Bifurcations and Patterns in Opinion Dynamics

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thanks: Michael Cross (Caltech), Lev Tsimring (San Diego)

Talk, papers available from: http://cnls.lanl.gov/~ebn

Kinetic and Mean-field models in the Socio-Economic Sciences, Edinburgh, July 27, 2009

Plan

I. Pure compromise dynamics
   A. Continuous opinions
   B. Discrete opinions

II. Noisy compromise dynamics
   A. Single-party dynamics
   B. Two-party dynamics
   C. Multi-party dynamics

Themes

1. Bifurcations
2. Pattern Formation
3. Scaling
4. Coarsening

1. Pure compromise dynamics

The compromise process

- Opinion measured by a continuum variable

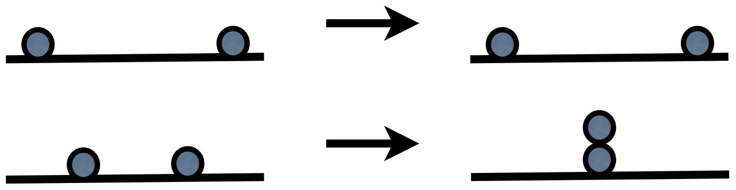
$$-\Delta < x < \Delta$$

1. **Compromise**: reached by pairwise interactions

$$(x_1, x_2) \to \left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}\right)$$

2. **Conviction**: restricted interaction range

$$|x_1 - x_2| < 1$$



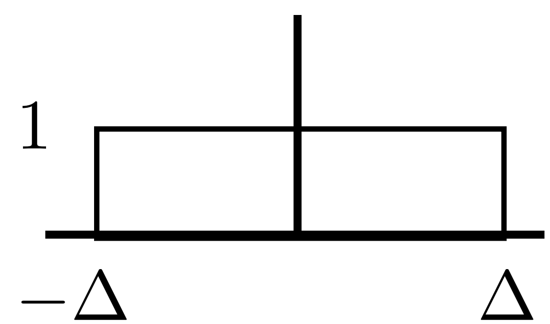
- Minimal, one parameter model
- Mimics competition between compromise and conviction

R Axelrod, J Conf. Res. 41, 203 (1997)
G. Deffuant, G Weisbuch et al, Adv. Comp. Sys 3, 87 (2000)

Problem set-up

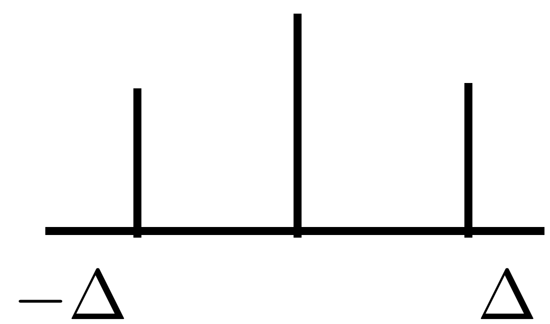
- Given uniform initial (un-normalized) distribution

$$P_0(x) = \begin{cases} 1 & |x| < \Delta \\ 0 & |x| > \Delta \end{cases}$$



- Find final distribution

$$P_\infty(x) = ?$$



- Multitude of final steady-states

$$P_0(x) = \sum_{i=1}^{N} m_i\,\delta(x - x_i) \qquad |x_i - x_j| > 1$$

- Dynamics selects one (deterministically!)

Multiple localized clusters

Further details

- Dynamic treatment

  Each individual interacts once per unit time

- Random interactions

  Two interacting individuals are chosen randomly

- Infinite particle limit is implicitly assumed

$$N \to \infty$$

- Process is galilean invariant $x \to x + x_0$

  Set average opinion to zero $\langle x \rangle = 0$

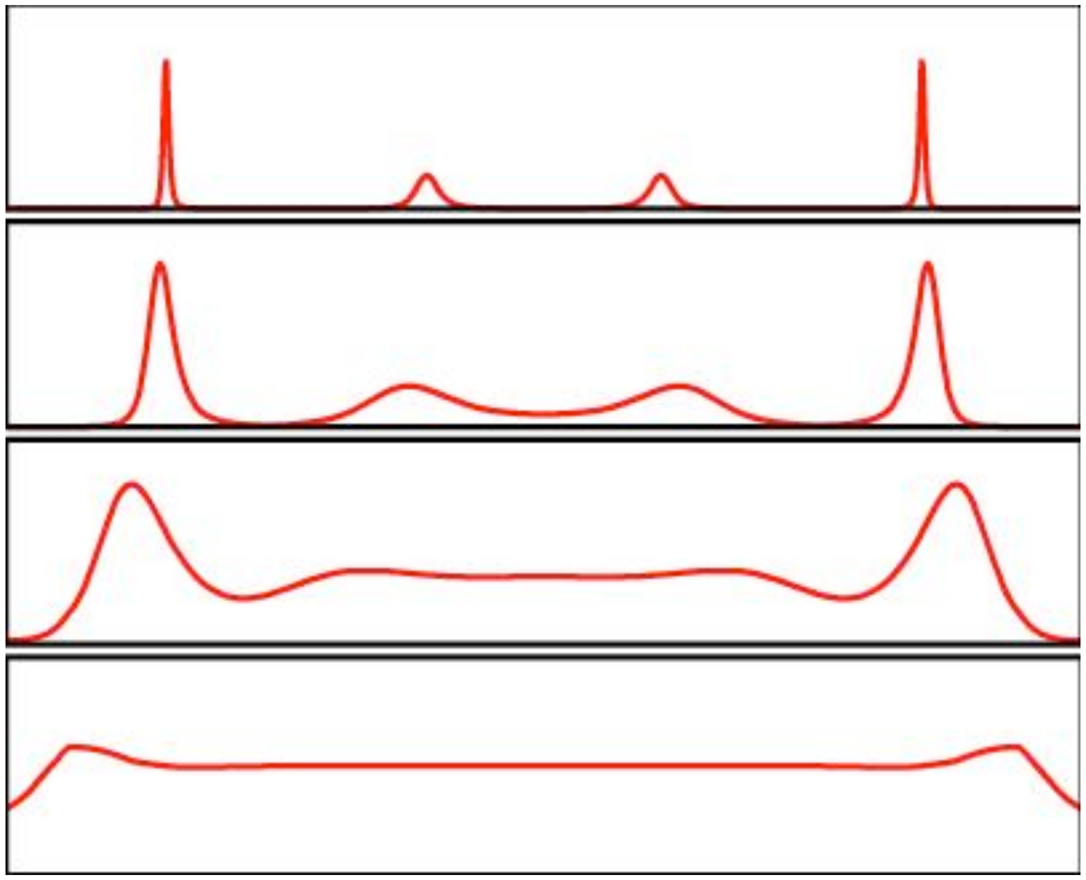
Numerical methods, kinetic theory

- Same master equation, restricted integration

$$\frac{\partial P(x,t)}{\partial t} = \iint_{|x_1 - x_2| < 1} dx_1 dx_2 P(x_1,t)P(x_2,t)\left[\delta\left(x - \frac{x_1+x_2}{2}\right) - \delta(x - x_1)\right]$$

☐ Direct Monte Carlo simulation of stochastic process

☑ Numerical integration of rate equations



Two Conservation Laws

- Total population is conserved

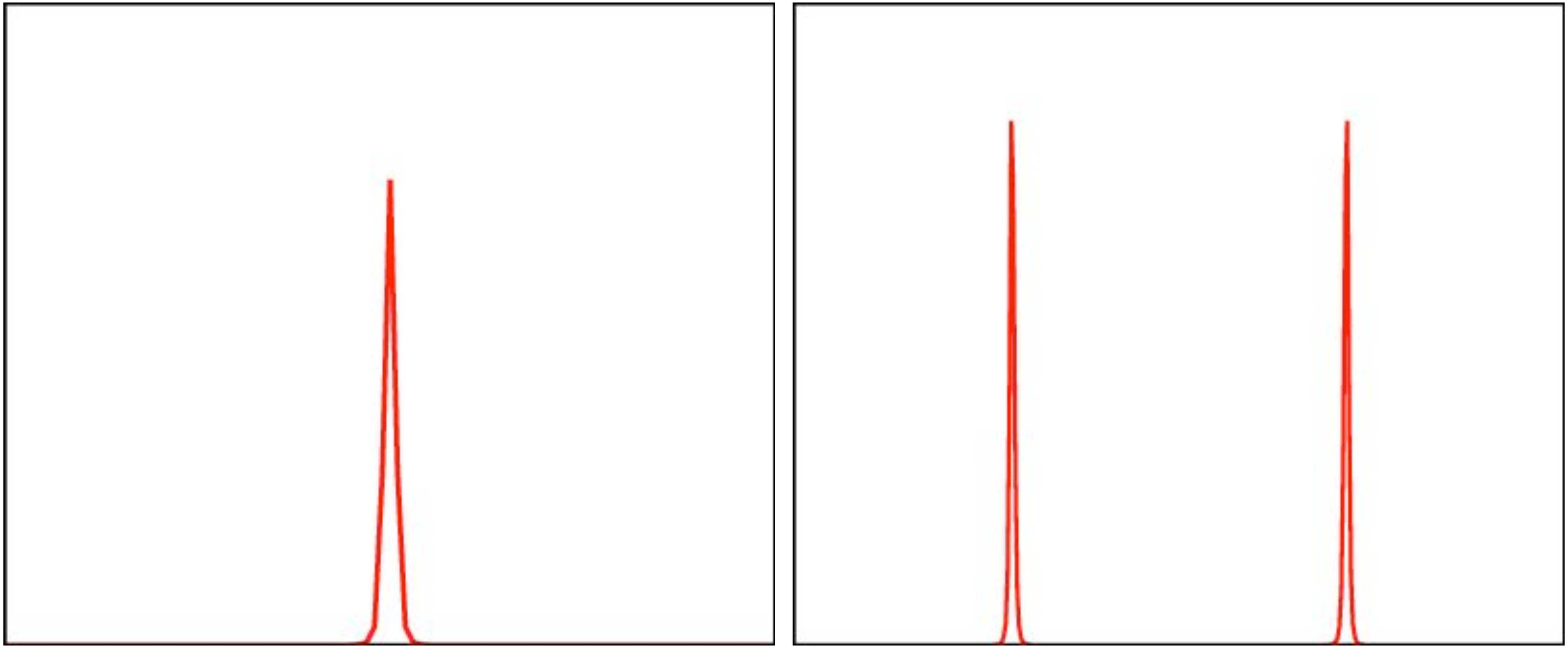
$$\int_{-\Delta}^{\Delta} dx\, P(x) = 2\Delta$$

- Average opinion is conserved

$$\int_{-\Delta}^{\Delta} dx\, x\, P(x) = 0$$

Rise and fall of central party

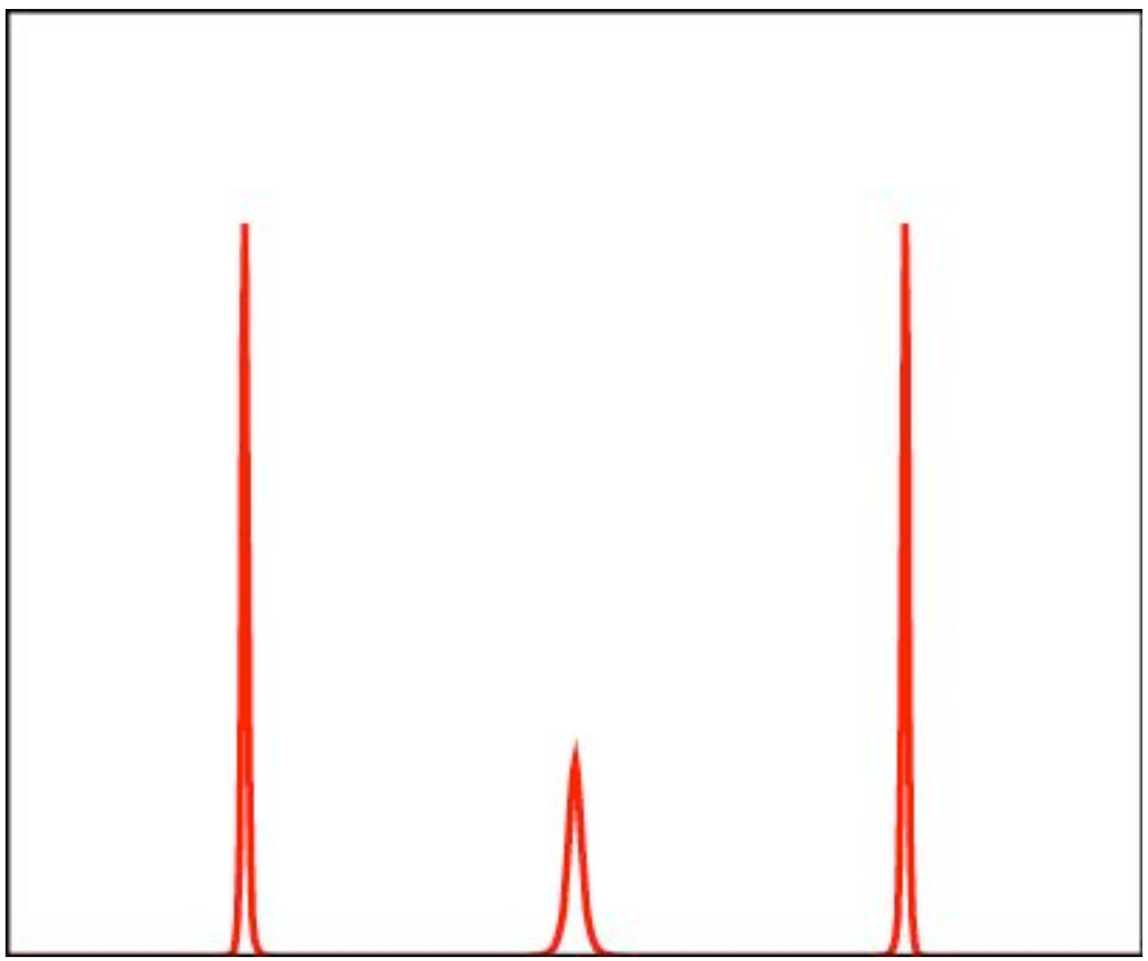
$0 < \Delta < 1.871$ $\quad\quad$ $1.871 < \Delta < 2.724$



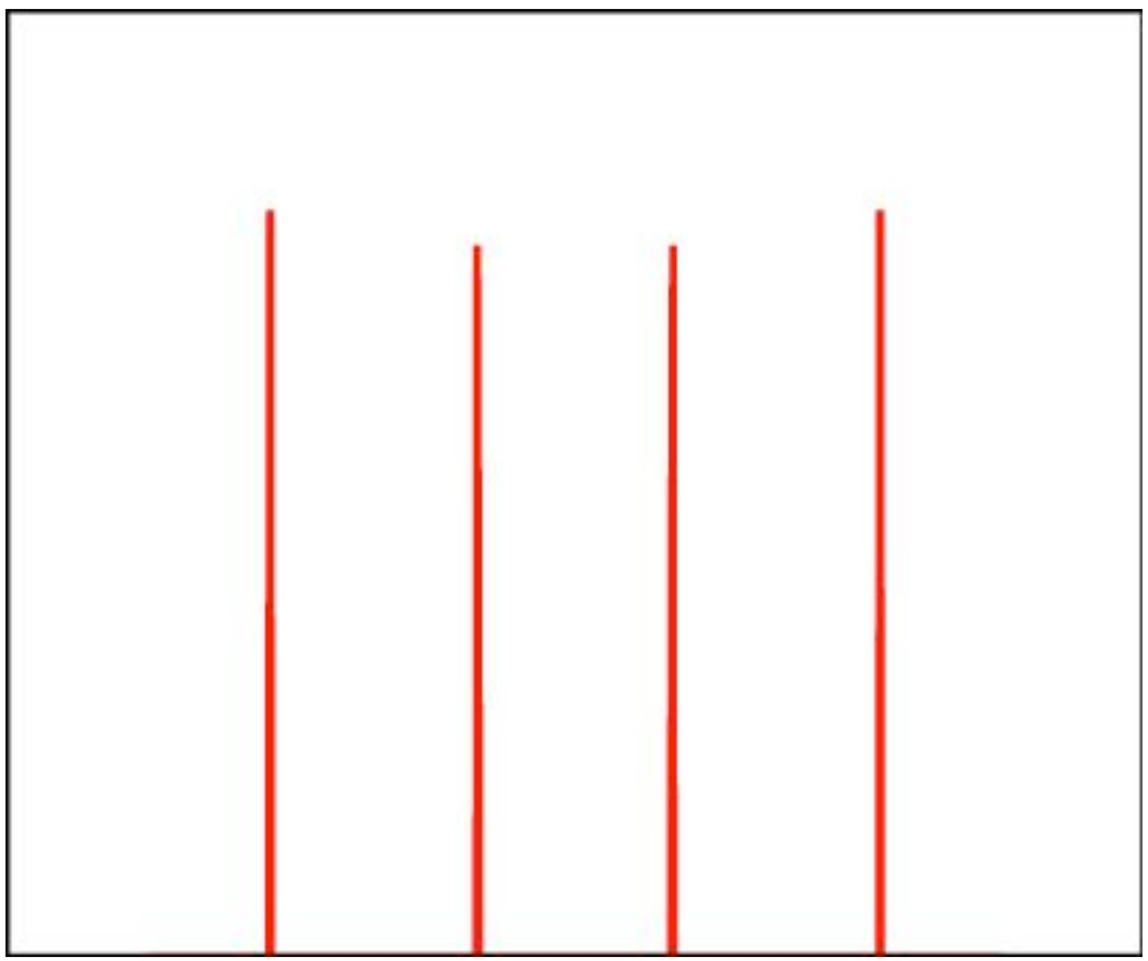
Central party may or may not exist!

Resurrection of central party

$$2.724 < \Delta < 4.079$$

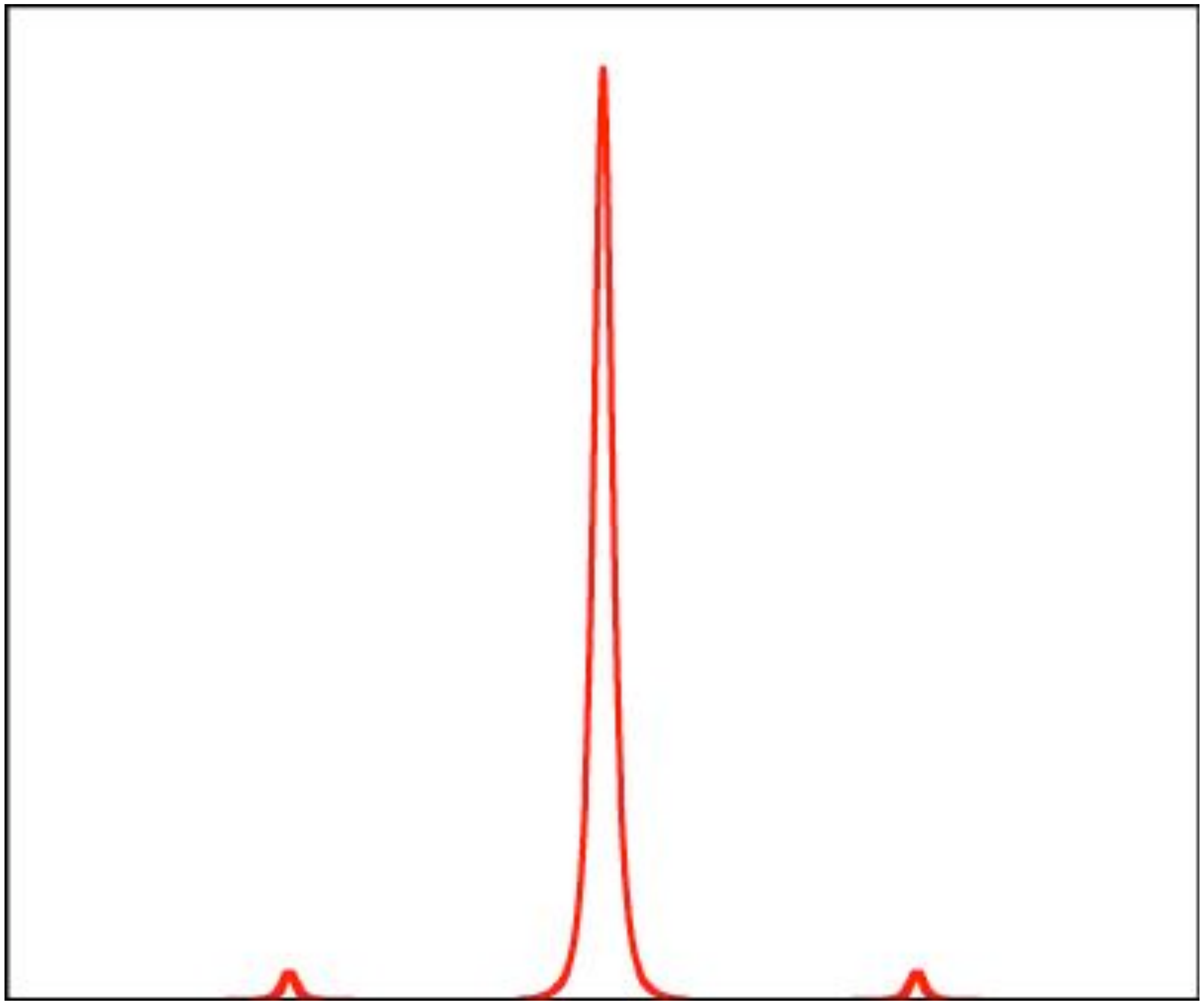


$$4.079 < \Delta < 4.956$$



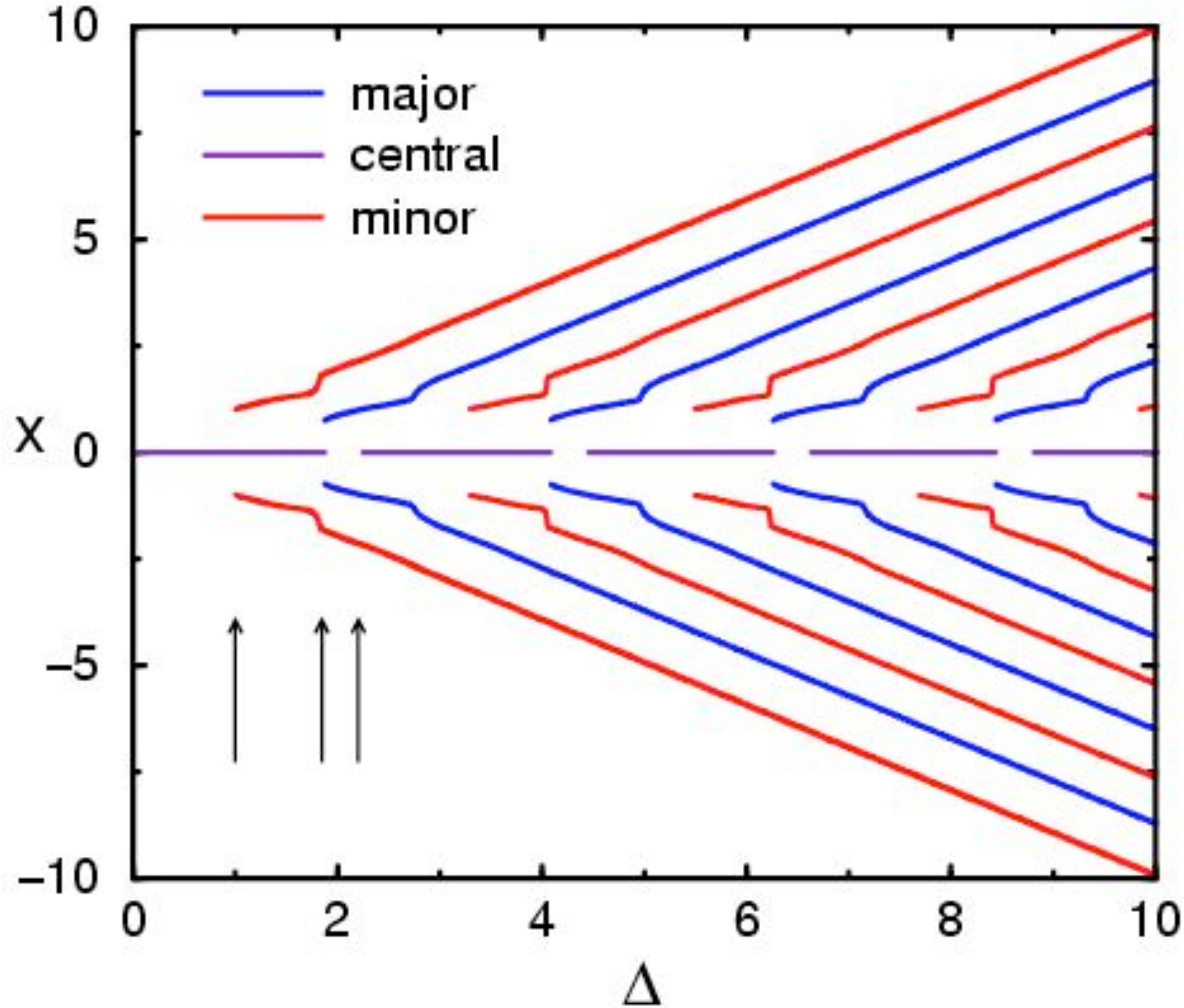
Parties may or may not be equal in size

Emergence of extremists



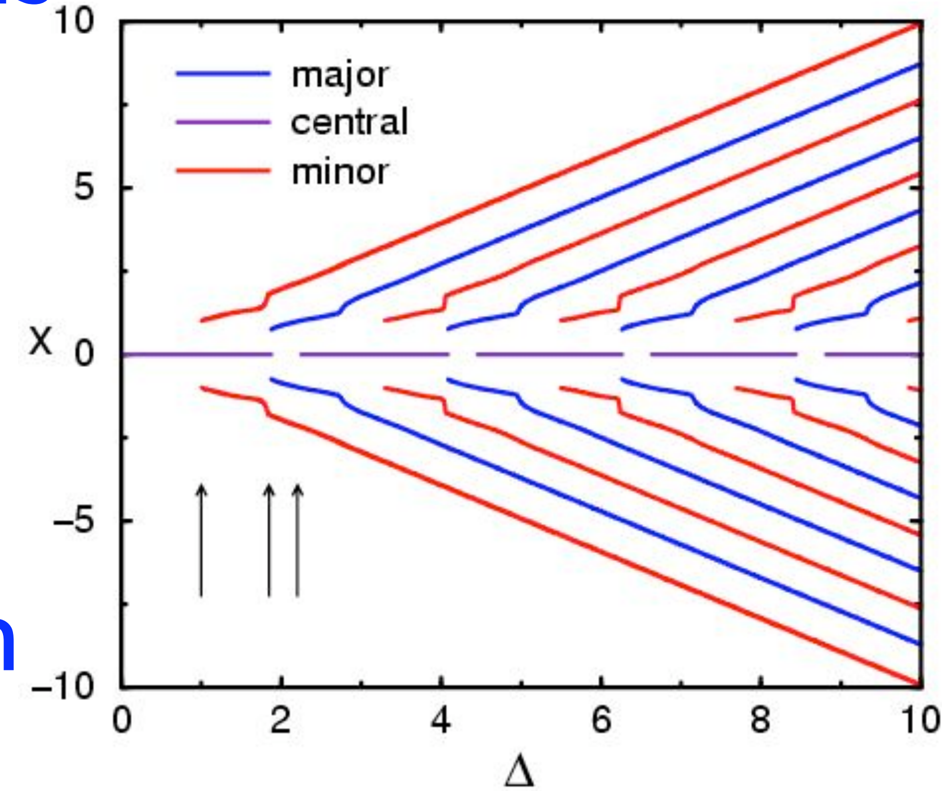
Tiny fringe parties ($m \sim 10^{-3}$)

Bifurcations and Patterns



Self-similar structure, universality

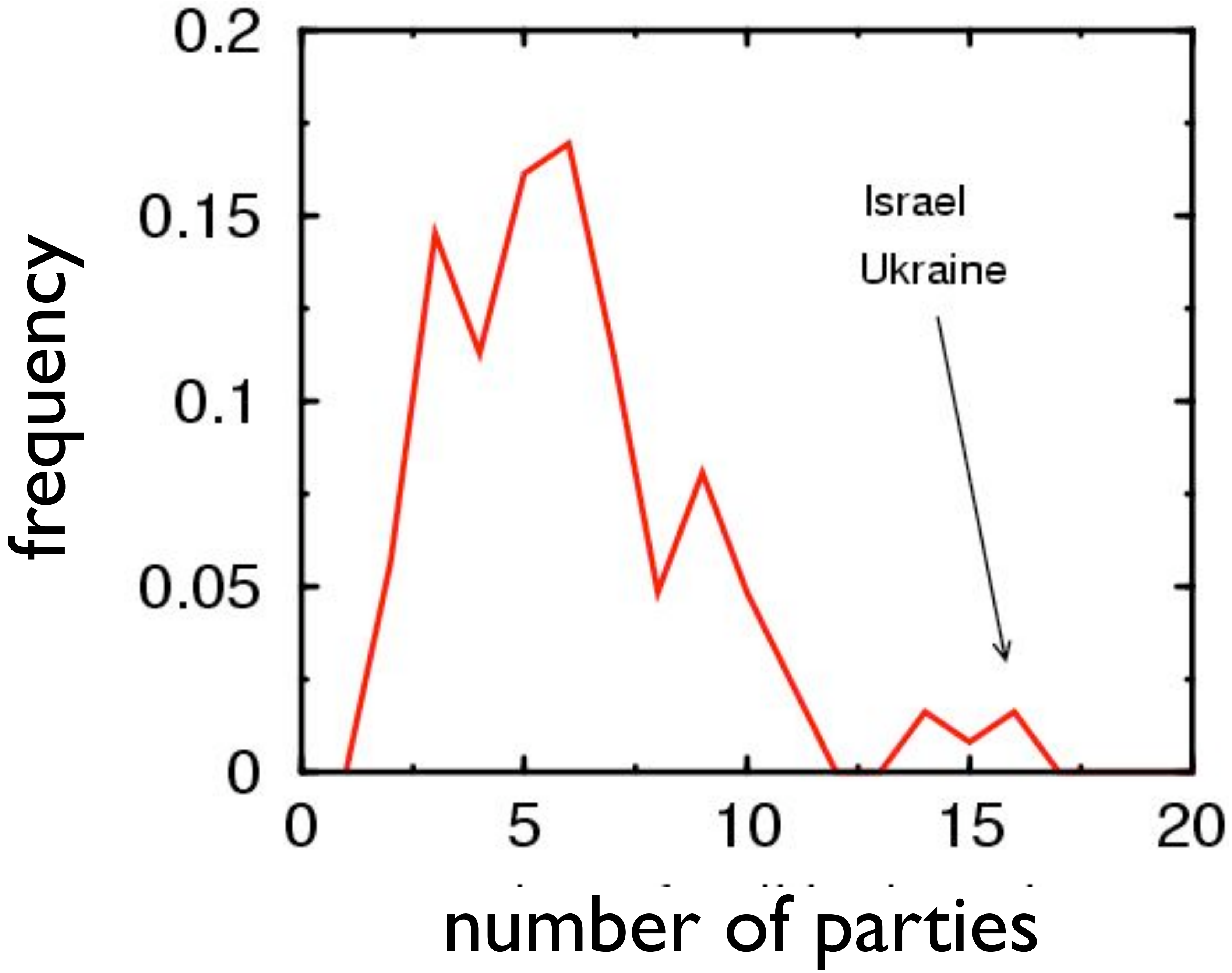
- Periodic sequence of bifurcations
    1. Nucleation of minor cluster branch
    2. Nucleation of major cluster brunch
    3. Nucleation of central cluster



- Alternating major-minor pattern
- Clusters are equally spaced
- Period L gives major cluster mass, separation

$$x(\Delta) = x(\Delta) + L \qquad L = 2.155$$

How many political parties?



- Data: CIA world factbook 2002
- 120 countries with multi-party parliaments
- Average=5.8; Standard deviation=2.9

Cluster mass

- Masses are periodic

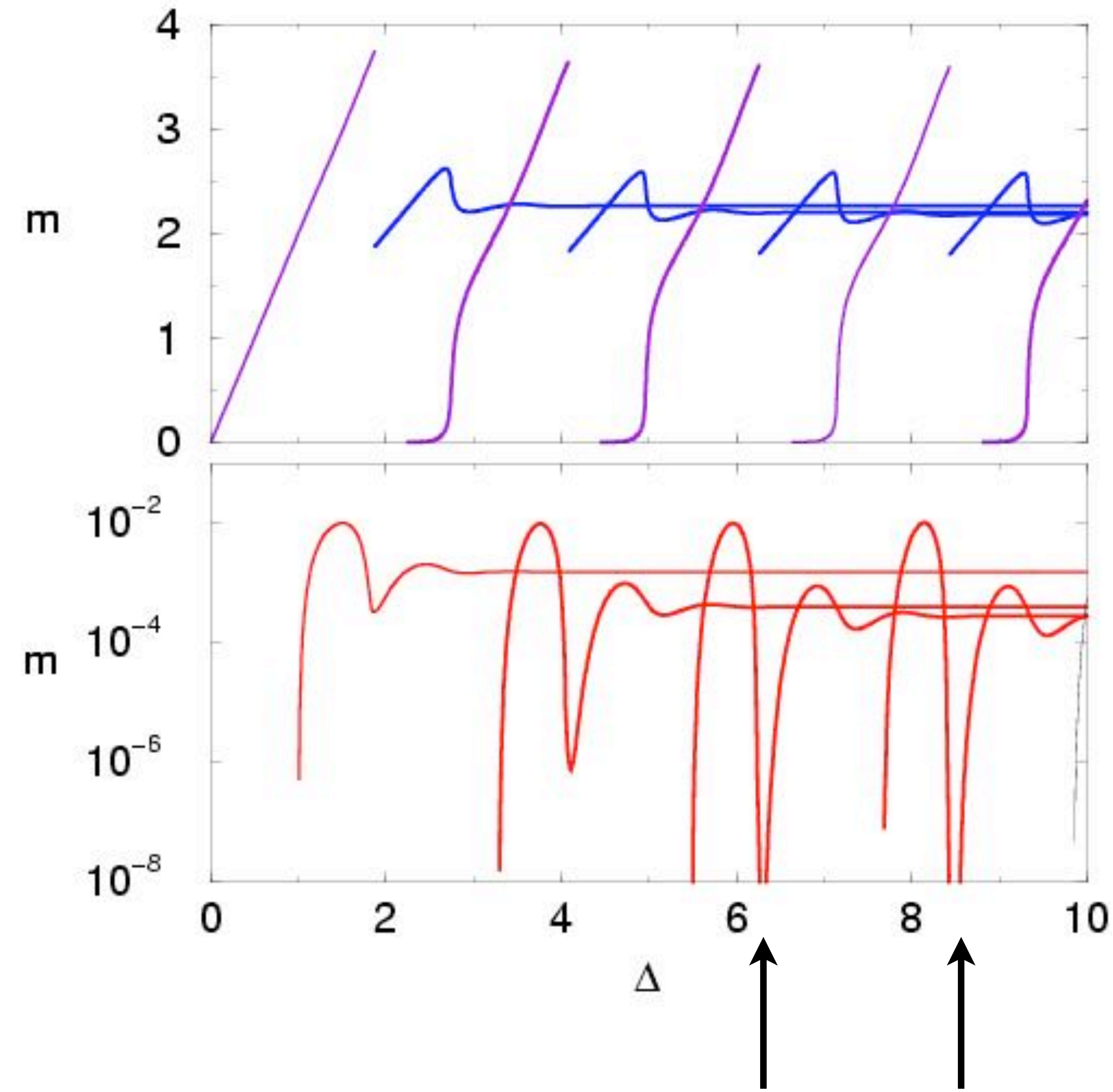
$$m(\Delta) = m(\Delta + L)$$

- Major mass

$$M \to L = 2.155$$

- Minor mass

$$m \to 3 \times 10^{-4}$$



Why are the minor clusters so small?

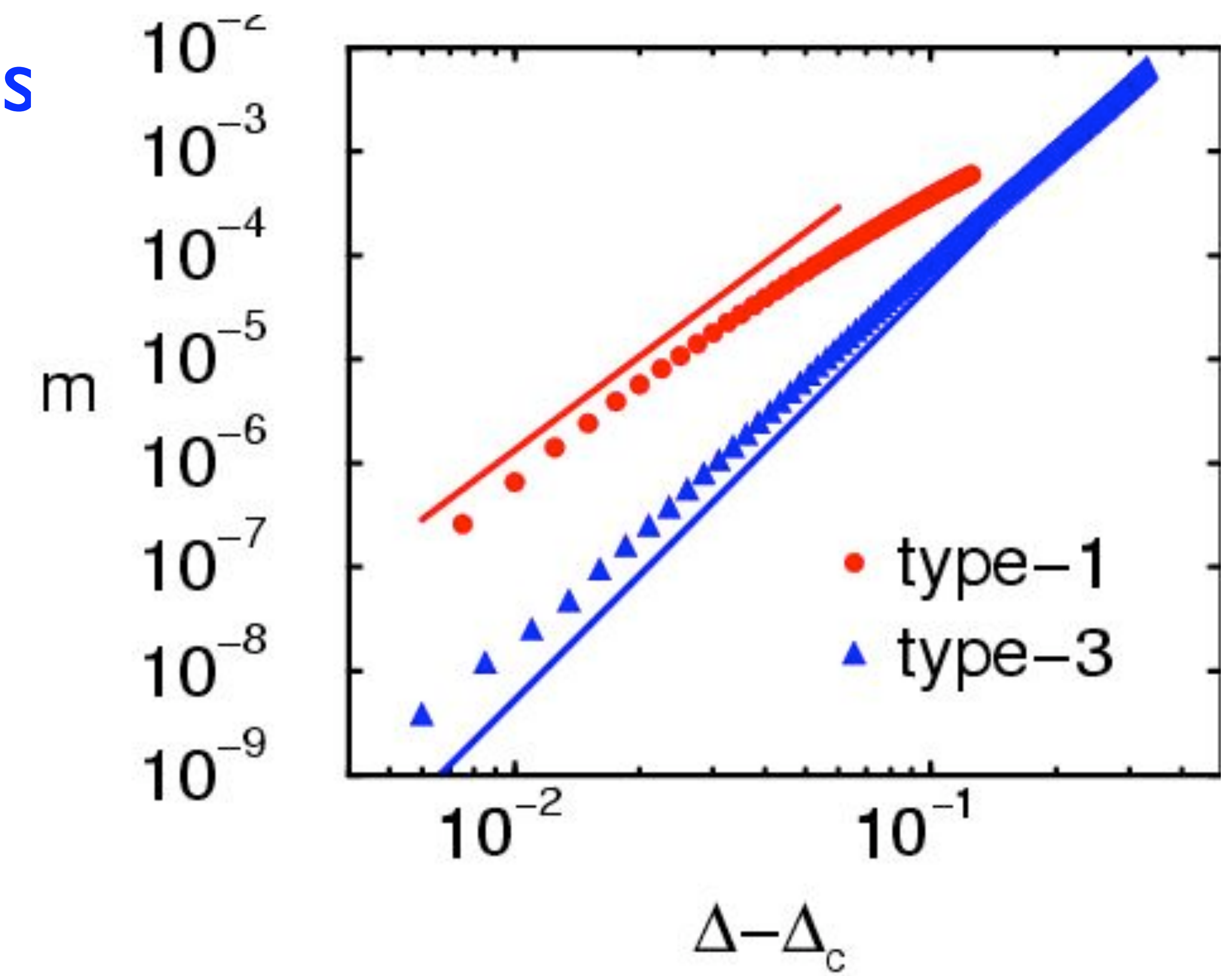
gaps?

Scaling near bifurcation points

- Minor mass vanishes

$$m \sim (\Delta - \Delta_c)^\alpha$$

- Universal exponent



$$\alpha = \begin{cases} 3 & \text{type } 1 \\ 4 & \text{type } 3 \end{cases}$$

L-2 is the small parameter
explains small saturation mass

Consensus dynamics

- Integrable for $\Delta < 1/2$

$$\langle x^2(t)\rangle = \langle x^2(0)\rangle\, e^{-\Delta t}$$

- Final state: localized

$$P_\infty(x) = 2\Delta\,\delta(x)$$

- Rate equations in Fourier space

$$P_t(k) + P(k) = P^2(k/2)$$

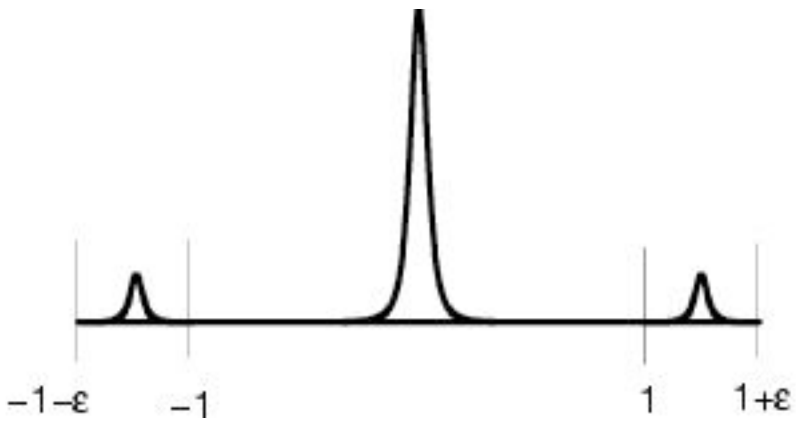
- Self-similar collapse dynamics

$$\Phi(z) \propto \left(1+z^2\right)^{-2} \qquad z = x/\sqrt{\langle x^2\rangle}$$

The Inelastic Maxwell Model, EB & PL Krapivsky, Lecture Notes in Physics **624,** 65 (2003)

Heuristic derivation of exponent

- Perturbation theory $\Delta = 1 + \epsilon$
- Major cluster $x(\infty) = 0$
- Minor cluster $x(\infty) = \pm(1 + \epsilon/2)$



- Rate of transfer from minor cluster to major cluster

$$\frac{dm}{dt} = -m\,M \longrightarrow m \sim \epsilon\, e^{-t}$$

- Process stops when

$$x \sim e^{-t_f/2} \sim \epsilon \qquad \langle x^2 \rangle \sim e^{-t}$$

- Final mass of minor cluster

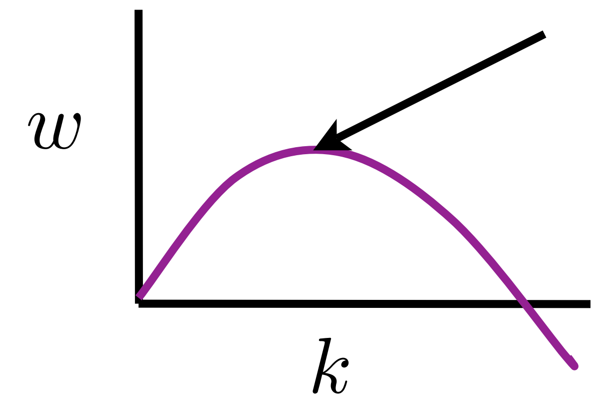
$$m(\infty) \sim m(t_f) \sim \epsilon^3 \qquad \alpha = 3$$

Pattern selection

- Linear stability analysis

$$P - 1 \propto e^{i(kx+wt)} \quad \Longrightarrow \quad w(k) = \frac{8}{k}\sin\frac{k}{2} - \frac{2}{k}\sin k - 2$$

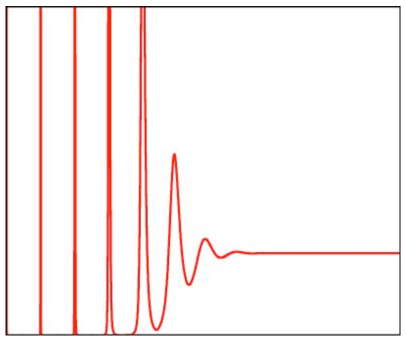
- Fastest growing mode



$$\frac{dw}{dk} \quad \Longrightarrow \quad L = \frac{2\pi}{k} = 2.2515$$

- Traveling wave (FKPP saddle point analysis)

$$\frac{dw}{dk} = \frac{\mathrm{Im}(w)}{\mathrm{Im}(k)} \quad \Longrightarrow \quad L = \frac{2\pi}{k} = 2.0375$$



Patterns induced by wave propagation from boundary
However, emerging period is different

$$2.0375 < L < 2.2515$$

Pattern selection is intrinsically nonlinear

Discrete opinions

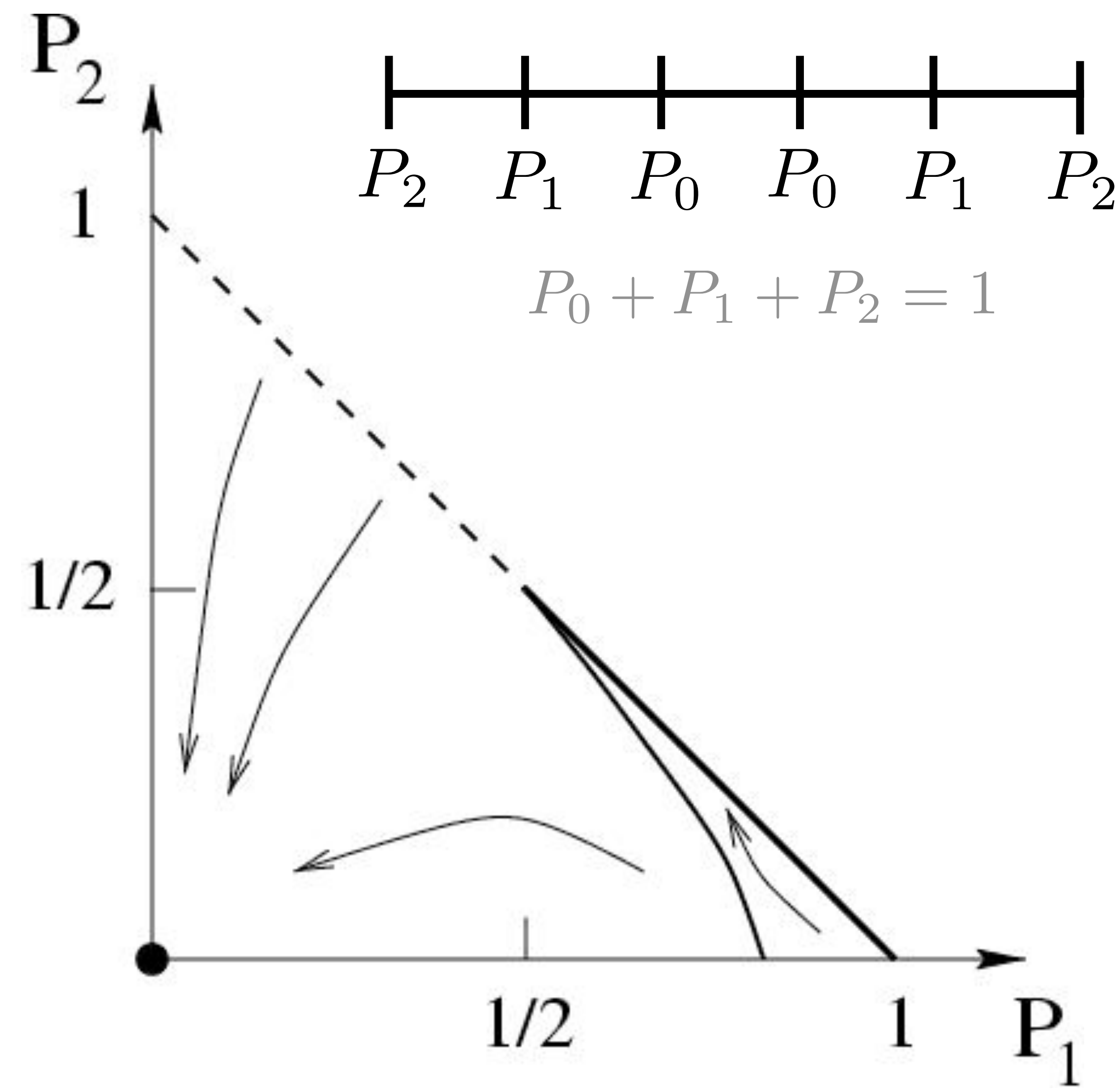
- Compromise process

$$(n-1, n+1) \to (n, n)$$

- Master equation

$$\frac{dP_n}{dt} = 2P_{n-1}P_{n+1} - P_n(P_{n-2} + P_{n+2})$$

- Simplest example: 6 states
- Symmetry + normalization:
- Two-dimensional problem

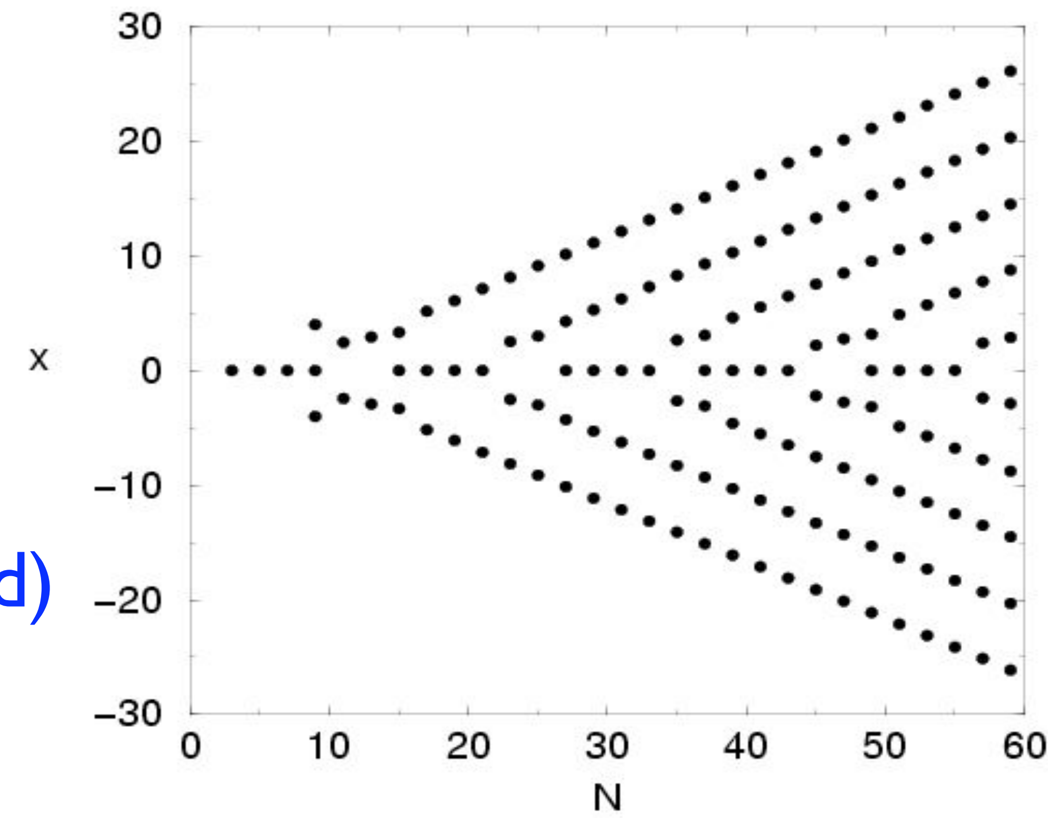


Initial condition determines final state

Isolated fixed points, lines of fixed points

Discrete opinions

- Dissipative system, volume contracts
- Energy (Lyapunov) function exists
- No cycles or strange attractors
- Uniform state is unstable (Cahn-Hilliard)



$$P_i = 1 + \phi_i \qquad \phi_t + (\phi + a\,\phi_{xx} + b\,\phi^2)_{xx}$$

Discrete case yields useful insights

Pattern selection

- Linear stability analysis

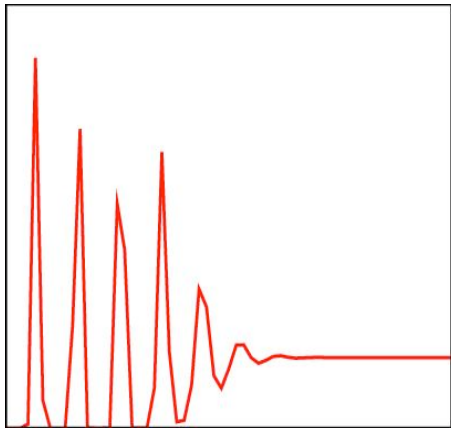
$$P - 1 \propto e^{i(kx+wt)} \longrightarrow w(k) = 4\cos k - 4\cos 2k - 2$$

- Fastest growing mode

$$\frac{dw}{dk} \quad \Longrightarrow \quad L = \frac{2\pi}{k} = 6$$

- Traveling wave (FKPP saddle point analysis)

$$\frac{dw}{dk} = \frac{\mathrm{Im}(w)}{\mathrm{Im}(k)} \quad \Longrightarrow \quad L = \frac{2\pi}{k} = 5.31$$



Again, linear stability gives useful upper and lower bounds

$$5.31 < L < 6 \qquad \text{while} \quad L_{\text{select}} = 5.67$$

Pattern selection is intrinsically nonlinear

I. Conclusions

- Clusters form via bifurcations
- Periodic structure
- Alternating major-minor pattern
- Central party does not always exist
- Power-law behavior near transitions
- Nonlinear pattern selection

I. Outlook

- Pattern selection criteria
- Gaps
- Role of initial conditions, classification
- Role of spatial dimension, correlations
- Disorder, inhomogeneities
- Tiling/Packing in 2D
- Discord dynamics (seceder model, Halpin-Healy 03)

Many open questions

II. Noisy compromise dynamics

Diffusion (noise)

- **Diffusion:** Individuals change opinion spontaneously

$$n \xrightarrow{D} n \pm 1$$

- Adds noise (“temperature”)
- Linear process: no interaction
- Mimics unstable, varying opinion
- Influence of environment, news, editorials, events

Rate equations

- Compromise: reached through pairwise interactions

$$(n-1, n+1) \to (n, n)$$

- Conserved quantities: total population, average opinion
- Probability distribution $P_n(t)$
- Kinetic theory: nonlinear rate equations

$$\frac{dP_n}{dt} = 2P_{n-1}P_{n+1} - P_n(P_{n-2} + P_{n+2}) + D(P_{n-1} + P_{n+1} - 2P_n)$$

☐ Direct Monte Carlo simulations of stochastic process

☑ Numerical integration of rate equations

Single-party dynamics

- Initial condition: large isolated party

$$P_n(0) = m(\delta_{n,0} + \delta_{n,-1})$$

- Steady-state: compromise and diffusion balance

$$DP_n = P_{n-1}P_{n+1}$$

- Core of party: localized to a few opinion states

$$P_0 = m \qquad P_1 = D \qquad P_2 = D^2 m^{-1}$$

- Compromise negligible for n>2

Party has a well defined core

The tail

- Diffusion dominates outside the core

$$\frac{dP_n}{dt} = D(P_{n-1} + P_{n+1} - 2P_n) \qquad P \ll D$$

- Standard problem of diffusion with source

$$P_n \sim m^{-1}\Psi(n\,t^{-1/2})$$

- Tail mass

$$M_{\text{tail}} \sim m^{-1}t^{1/2}$$

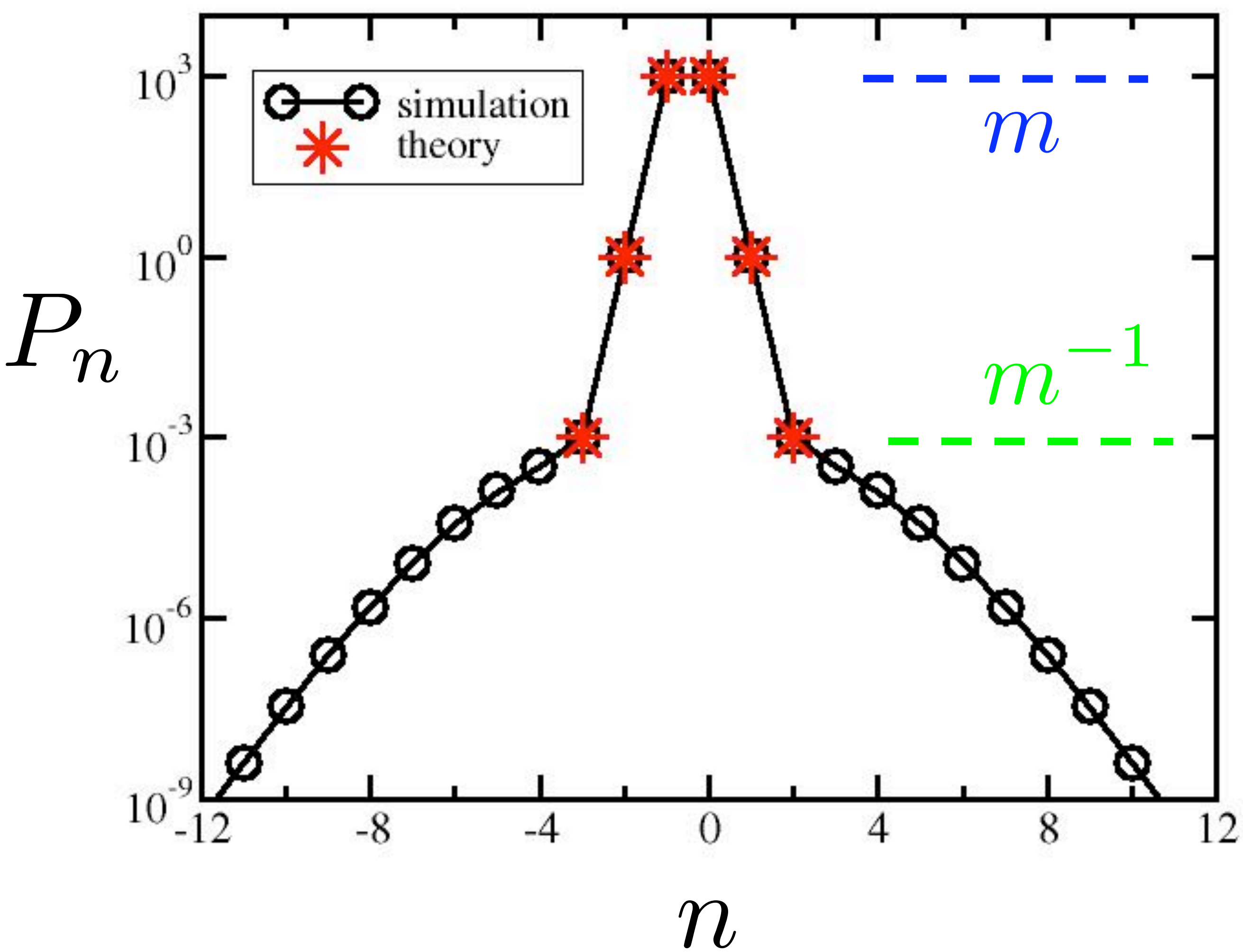
- Party dissolves when

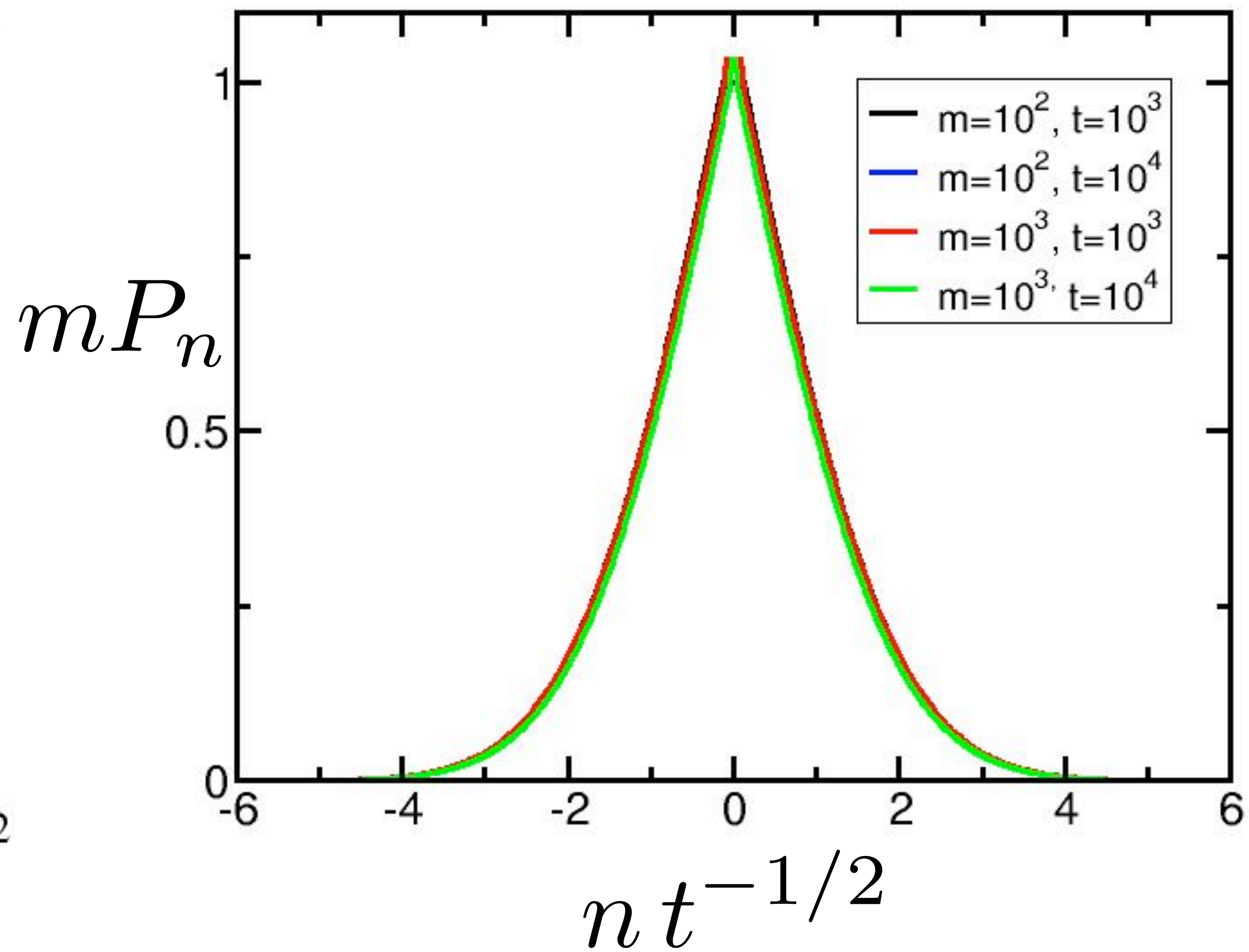
$$M_{\text{tail}} \sim m \qquad \Longrightarrow \qquad \tau \sim m^4$$

Party lifetime grows dramatically with its size

Core versus tail

$m = 10^3$





Party height=m

Party depth~m^{-1}

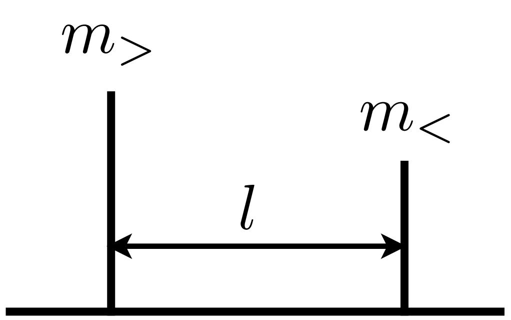
Self-similar shape

Gaussian tail

Qualitative features

- Exists in a quasi-steady state
- Tight core localized to a few sites
- Random opinion changes of members do not affect party position
- Party lifetime grows very fast with size
- Ultimate fate of a party: demise
- Its remnant: a diffusive cloud
- Depth inversely proportional to size, the larger the party the more stable

Two party dynamics



- Initial condition: two large isolated parties

$$P_n(0) = m_> \left(\delta_{n,0} + \delta_{n,-1}\right) + m_< \left(\delta_{n,l} + \delta_{n,l+1}\right)$$

- Interaction between parties mediated by diffusion

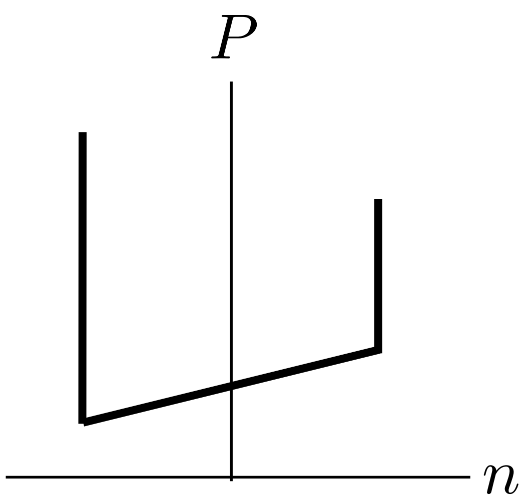
$$0 = P_{n-1} + P_{n+1} - 2P_n$$

- Boundary conditions set by parties depths

$$P_0 = \frac{1}{m_>} \qquad P_l = \frac{1}{m_<}$$

- Steady state: linear profile

$$P_n = \frac{1}{m_<} + \left(\frac{1}{m_<} - \frac{1}{m_>}\right)\frac{n}{l}$$



Merger

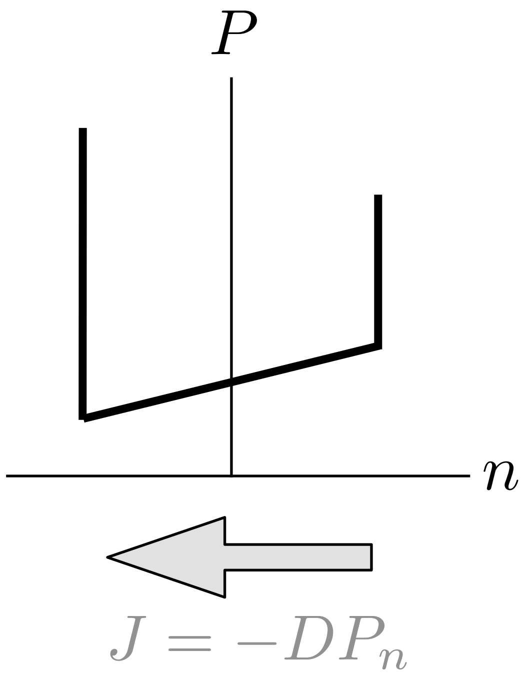
- Steady flux from small party to larger one

$$J \sim \frac{1}{l}\left(\frac{1}{m_<} - \frac{1}{m_>}\right) \sim \frac{1}{l m_<}$$

- Merger time

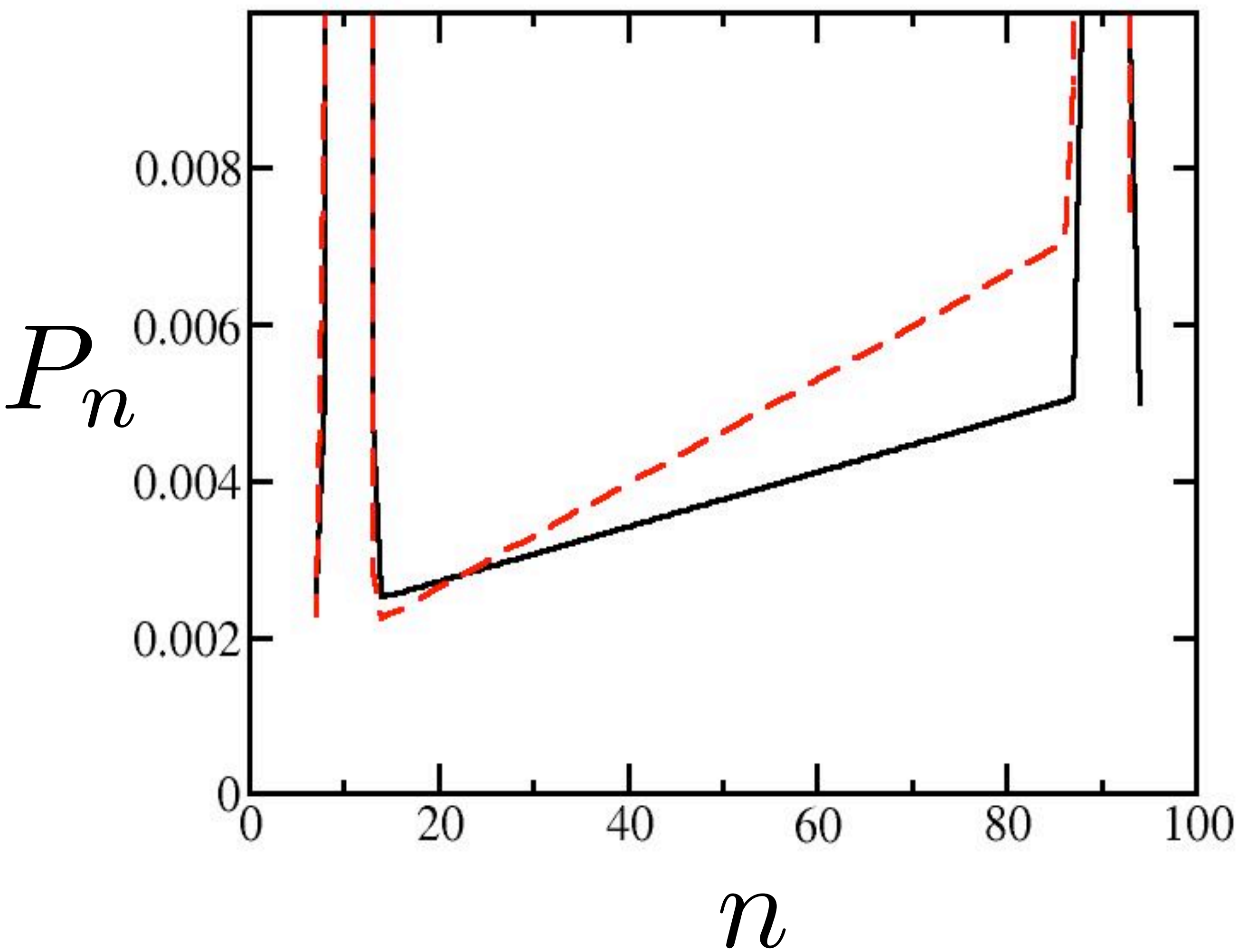
$$T \sim \frac{m_<}{J} \sim l m_<^2$$

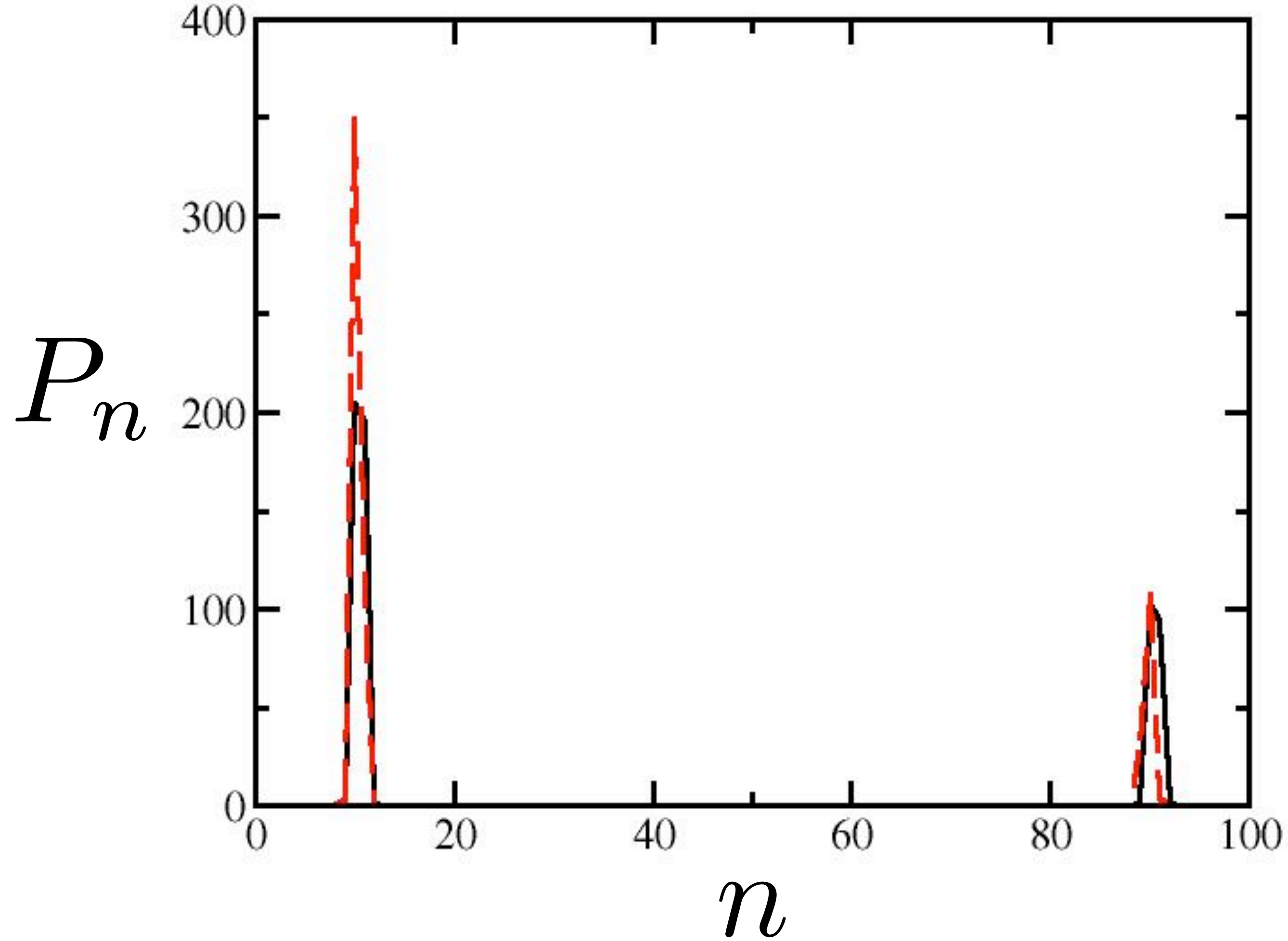
- Lifetime grows with separation (“niche”)
- Outcome of interaction is deterministic
- Larger party position remains fixed throughout merger process



Small party absorbed by larger one

Merger: numerical results





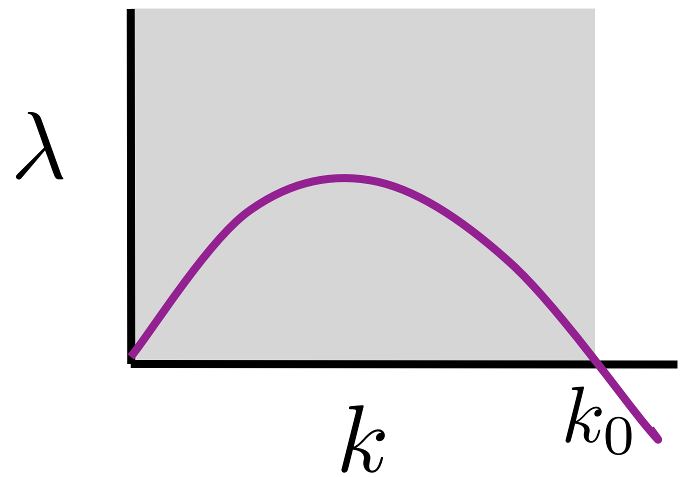
Multiple party dynamics

- Initial condition: large isolated party

$$P_n(0) = \text{randomly chosen number in } [1-\epsilon : 1+\epsilon]$$

- Linear stability analysis

$$P_n - 1 \sim e^{ikn+\lambda t}$$



- Growth rate of perturbations

$$\lambda(k) = (4\cos k - 4\cos 2k - 2) - 2D(1-\cos 2k)$$

- Long wavelength perturbations unstable

$$k < k_0 \qquad \cos k_0 = D/2$$

P=1 stable only for strong diffusion $D>D_c=2$

Strong noise ($D>D_c$)

- Regardless of initial conditions

$$P_n \to \langle P_n(0) \rangle$$

- Relaxation time

$$\lambda \approx (D_c - D)k^2 \quad \Longrightarrow \quad \tau \sim (D - D_c)^{-2}$$

No parties, disorganized political system

Weak noise ($D<D_c$): Coarsening

- Smaller parties merge into large parties
- Party size grows indefinitely
- Assume a self-similar process, size scale m
- Conservation of populations implies separation

$$l \sim m$$

- Use merger time to estimate size scale

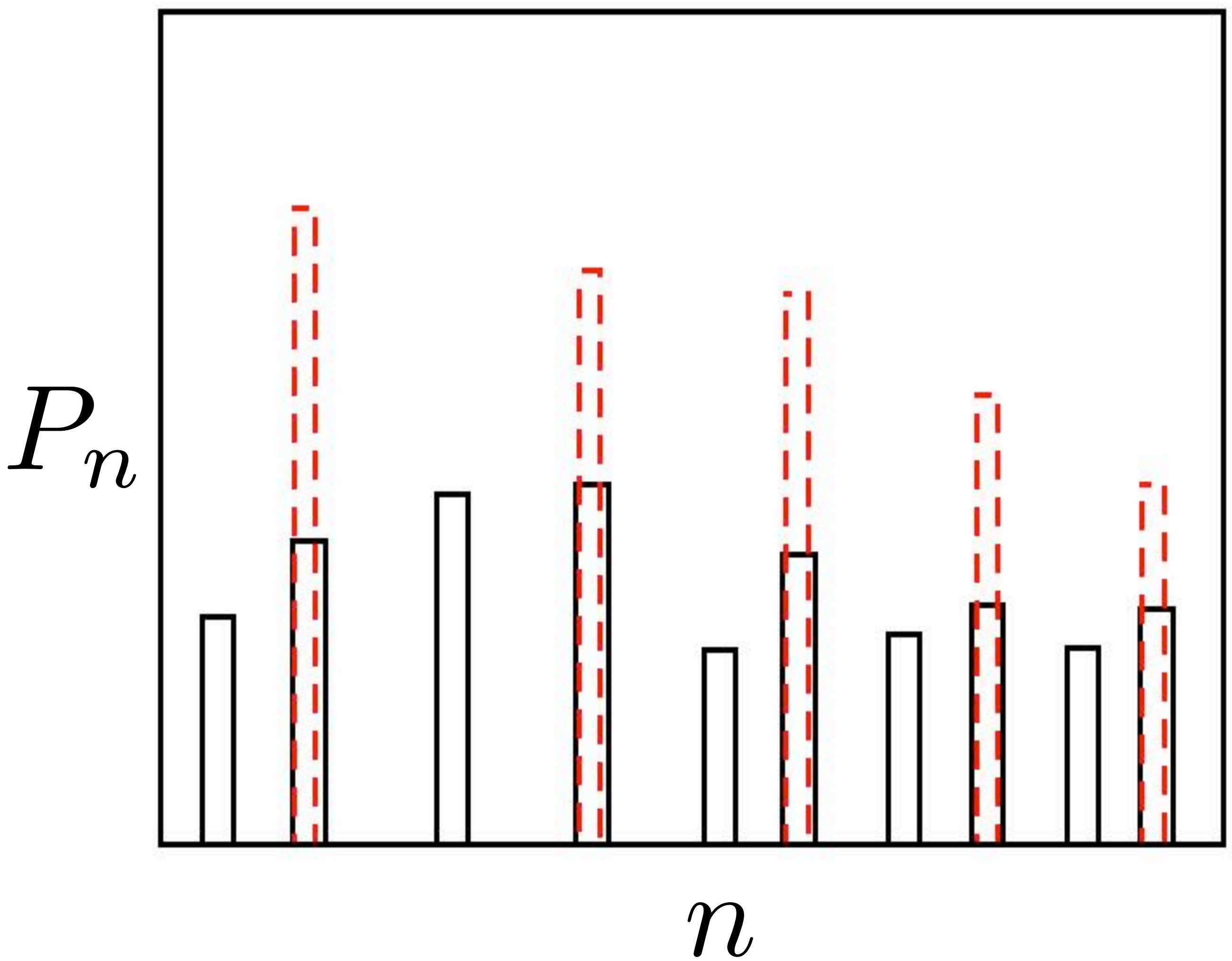
$$t \sim l m^2 \sim m^3 \qquad \Longrightarrow \qquad m \sim t^{1/3}$$

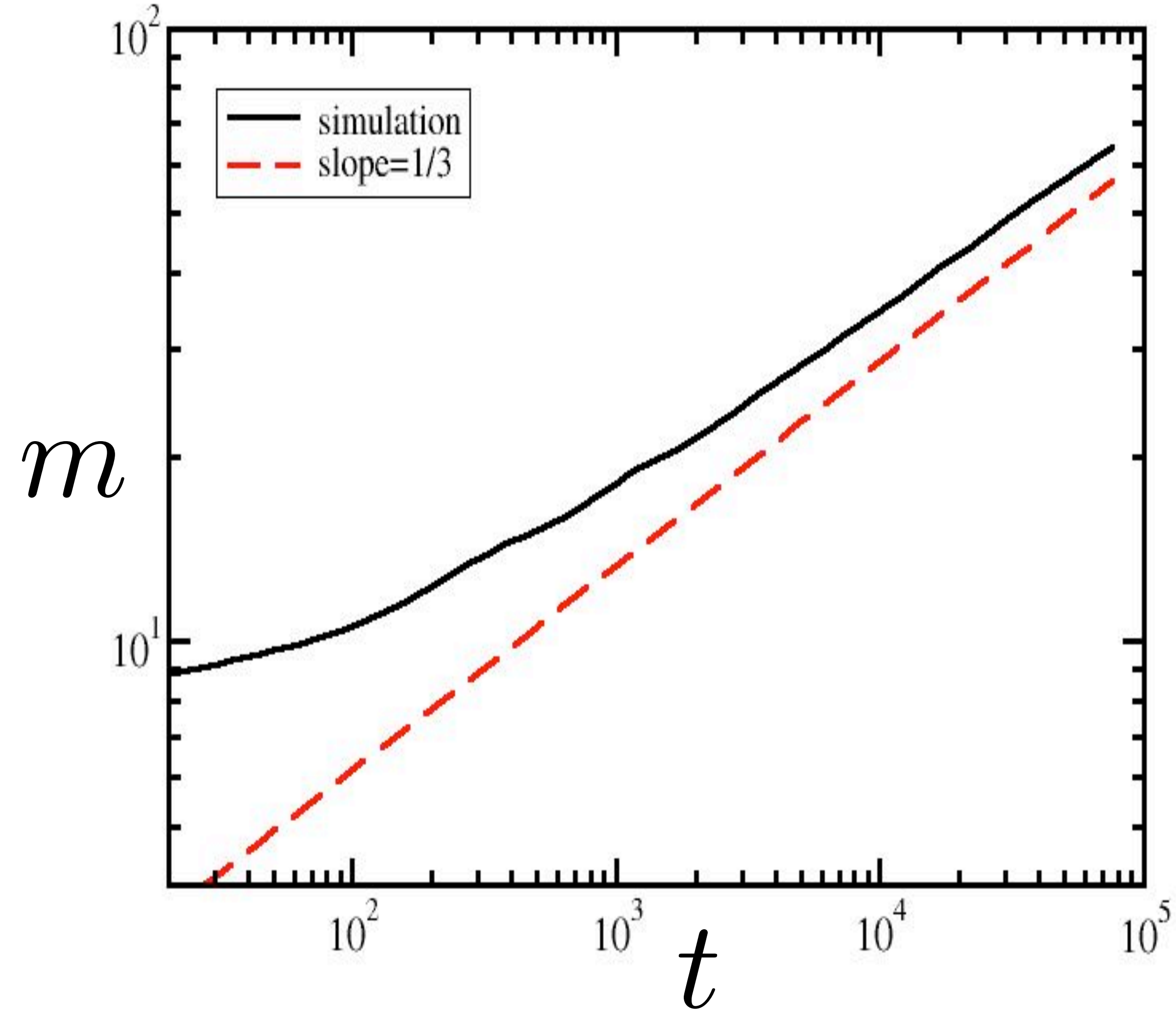
- Self-similar size distribution

$$P_m \sim t^{-1/3} F(m\, t^{-1/3})$$

Lifshitz-Slyozov coarsening

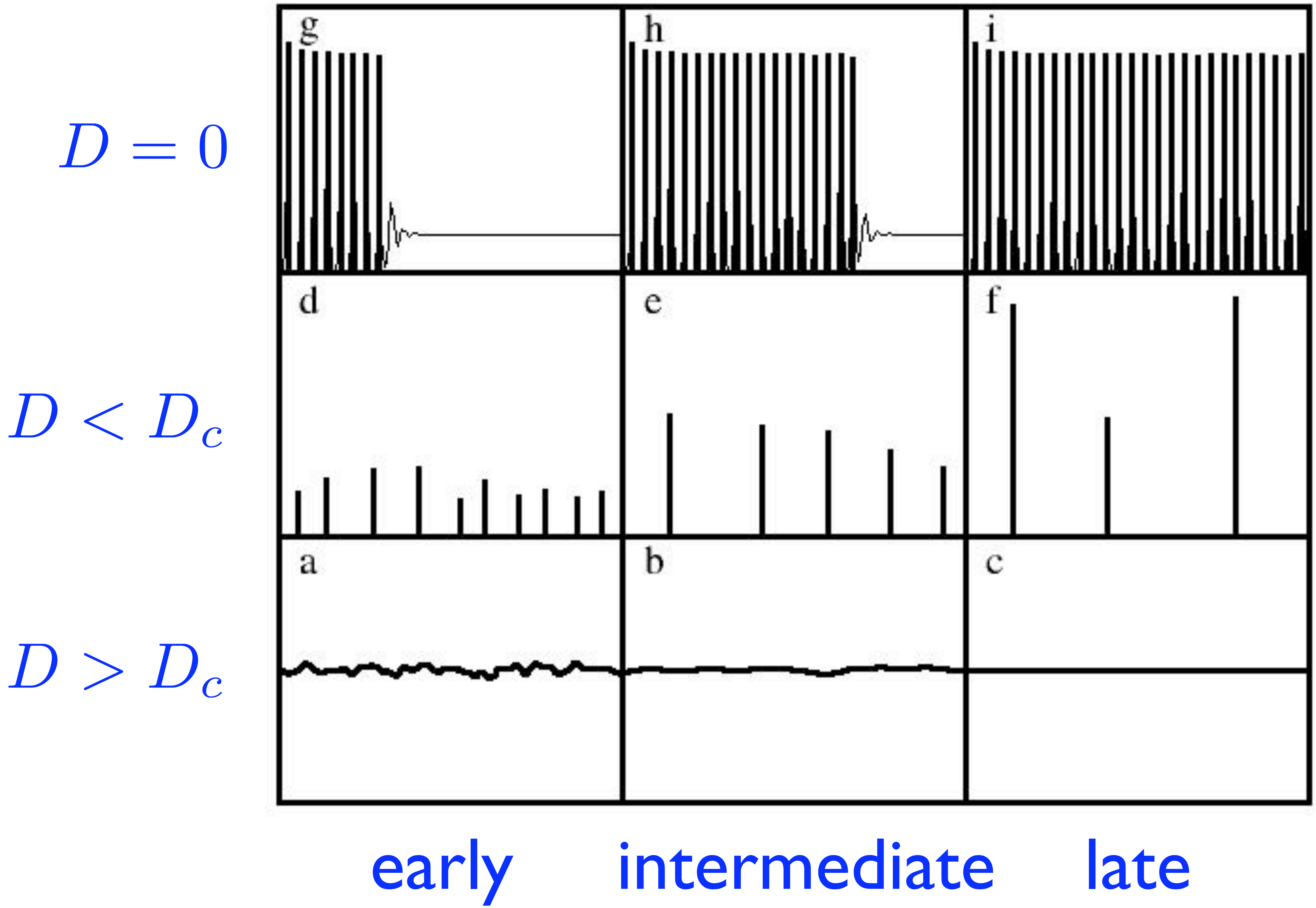
Coarsening: numerical results





- Parties are static throughout process
- A small party with a large niche may still outlast a larger neighbor!

Three scenarios



II. Conclusions

- Isolated parties
  - Tight, immobile core and diffusive tail
  - Lifetime grows fast with size
- Interaction between two parties
  - Large party grows at expense of small one
  - Deterministic outcome, steady flux
- Multiple parties
  - Strong noise: disorganized political system, no parties
  - Weak noise: parties form, coarsening mosaic
  - No noise: pattern formation

Publications

1. E. Ben-Naim, P.L. Krapivsky, and S. Redner, Physica D **183**, 190 (2003).
2. E. Ben-Naim, Europhys. Lett. **69**, 671 (2005).

“I can calculate the motions of heavenly bodies, but not the madness of people.”

Isaac Newton