Popularity-Driven Networking

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E. Ben-Naim and P.L. Krapivsky, EPL **97**, 48003 (2012)
Talk, paper available from: http://cnls.lanl.gov/~ebn

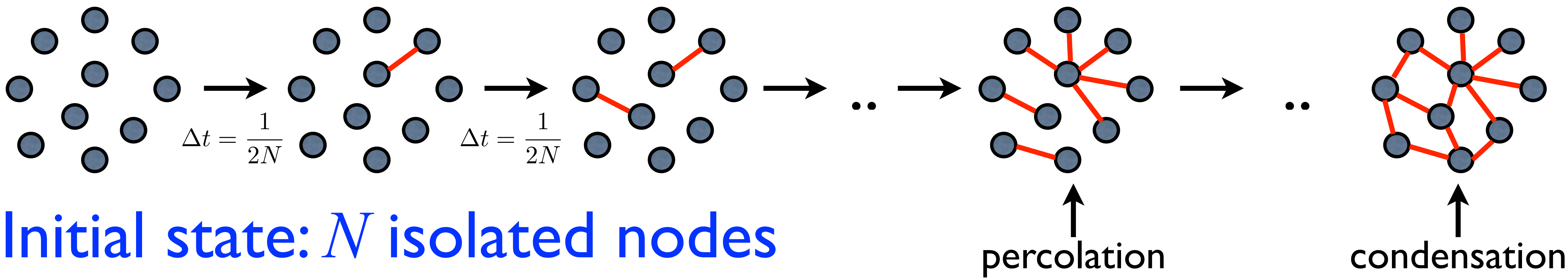
APS March Meeting, March 19, 2013

Plan

- Growing random graphs: uniform linking
  - Degree distribution
  - Component size distribution
- Growing random graphs: popularity-driven linking
  - Degree distribution
  - Component size distribution

Erdos, Renyi 60

Random Graphs: Uniform Linking



- Initial state: N isolated nodes
- Dynamical linking
  1. Pick two nodes at random
  2. Connect the two nodes with a link
  3. Augment time $\quad t \to t + \frac{1}{2N}$
- Each node experiences one linking event per unit time

Percolation: one component contains <u>fraction</u> of mass

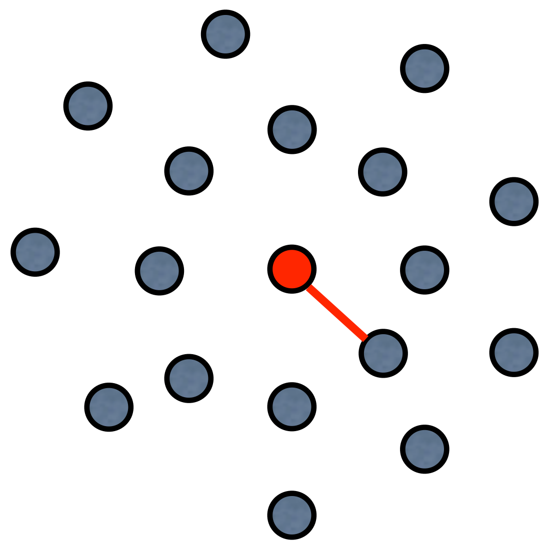
Condensation: one component contains <u>all</u> mass

Percolation time is finite; Condensation time is divergent

Degree Distribution & Condensation

- Distribution of nodes with degree j at time t is $n_j(t)$
- Linking process is simple augmentation

$$j \to j+1$$



- Linear evolution equation

$n_j(t=0) = \delta_{j,0}$

$$\frac{dn_j}{dt} = n_{j-1} - n_j$$

- Degree distribution is Poissonian

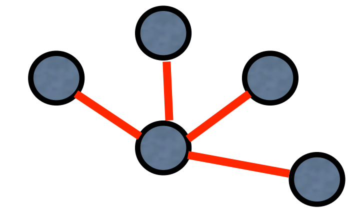
$$n_j(t) = \frac{t^j}{j!} e^{-t}$$

- Isolated nodes disappear when $N\, n_0(t_c) = 1$

$$t_c \simeq \ln N$$

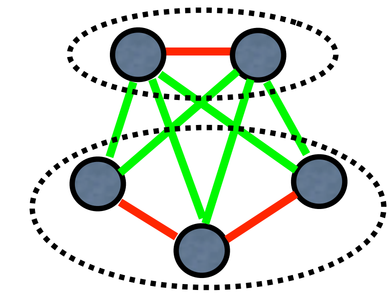
Condensation time diverges with system size

Component Size Distribution & Percolation



- Component = a connected set of nodes
- Merger rate = product of component sizes

$$[i] + [j] \xrightarrow{K_{i,j}} [i,j] \qquad K_{ij} = ij$$



- Nonlinear evolution equation

$$\frac{dc_k}{dt} = \frac{1}{2}\sum_{i+j=k} ij c_i c_j - k c_k \qquad c_k(t=0) = \delta_{k,1}$$

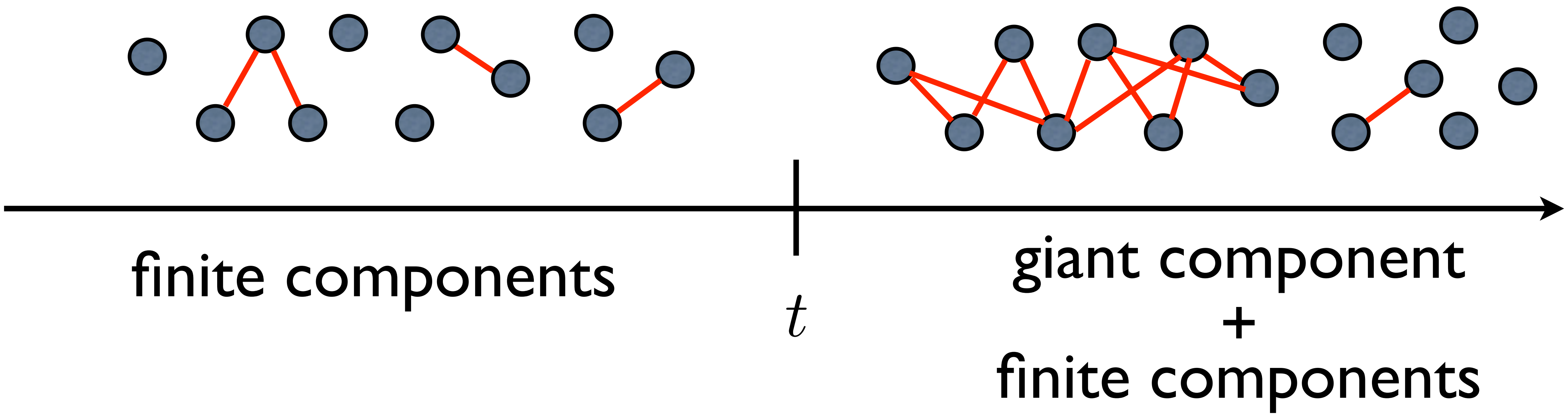
- Component size distribution

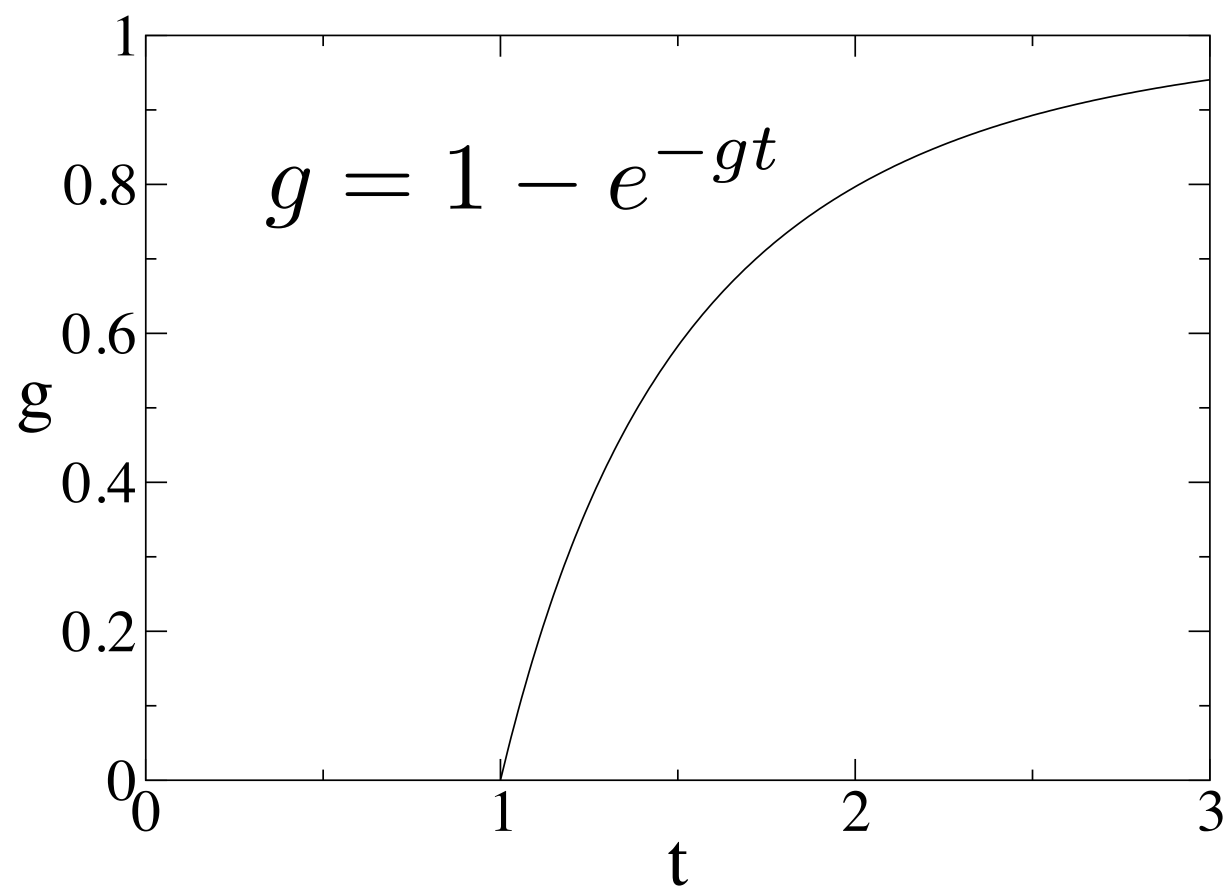
$$c_k(t) = \frac{1}{k \cdot k!}(kt)^{k-1}e^{-kt} \qquad \sum_k k\, c_k = \begin{cases} 1 & t < 1 \\ 1-g & t > 1 \end{cases}$$

- Percolation: finite clusters contain only fraction of mass
- Giant component with macroscopic size emerges

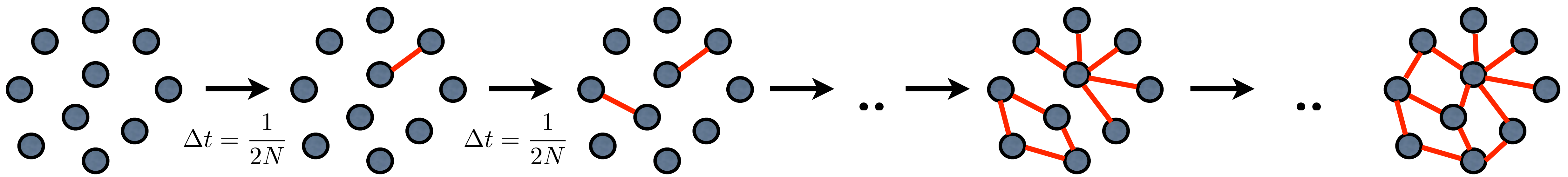
Percolation time is finite, independent of N

Two Phases


finite components
t
giant component
+
finite components


$g = 1 - e^{-gt}$
g
t
0
0.2
0.4
0.6
0.8
1
0
1
2
3

Random Graphs: Popularity-Driven Linking



- Initial state: N isolated nodes
- Dynamical "popularity-driven" linking,
    1. Pick 2 nodes, each with <u>probability proportional to degree</u>
    2. Connect the 2 nodes with a link
- Motivation: online social networks, friends seek and accept friends according to popularity (facebook)

Barabasi-Albert 99

- Rich-gets-richer mechanism as in preferential attachment
- Hybrid between random graph and preferential attachment

Nature of percolation and condensation transitions?

Degree Distribution

- Distribution of nodes with degree *j* is n_j
- Linking process with linear linking rate

$$(i,j) \xrightarrow{C_{i,j}} (i+1, j+1) \qquad C_{i,j} = (i+1)(j+1)$$

- Linear evolution equation

$$\frac{dn_j}{dt} = \big(1 + \langle j \rangle\big)\Big[j\, n_{j-1} - (j+1)\, n_j\Big]$$

- Exponential degree distribution

$$\langle j \rangle = \frac{1}{1-t}$$

$$n_j = (1-t)\, t^j$$

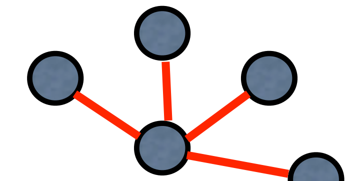
$$t_c = 1$$

- Isolated nodes disappear in finite time!
- Rich gets richer may not produce broad distribution

Condensation in finite time!

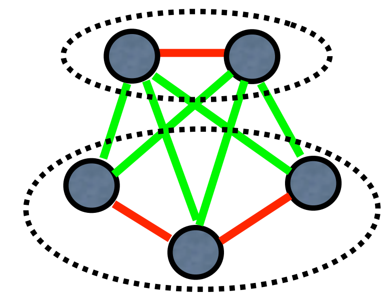
Component Size Distribution

- Components are trees: total degree gives total links
- Merger rate = product of number of links



$$[l] + [m] \xrightarrow{K_{l,m}} [l+m] \qquad K_{l,m} = (3l-2)(3m-2)$$

- <u>Closed nonlinear</u> evolution equation



$$\frac{dc_k}{dt} = \frac{1}{2}\sum_{l+m=k}(3l-2)(3m-2)c_l c_m - \langle j+1\rangle(3k-2)c_k,$$

$$c_k(t=0) = \delta_{k,1}$$

- Component size distribution

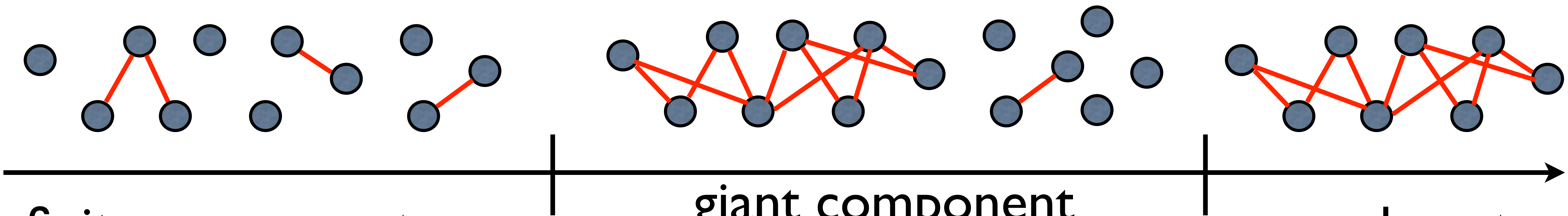
$$c_k = \frac{(3k-3)!}{k!(2k-1)!}\, t^{k-1}(1-t)^{2k-1} \qquad \sum_k k\, c_k = \begin{cases} 1 & t < 1/3 \\ 1-g & t > 1/3 \end{cases}$$

- Percolation: finite clusters contain only fraction of mass
- Second moment diverges

$$\sum_k k^2 c_k = \frac{1-2t}{1-3t} \qquad t_g = 1/3$$

Percolation time is smaller than condensation time!

Three Phases



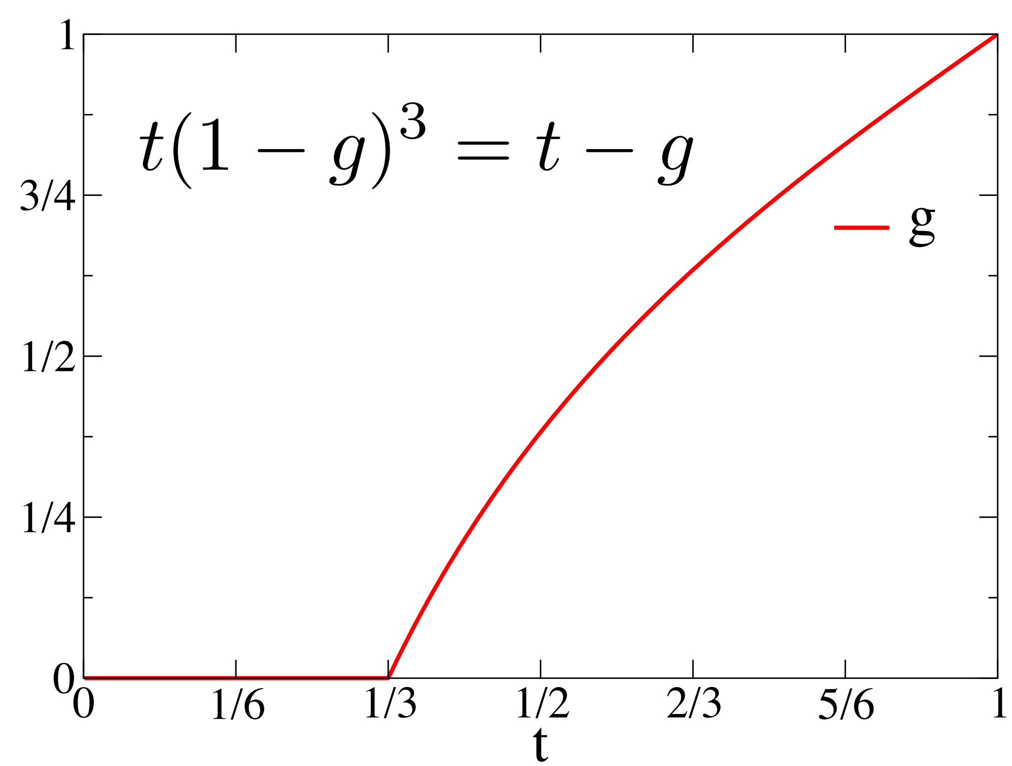
finite components t_g giant component + finite components t_c condensate

Moments of size distribution diverge prior to percolation

$$\langle k^2 \rangle = \frac{1-2t}{1-3t}$$

Average degree diverges prior to condensation

$$\langle j \rangle = \frac{1}{1-t}$$



size distribution

$$c_k \sim k^{-5/2} e^{-k/k_*}$$

precisely at critical point

$$c_k \sim k^{-5/2}$$

Two successive finite time singularities!

Generalized linking rates

- LInking rate is a general power of degree

$$C_{ij} = (ij)^{\alpha}$$

- Average degree

condensation time

$$\langle j \rangle \sim \begin{cases} t^{1/(1-2\alpha)} & \alpha < 1/2, \\ e^{\text{const.}\times t} & \alpha = 1/2, \\ (t_c - t)^{-1/(2\alpha-1)} & 1/2 < \alpha \leq 1. \end{cases}$$

divergent

finite

- Instantaneous condensation

$$t_c \sim (\ln N)^{-\gamma} \quad \text{when} \quad \alpha > 1$$

Condensation time: divergent, finite, or, vanishing

Summary

- Popularity-driven evolution of a random graphs
- Linking rate is proportional to degree (rich-gets-richer)
- Degree distribution is exponential
- Percolation time is finite
- Condensation time is finite

Outlook

- Cycle structure: number, size, distribution, etc.
- Analysis of condensate: cycles, shortest paths