$

- Long wavelength perturbations unstable

$$k < k_0 \qquad \cos k_0 = D/2$$

P=1 stable only for strong diffusion $D > D_c = 2$

Strong noise ($D>D_c$)

- Regardless of initial conditions

$$P_n \rightarrow \langle P_n(0) \rangle$$

- Relaxation time

$$\lambda \approx (D_c - D)k^2 \quad \Longrightarrow \quad \tau \sim (D - D_c)^{-2}$$

No parties, disorganized political system

Weak noise ($D<D_c$): Coarsening

- Smaller parties merge into large parties
- Party size grows indefinitely
- Assume a self-similar process, size scale m
- Conservation of populations implies separation

$$l \sim m$$

- Use merger time to estimate size scale

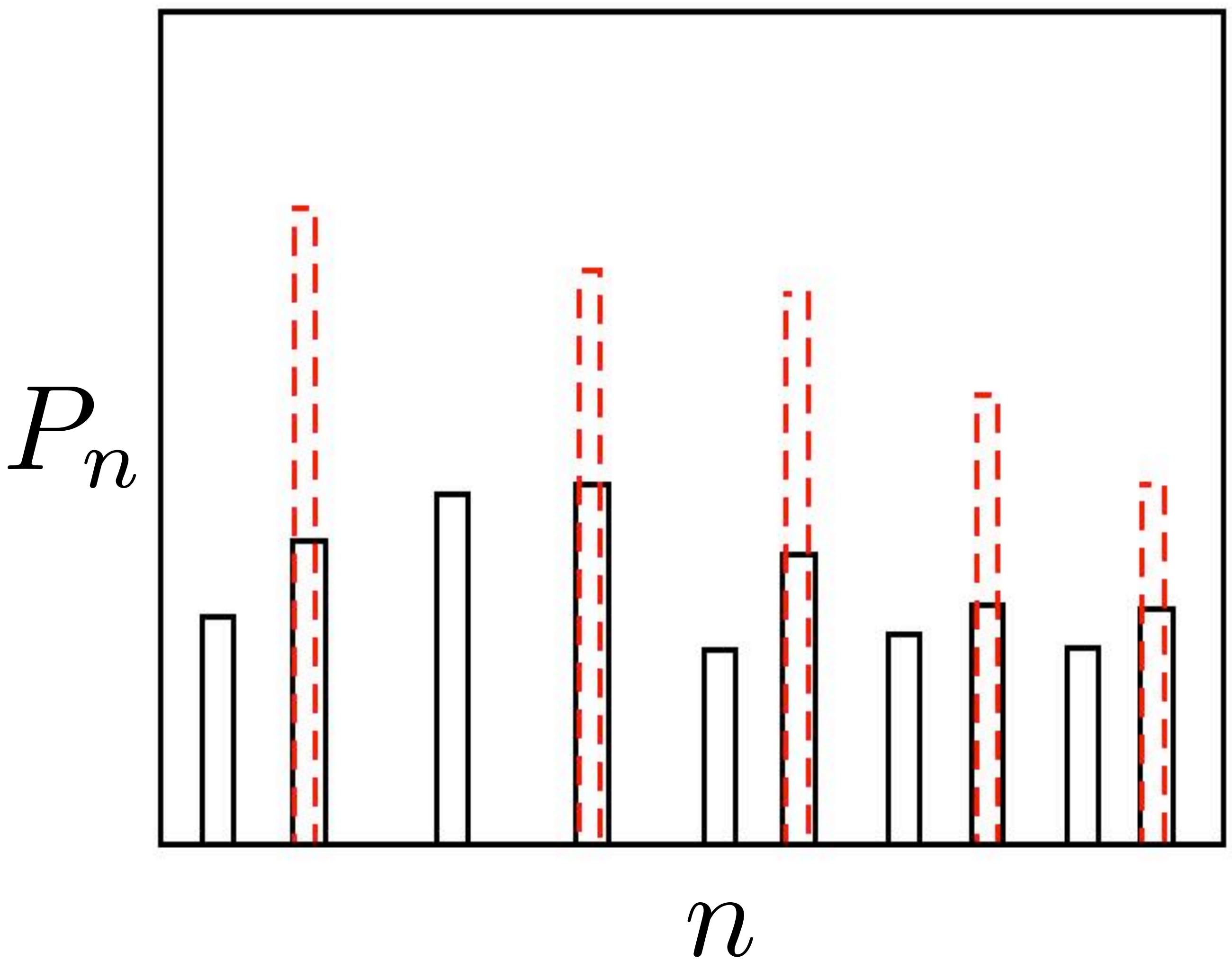
$$t \sim l m^2 \sim m^3 \qquad \Longrightarrow \qquad m \sim t^{1/3}$$

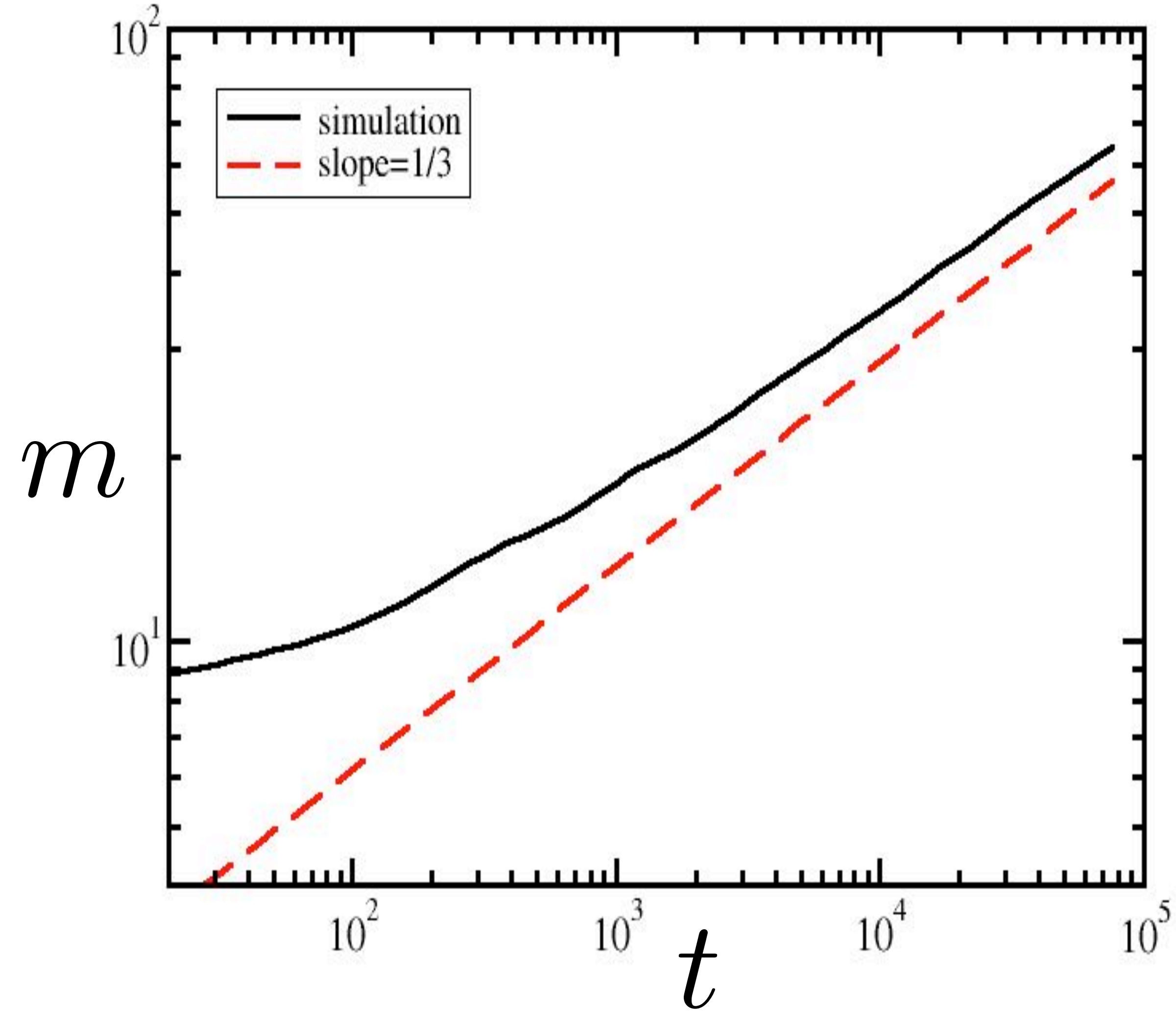
- Self-similar size distribution

$$P_m \sim t^{-1/3} F(m\, t^{-1/3})$$

Lifshitz-Slyozov coarsening

Coarsening: numerical results





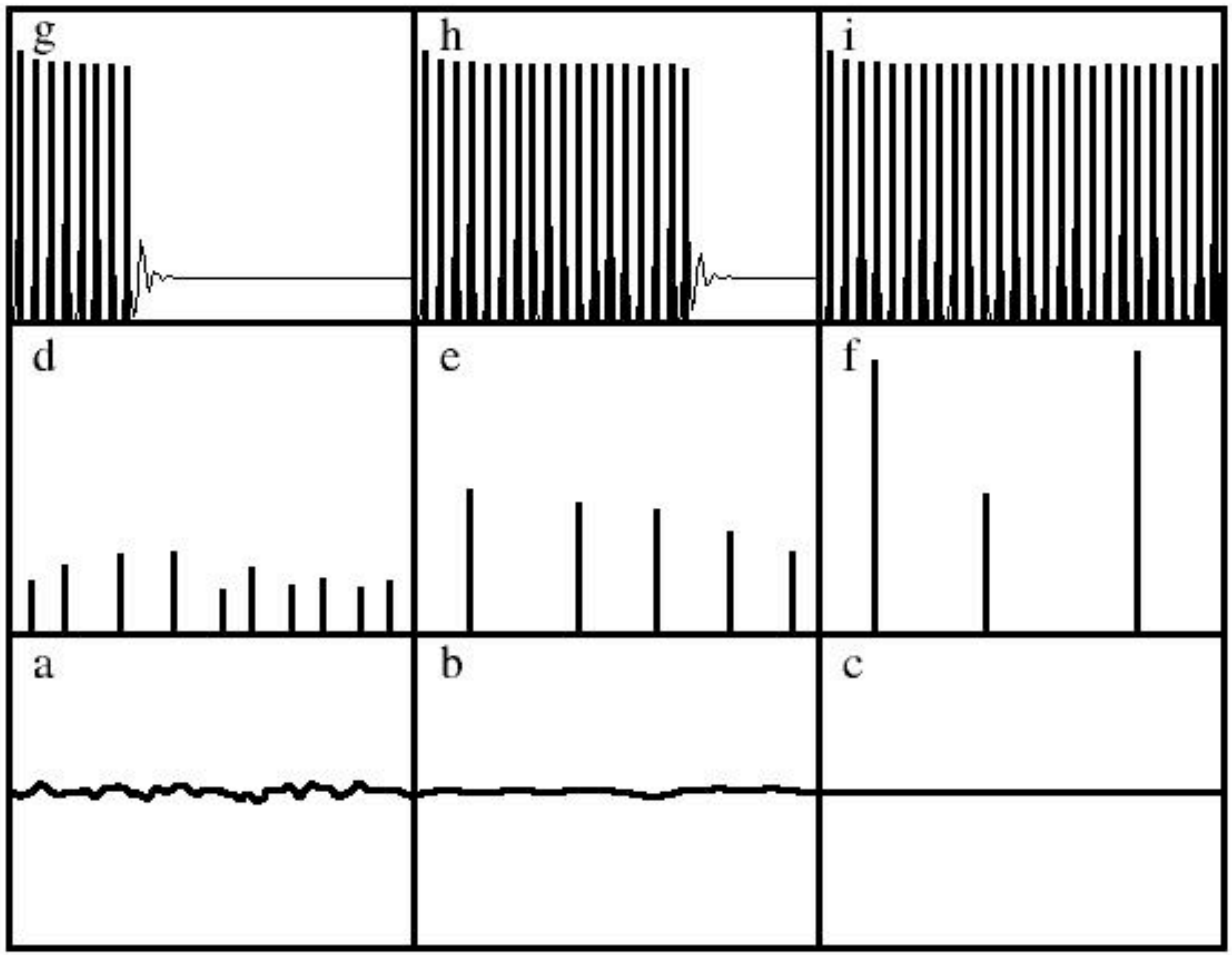
- Parties are static throughout process
- A small party with a large niche may still outlast a larger neighbor!

Three scenarios

$D = 0$

$D < D_c$

$D > D_c$



early intermediate late

II. Conclusions

- Isolated parties
  - Tight, immobile core and diffusive tail
  - Lifetime grows fast with size
- Interaction between two parties
  - Large party grows at expense of small one
  - Deterministic outcome, steady flux
- Multiple parties
  - Strong noise: disorganized political system, no parties
  - Weak noise: parties form, coarsening mosaic
  - No noise: pattern formation

Publications

1. E. Ben-Naim, P.L. Krapivsky, and S. Redner, Physica D **183**, 190 (2003).
2. E. Ben-Naim, Europhys. Lett. **69**, 671 (2005).
3. E. Ben-Naim and A. Scheel, unpublished (2010).