Kinetics of Brownian Maxima

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arXiv: 1405.0533

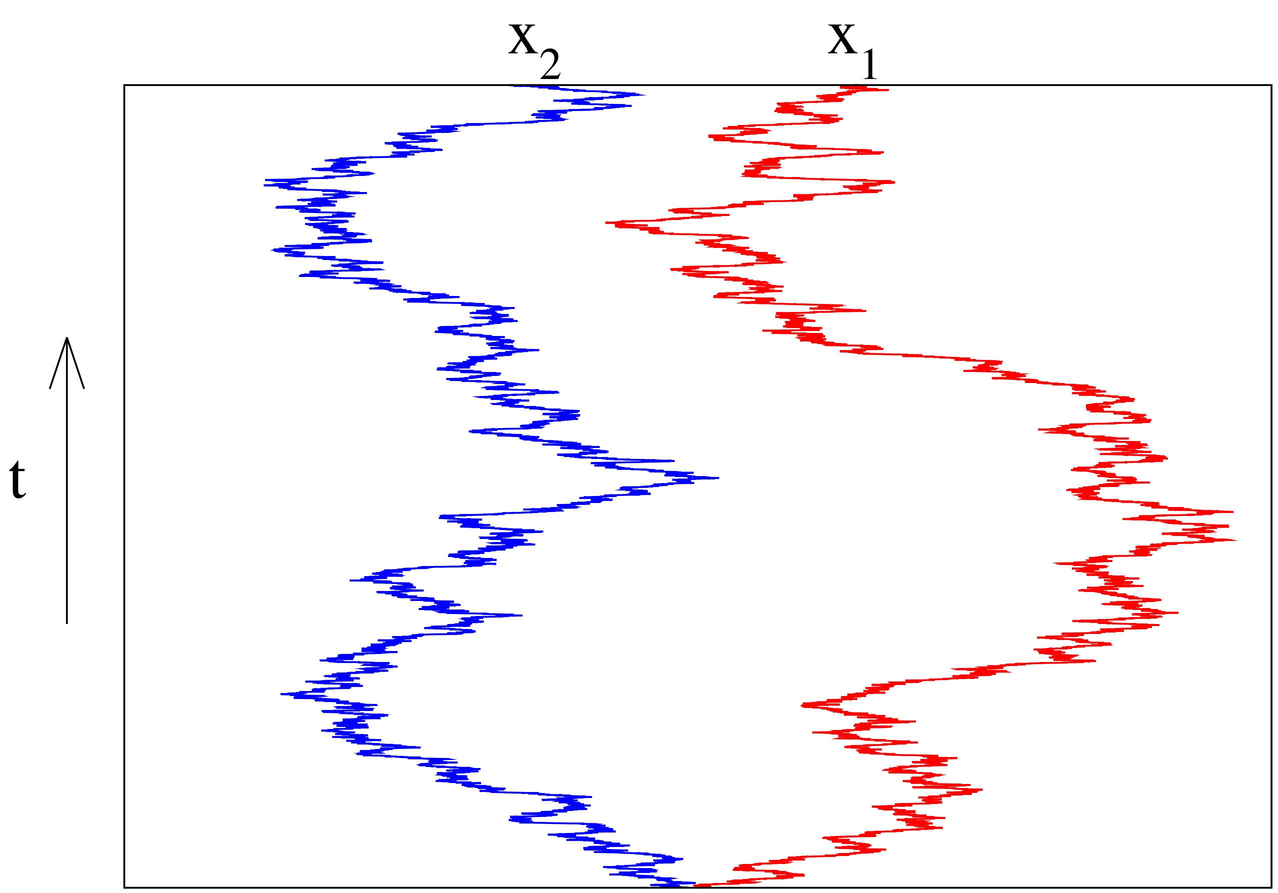
Talk, publications available from: http://cnls.lanl.gov/~ebn

Sidney Redner Fest, Boston University, May 10, 2014

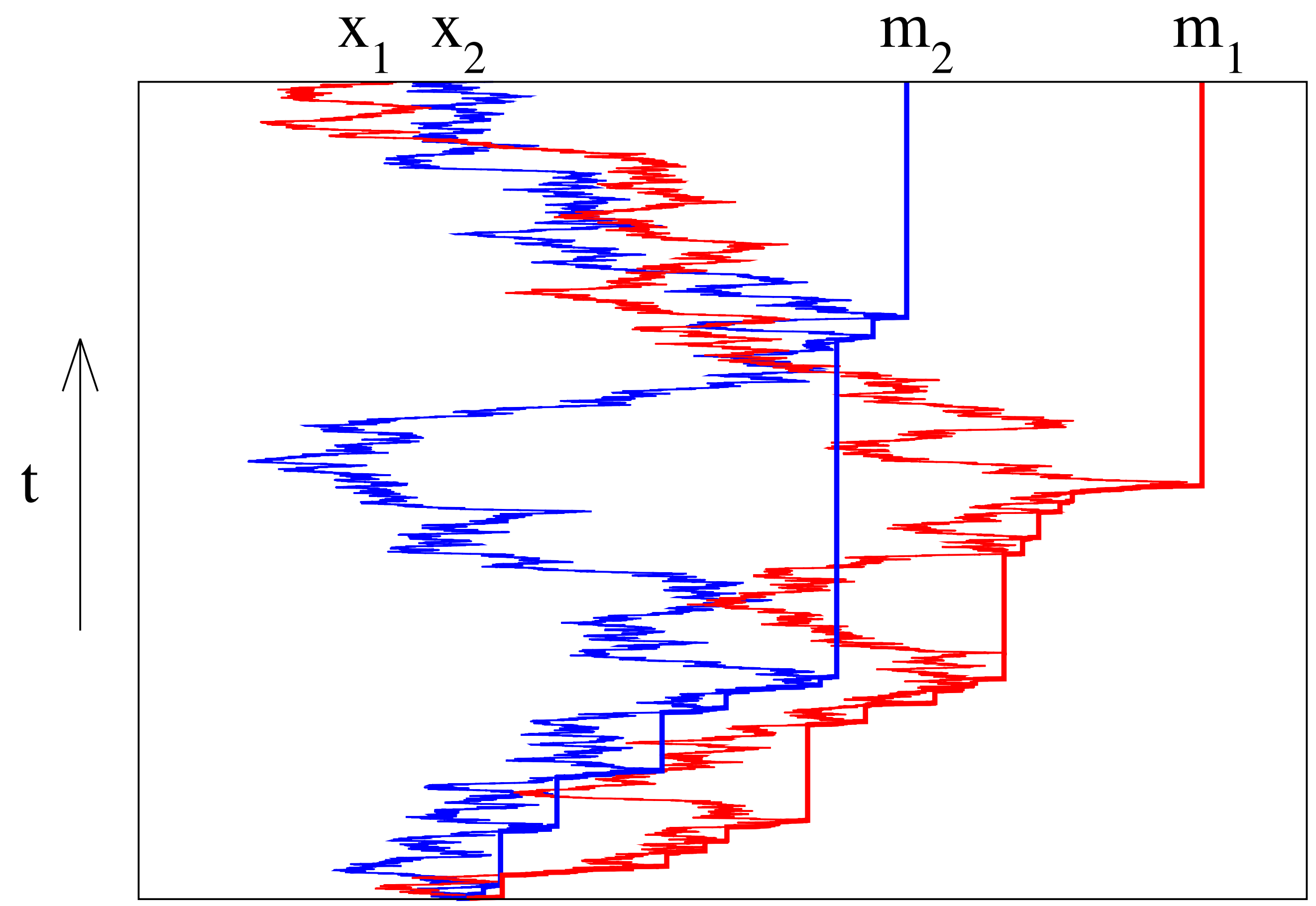
Extreme value statistics
New frontier in nonequilibrium statistical physics

- Brownian motion — Comtet & Majumdar, Krug, Redner
- Surface growth — Spohn, Halpin-Healy, Majumdar & Schehr
- Transport — Derrida & Lebowitz & Speer
- Climate — Bunde & Havlin, Krug & Wergen, Redner
- Earthquakes — Davidesn, Newman & Turcotte, EB
- Finance — Bouchaud, Stanley, Majumdar

Brownian Positions

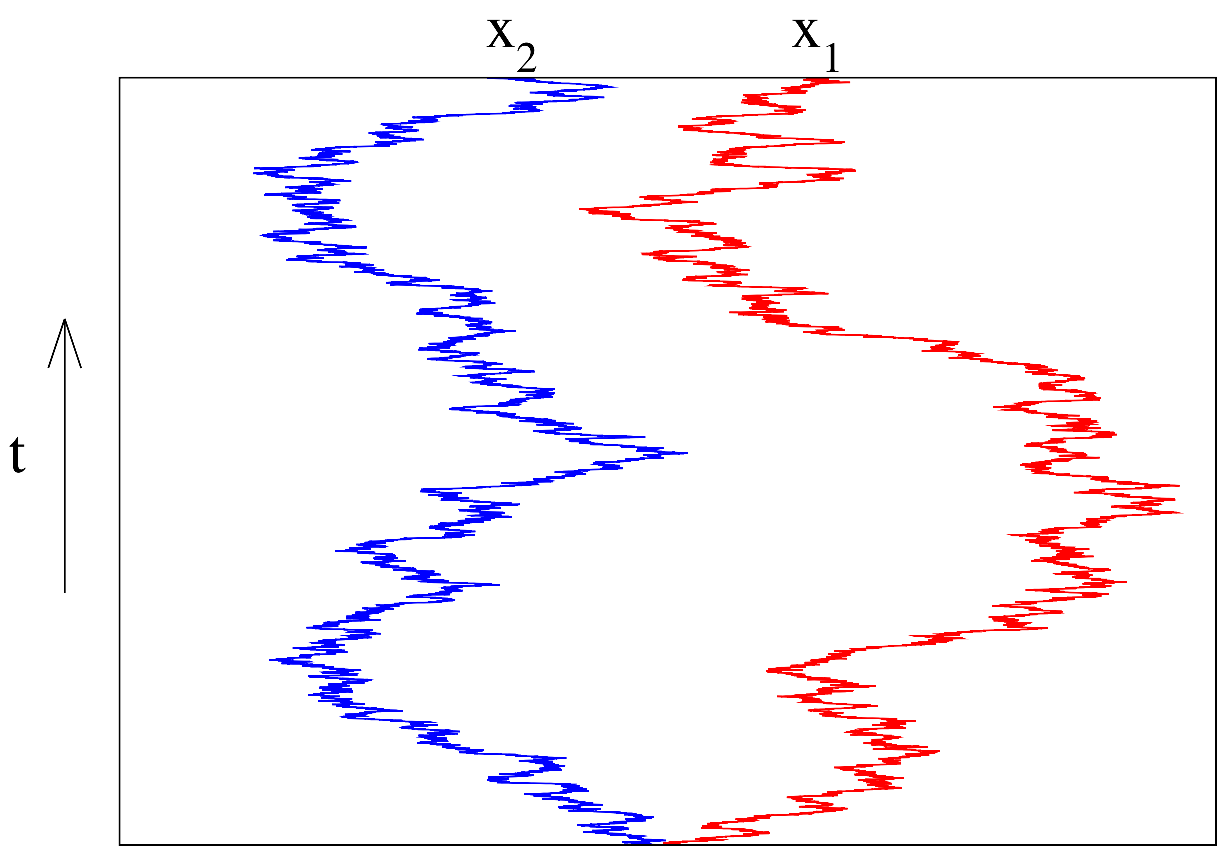


Brownian Maxima



First-Passage Kinetics: Brownian Positions

Probability two Brownian particle do not meet



- Universal probability

Sparre Andersen 53

$$S_t = \binom{2t}{t} 2^{-t}$$

- Asymptotic behavior

Feller 68

$$S \sim t^{-1/2}$$

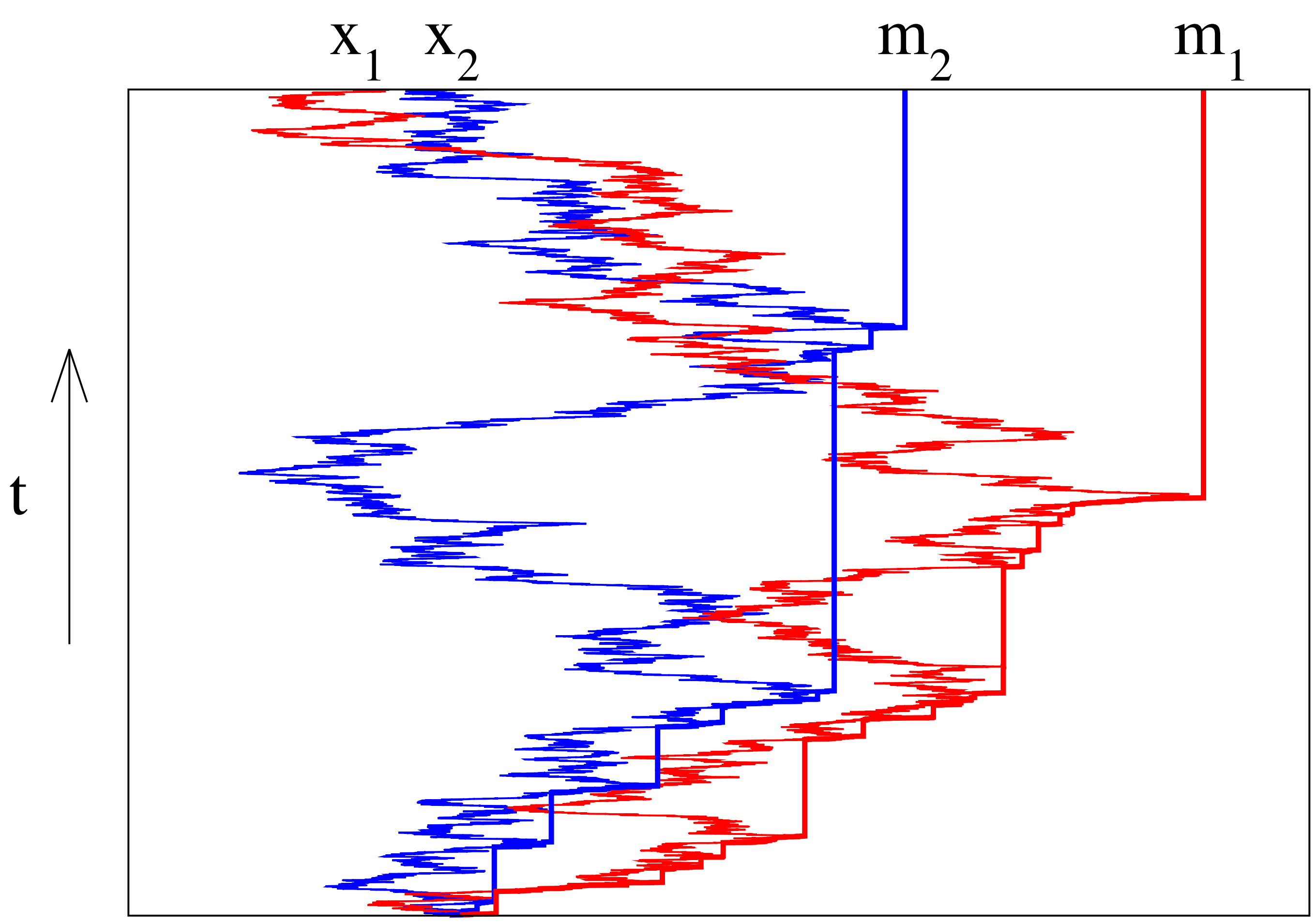
Behavior holds for Levy flights, different mobilities, etc

Universal first-passage exponent

S. Redner, A guide to First-Passage Processes 2001

First-Passage Kinetics: Brownian Maxima

Probability maximal positions remain ordered



- Numerical simulations

$$S \sim t^{-\beta}$$

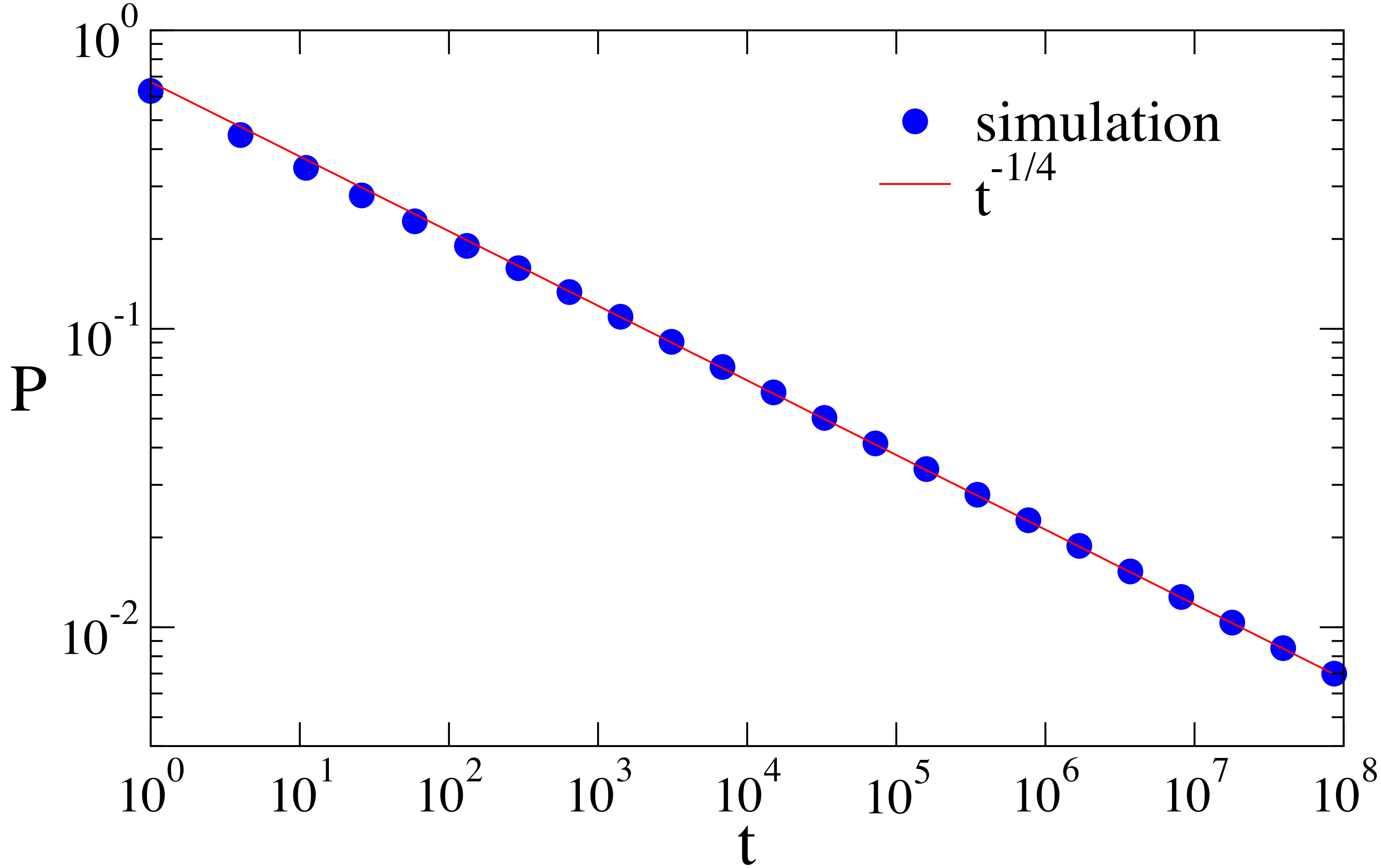
- First-passage exponent

$$\beta = 0.2503 \pm 0.0005$$

Is ¼ exact? and if it is, does ¼=½×½?

Is exponent universal?

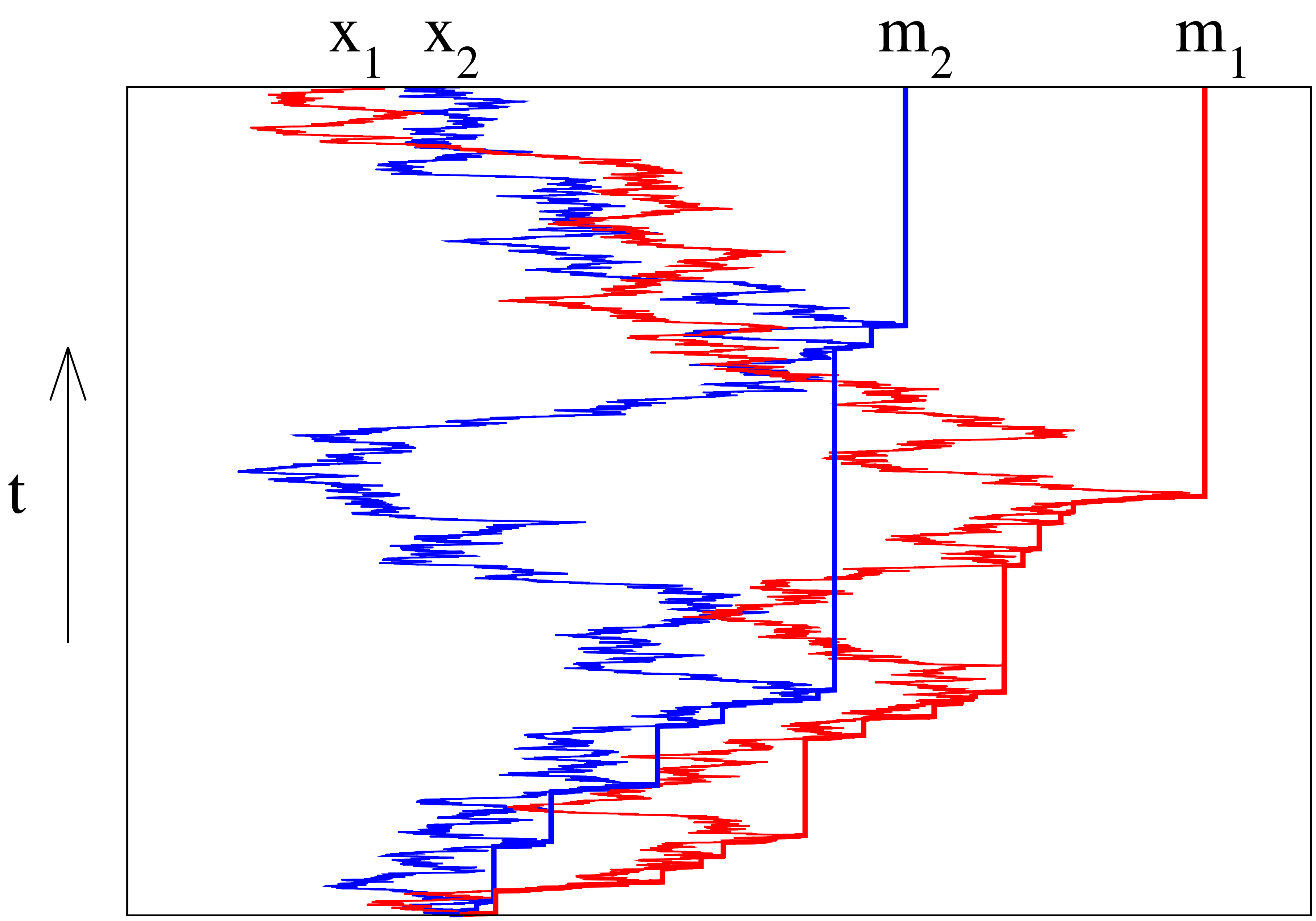
Monte Carlo Simulations



Hints at a rational exponent

Inconclusive due to slow convergence

$m_1 > m_2$ if and only if $m_1 > x_2$



From four variables to three

- Four variables: two positions, two maxima

$$m_1 > x_1 \quad \text{and} \quad m_2 > x_2$$

- The two maxima must always be ordered

$$m_1 > m_2$$

- Key observation: trailing maximum is irrelevant!

$$m_1 > m_2 \quad \text{if and only if} \quad m_1 > x_2$$

- Three variables: two positions, one maximum

$$m_1 > x_1 \quad \text{and} \quad m_1 > x_2$$

From three variables to two

- Introduce two distances from the maximum

$$u = m_1 - x_1 \quad \text{and} \quad v = m_1 - x_2$$

- Both distances undergo Brownian motion

$$\frac{\partial \rho(u,v,t)}{\partial t} = D\nabla^2 \rho(u,v,t)$$

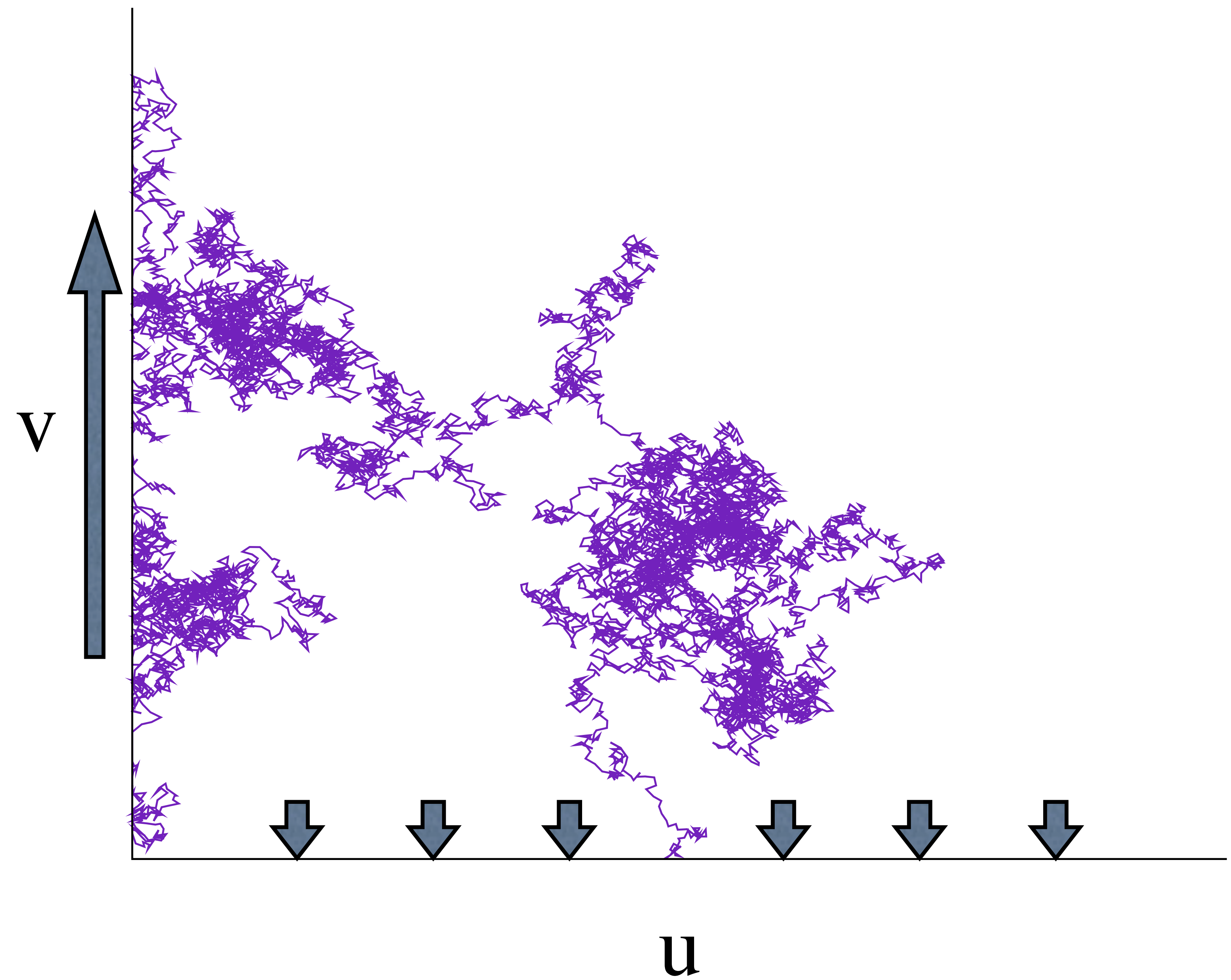
- Boundary conditions: (i) absorption (ii) advection

$$\rho\Big|_{v=0} = 0 \quad \text{and} \quad \left(\frac{\partial \rho}{\partial u} - \frac{\partial \rho}{\partial v}\right)\Big|_{u=0} = 0$$

- Probability maxima remain ordered

$$P(t) = \int_0^\infty \int_0^\infty du\, dv\, \rho(u,v,t)$$

Diffusion in corner geometry



“Backward” evolution

- Study evolution as function of **initial conditions**

$$P \equiv P(u_0, v_0, t)$$

- Obeys backward diffusion equation

$$\frac{\partial P(u_0, v_0, t)}{\partial t} = D\nabla^2 P(u_0, v_0, t)$$

- Boundary conditions: (i) absorption (ii) advection

$$P\Big|_{v_0=0} = 0 \qquad \text{and} \qquad \left(\frac{\partial P}{\partial u_0} + \frac{\partial P}{\partial v_0}\right)\Big|_{u_0=0} = 0$$

- Advection boundary condition is conjugate!

Solution

- Use polar coordinates

$$r = \sqrt{u_0^2 + v_0^2} \quad \text{and} \quad \theta = \arctan \frac{v_0}{u_0}$$

- Laplace operator

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

- Boundary conditions: (i) absorption (ii) advection

$$P\big|_{\theta=0} = 0 \quad \text{and} \quad \left(r\,\frac{\partial P}{\partial r} - \frac{\partial P}{\partial \theta}\right)\Big|_{\theta=\pi/2} = 0$$

- dimensional analysis + power law + separable form

$$P(r,\theta,t) \sim \left(\frac{r^2}{Dt}\right)^{\beta} f(\theta)$$

Selection of exponent

- Exponent related to eigenvalue of angular part of Laplacian

$$f''(\theta) + (2\beta)^2 f(\theta) = 0$$

- Absorbing boundary condition selects solution

$$f(\theta) = \sin(2\beta\theta)$$

- Advection boundary condition selects exponent

$$\tan(\beta\pi) = 1$$

- First-passage probability

$$P \sim t^{-1/4}$$

considine & redner 88

General Diffusivities

ben Avraham
Leyvraz 88

- Particles have diffusion constants D_1 and D_2

$$(x_1, x_2) \to (\widehat{x}_1, \widehat{x}_2) \quad \text{with} \quad (\widehat{x}_1, \widehat{x}_2) = \left(\frac{x_1}{\sqrt{D_1}}, \frac{x_2}{\sqrt{D_2}}\right)$$

- Condition on maxima involves ratio of mobilities

$$\sqrt{\frac{D_1}{D_2}}\,\widehat{m}_1 > \widehat{m}_2$$

- Analysis straightforward to repeat

$$\sqrt{\frac{D_1}{D_2}}\tan\left(\beta\pi\right) = 1$$

- First-passage exponent: nonuniversal, mobility-dependent

$$\beta = \frac{1}{\pi}\arctan\sqrt{\frac{D_2}{D_1}}$$

Properties

- Depends on ratio of diffusion constants

$$\beta(D_1, D_2) \equiv \beta\left(\frac{D_1}{D_2}\right)$$

- Bounds: involve one immobile particle

$$\beta(0) = \frac{1}{2} \qquad \beta(\infty) = 0$$

- Rational for special values of diffusion constants

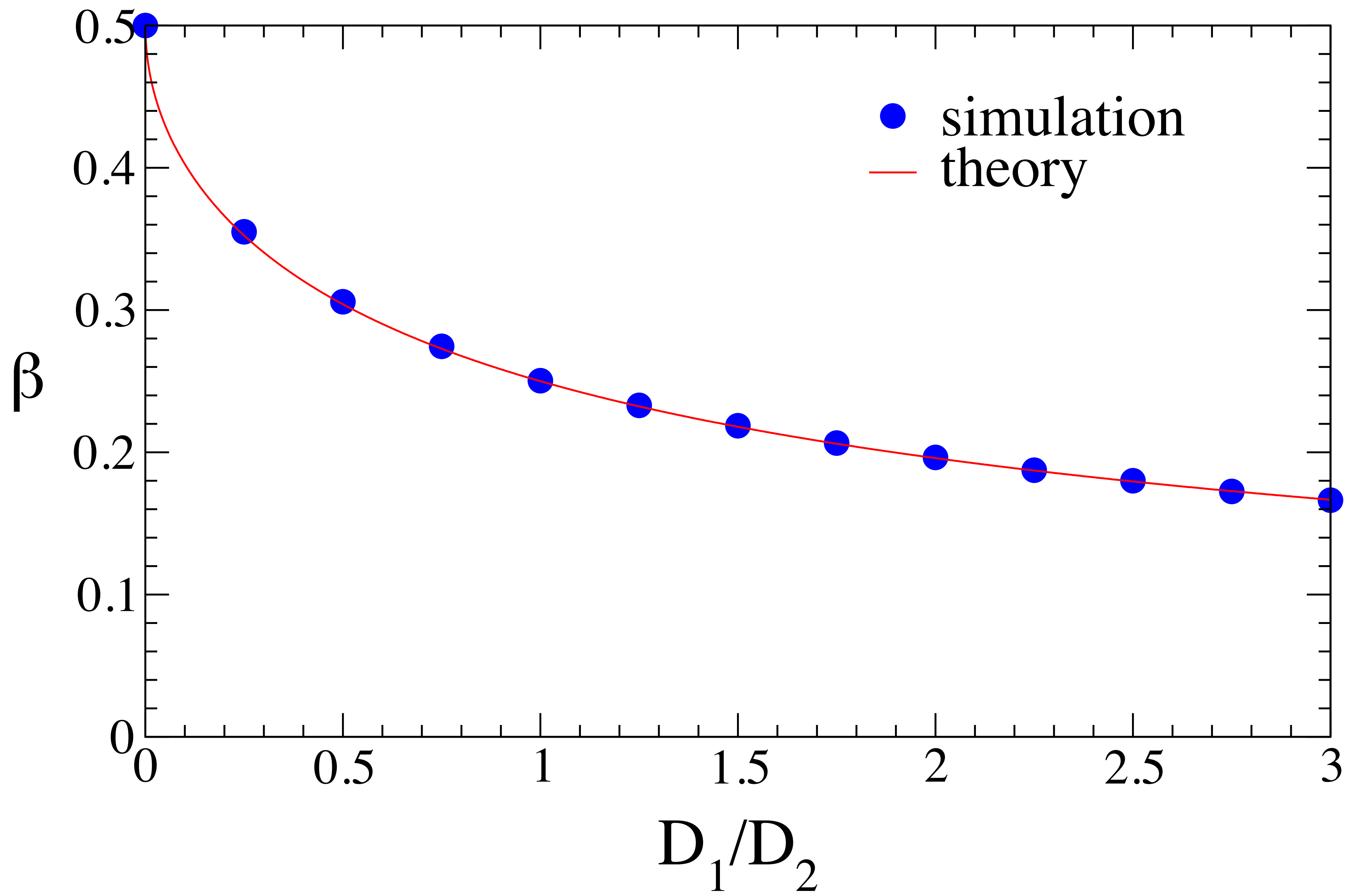
$$\beta(1/3) = 1/3 \qquad \beta(1) = 1/4 \qquad \beta(3) = 1/6$$

- Duality: between "fast chasing slow" and "slow chasing fast"

$$\beta\left(\frac{D_1}{D_2}\right) + \beta\left(\frac{D_2}{D_1}\right) = \frac{1}{2}$$

Alternating kinetics: slow-fast-slow-fast

Numerical verification

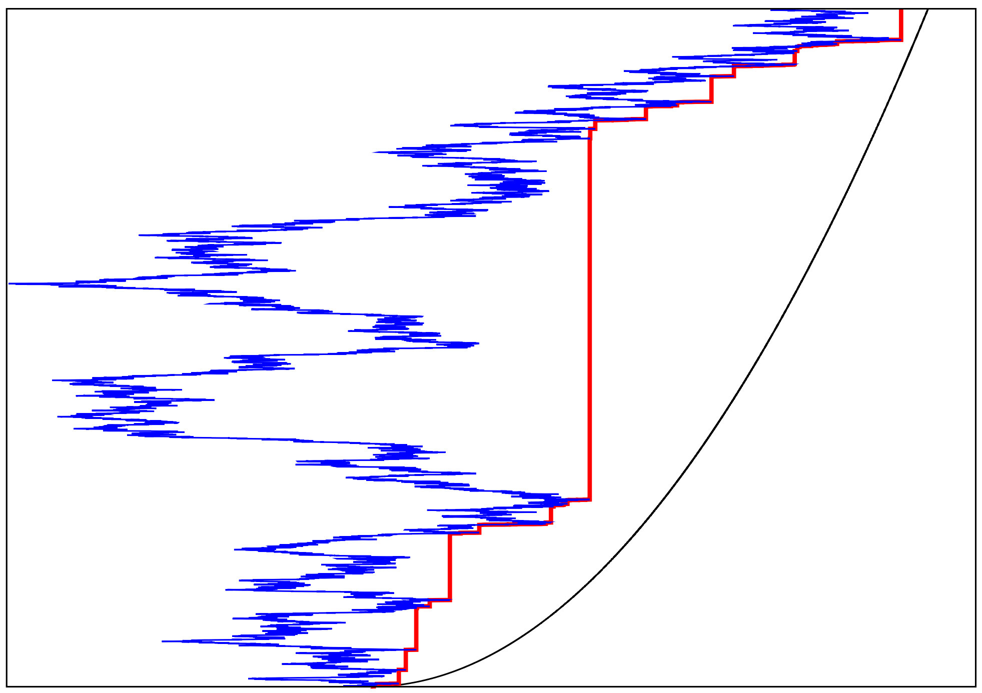


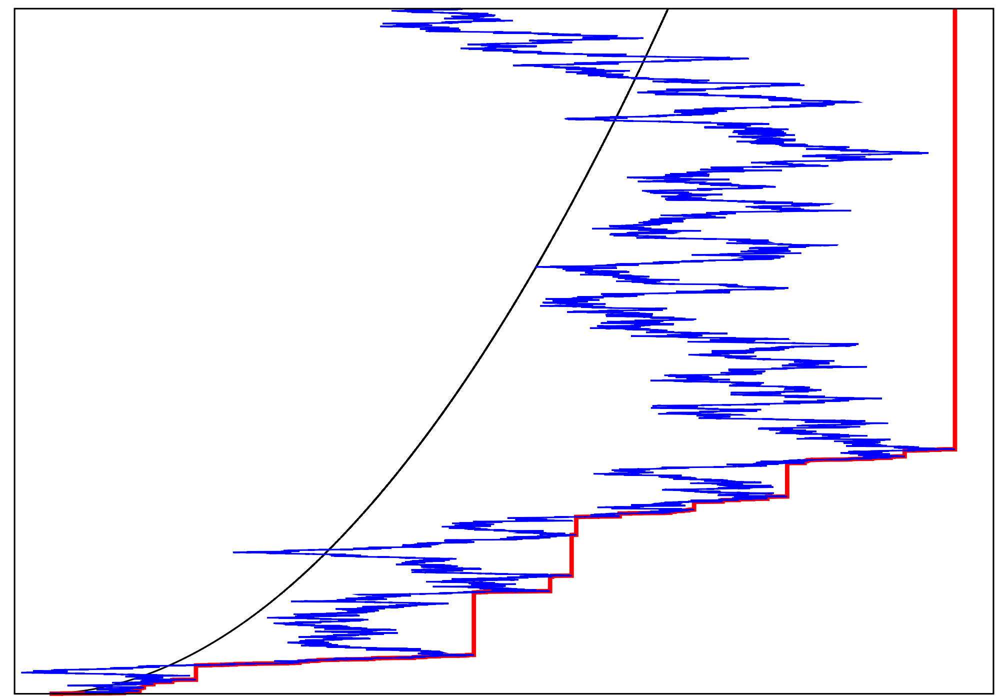
Perfect agreement
between theory and simulation

Inferior & Superior walks

Maximum is always behind or ahead of the a average maximum of a Brownian particle

Krapivsky & redner 95
EB & Krapivsky 14





$$D_{2\beta}\left(-\sqrt{2/\pi}\right)=0 \quad \beta=0.241608$$

$$D_{2\beta+1}\left(\sqrt{2/\pi}\right)=0 \quad \beta=0.382258$$

Different mobilities: neglect fluctuations in maximum of slower particle (represent maximum by its average) and obtain limits

$$\beta \simeq \begin{cases} \frac{1}{2}-\frac{1}{\pi}\sqrt{D_1/D_2} & D_1 \ll D_2 \\ \frac{1}{\pi}\sqrt{D_2/D_1} & D_2 \ll D_1 \end{cases}$$

Multiple particles

- All maxima perfectly ordered

Fisher & Huse 88

$$m_1 > m_2 > m_3 > \cdots > m_n$$

- Only one leader

Bramson & Griffith 91

$$m_1 > m_2 \quad m_1 > m_3 \quad \cdots \quad m_1 > m_n$$

- Only one laggard

ben Avraham & Redner 03

$$m_1 > m_n \quad m_2 > m_n \quad \cdots \quad m_{n-1} > m_n$$

- Three families of first-passage exponents

$$A_n \sim t^{-\alpha_n} \qquad B_n \sim t^{-\beta_n} \qquad C_n \sim t^{-\gamma_n}$$

Exponents are eigenvalues of "angular" component of Laplace in n dimensions

Three families of exponents

- Simulation results: maxima vs positions

Grassberger 03
ben Avraham 03
EB & Krapivsky 10

	maxima			positions		
n	α_n	β_n	γ_n	a_n	b_n	c_n
2	$1/4$	$1/4$	$1/4$	$1/2$	$1/2$	$1/2$
3	0.653	0.432	0.335	$3/2$	$3/4$	$3/8$
4	1.13	0.570	0.376	3	0.91	0.306
5	1.60	0.674	0.401	5	1.02	0.265
6	2.01	0.759	0.417	$15/2$	1.11	0.234

- Positions: one family is known

Fisher & Huse 88

$$b_n = \frac{n(n-1)}{4}$$

- Asymptotic behavior for large number of particles

$$\alpha_n \sim n \qquad \beta_n \simeq b_n \simeq \frac{1}{4}\ln n \qquad \gamma_n \to \frac{1}{2}$$

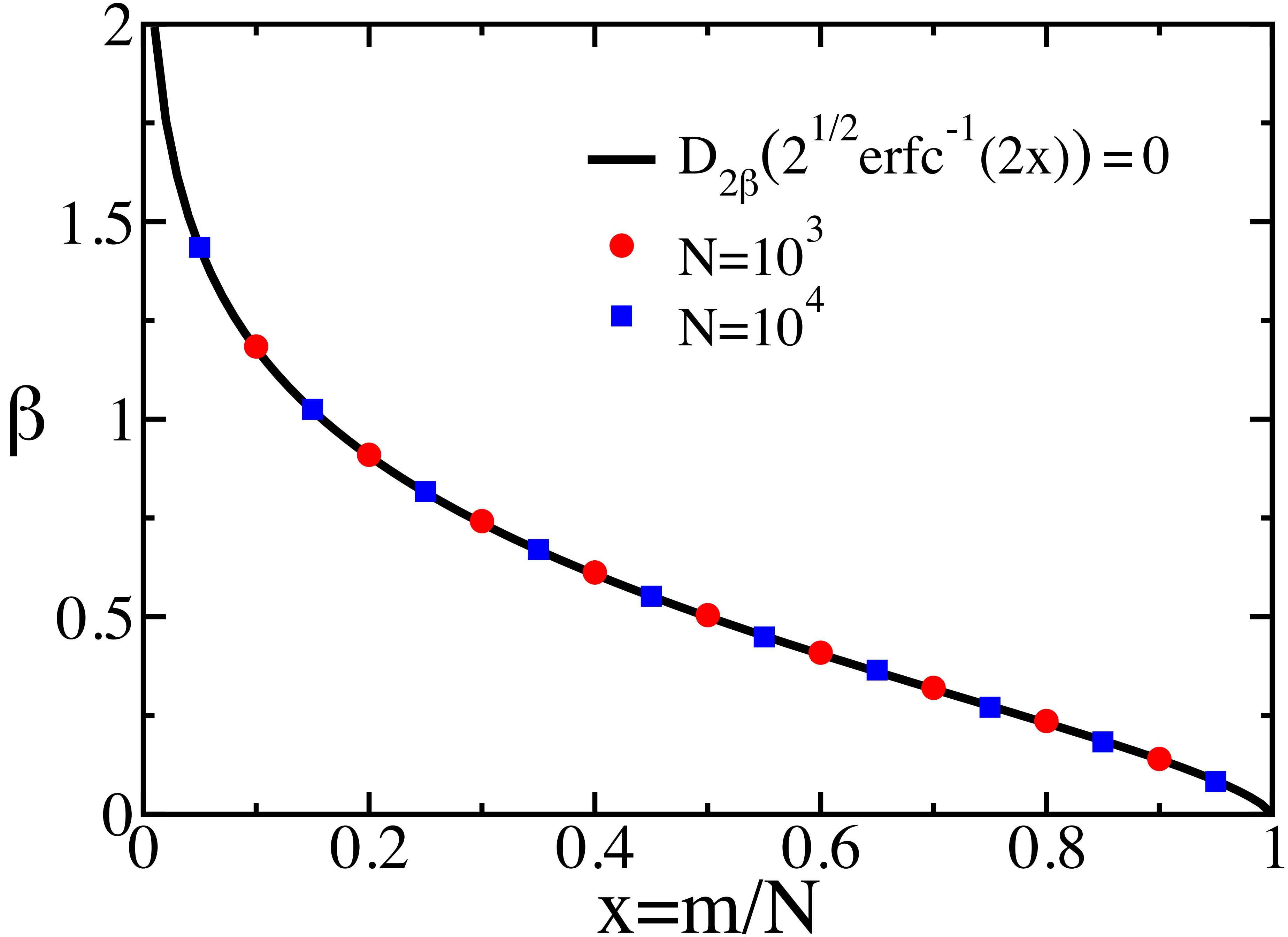
- And a conjecture!

$$\gamma_n = \frac{n-1}{2n}$$

$$\gamma_1 = 0$$
$$\gamma_2 = 1/4$$

Eigenvalues have scaling laws

in thermodynamic limit



Summary

- First-passage kinetics of extremes in Brownian motion
- Problem reduces to diffusion in a two-dimensional corner with mixed boundary conditions
- First-passage exponent obtained analytically
- Exponent is continuously varying function of mobilities
- Relaxation is generally slower compared with positions
- Open questions: multiple particles, higher dimensions
- Scaling of eigenvalues in thermodynamics limit?
- “Race between maxima” as a data analysis tool