Kinetics of Brownian Maxima

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Talk, publications available from: http://cnls.lanl.gov/~ebn

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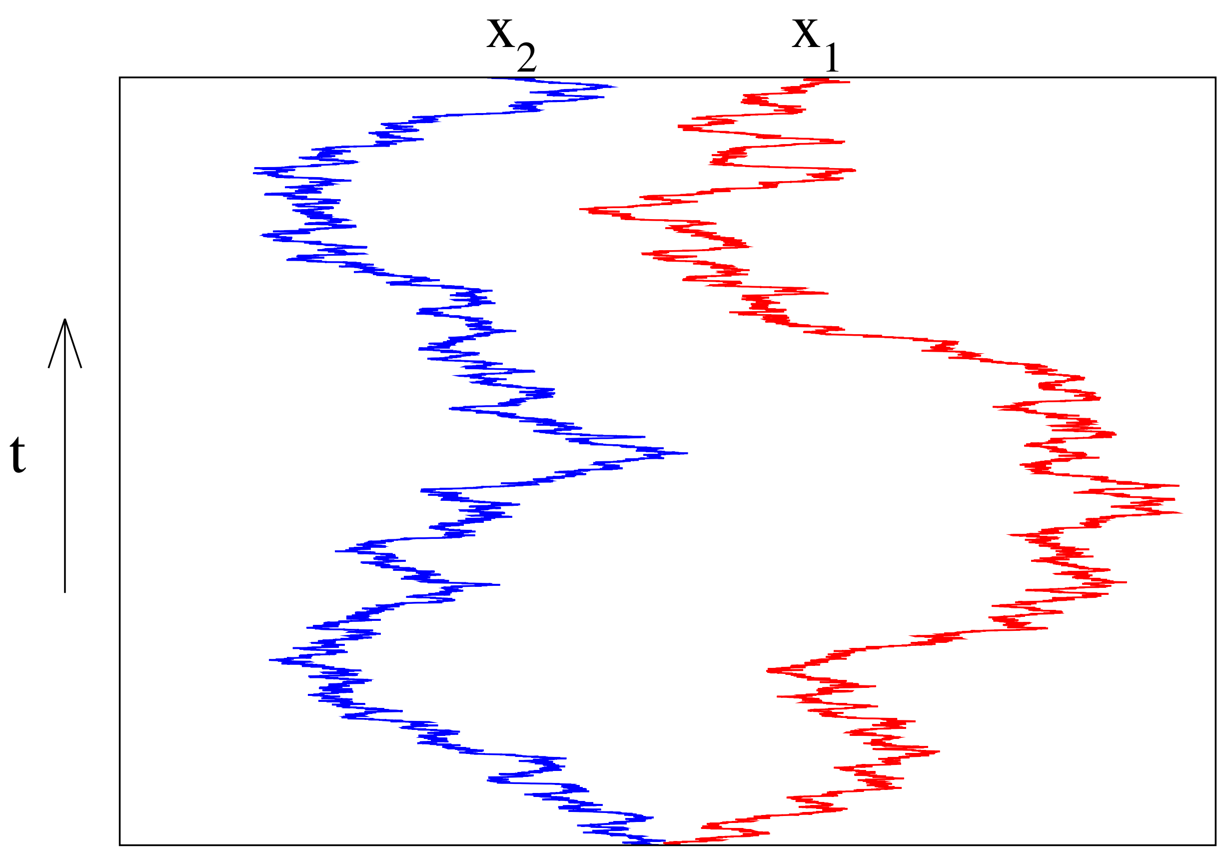
First-Passage Kinetics: Brownian Positions

Probability two Brownian particle do not meet



- Universal probability — Sparre Andersen 53

$$S_t = \binom{2t}{t} 2^{-t}$$

- Asymptotic behavior — Feller 68

$$S \sim t^{-1/2}$$

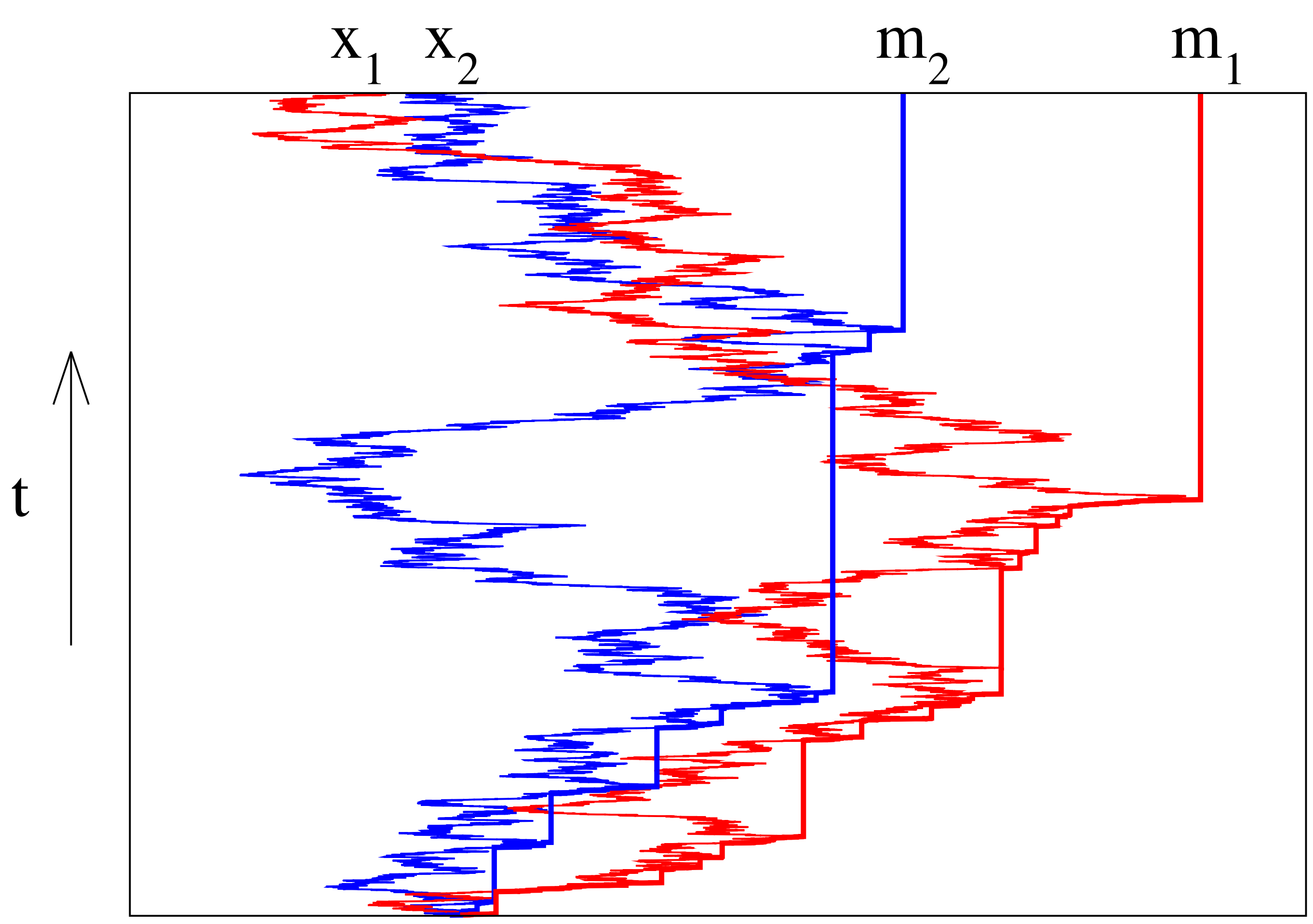
Behavior holds for Levy flights, different mobilities, etc

Universal first-passage exponent

S. Redner, A guide to First-Passage Processes 2001

First-Passage Kinetics: Brownian Maxima

Probability maximal positions remain ordered

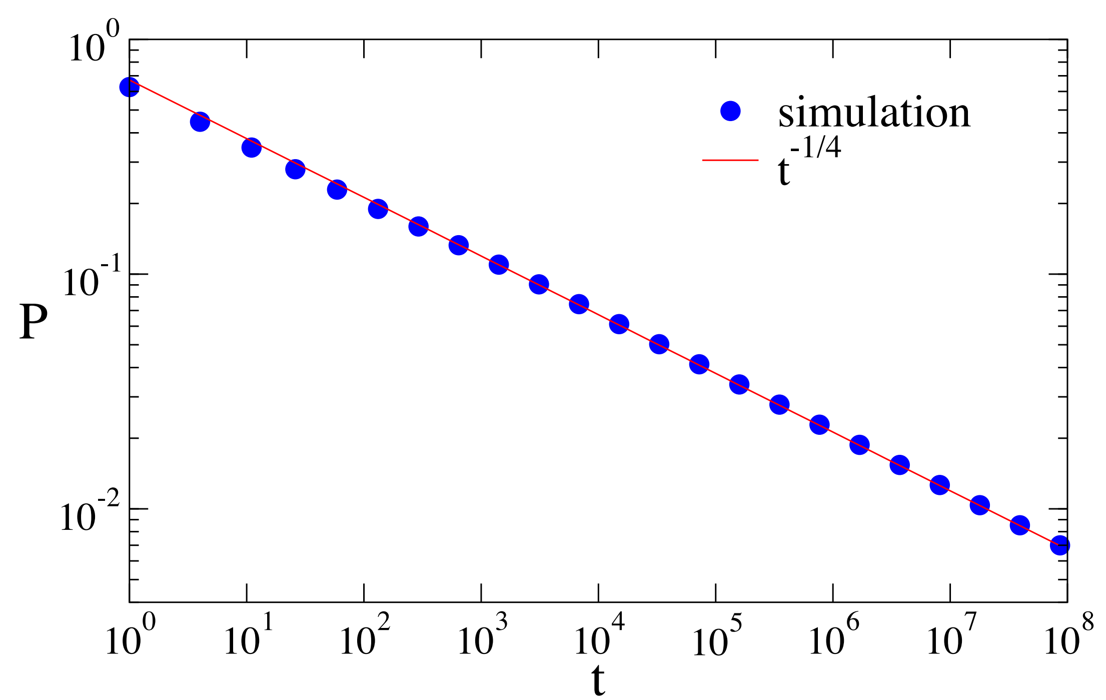


- Numerical simulations

$$S \sim t^{-\beta}$$

- First-passage exponent

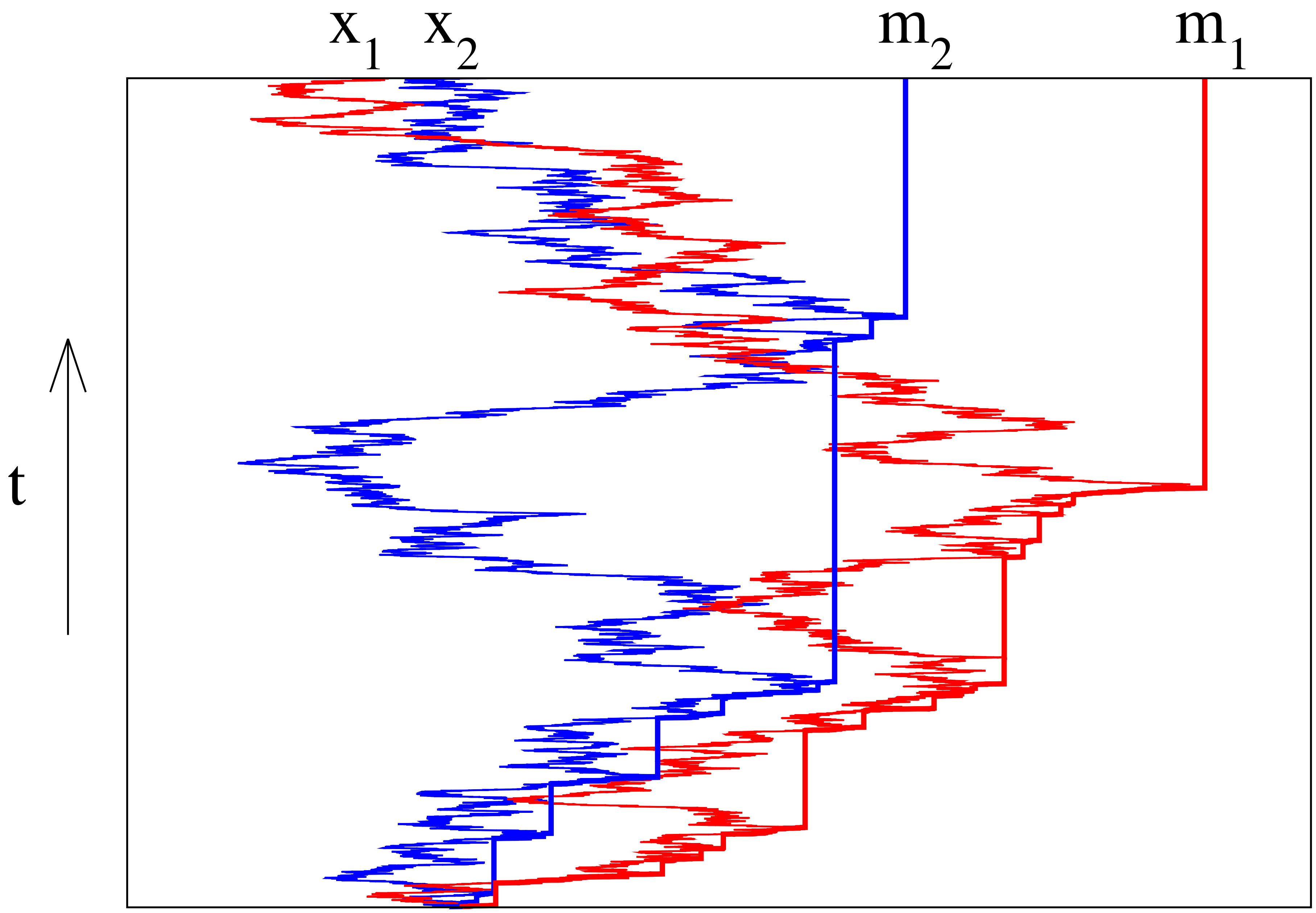
$$\beta = 0.2503 \pm 0.0005$$



Is ¼ exact?

Is exponent universal?

$m_1 > m_2$ if and only if $m_1 > x_2$



From four variables to three

- Four variables: two positions, two maxima

$$m_1 > x_1 \quad \text{and} \quad m_2 > x_2$$

- The two maxima must always be ordered

$$m_1 > m_2$$

- Key observation: trailing maximum is irrelevant!

$$m_1 > m_2 \quad \text{if and only if} \quad m_1 > x_2$$

- Three variables: two positions, one maximum

$$m_1 > x_1 \quad \text{and} \quad m_1 > x_2$$

From three variables to two

- Introduce two distances from the maximum

$$u = m_1 - x_1 \quad \text{and} \quad v = m_1 - x_2$$

- Both distances undergo Brownian motion

$$\frac{\partial \rho(u,v,t)}{\partial t} = D\nabla^2 \rho(u,v,t)$$

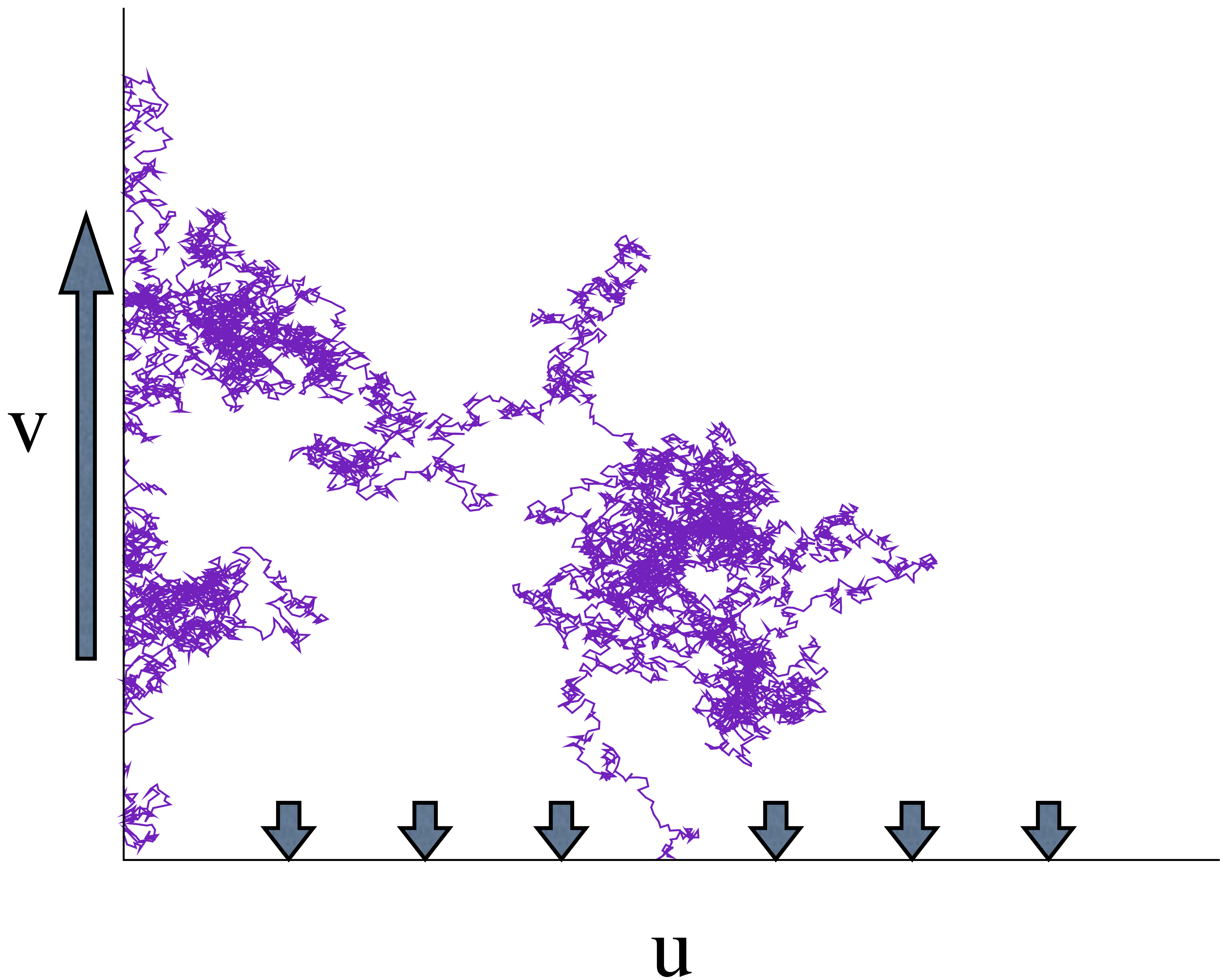
- Boundary conditions: (i) absorption (ii) advection

$$\rho\Big|_{v=0} = 0 \quad \text{and} \quad \left(\frac{\partial \rho}{\partial u} - \frac{\partial \rho}{\partial v}\right)\Bigg|_{u=0} = 0$$

- Probability maxima remain ordered

$$P(t) = \int_0^\infty \int_0^\infty du\, dv\, \rho(u,v,t)$$

Diffusion in corner geometry



"Backward" evolution

- Study evolution as function of **initial conditions**

$$P \equiv P(u_0, v_0, t)$$

- Obeys diffusion equation

$$\frac{\partial P(u_0, v_0, t)}{\partial t} = D\nabla^2 P(u_0, v_0, t)$$

- Boundary conditions: (i) absorption (ii) advection

$$P\Big|_{v_0=0} = 0 \qquad \text{and} \qquad \left(\frac{\partial P}{\partial u_0} + \frac{\partial P}{\partial v_0}\right)\Big|_{u_0=0} = 0$$

- Advection boundary condition is conjugate

Solution

- Use polar coordinates

$$r = \sqrt{u_0^2 + v_0^2} \quad \text{and} \quad \theta = \arctan \frac{v_0}{u_0}$$

- Laplace operator

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

- Boundary conditions: (i) absorption (ii) advection

$$P\big|_{\theta=0} = 0 \quad \text{and} \quad \left(r\,\frac{\partial P}{\partial r} - \frac{\partial P}{\partial \theta} \right) \Big|_{\theta=\pi/2} = 0$$

- dimensional analysis + power law + separable form

$$P(r,\theta,t) \sim \left(\frac{r^2}{Dt} \right)^{\beta} f(\theta)$$

Selection of exponent

- Exponent related to eigenvalue of angular part of Laplacian

$$f''(\theta) + (2\beta)^2 f(\theta) = 0$$

- Absorbing boundary condition selects solution

$$f(\theta) = \sin(2\beta\theta)$$

- Advection boundary condition selects exponent

$$\tan(\beta\pi) = 1$$

- First-passage probability

$$P \sim t^{-1/4}$$

General diffusivities

ben Avraham Leyvraz 88

- Particles have diffusion constants D_1 and D_2

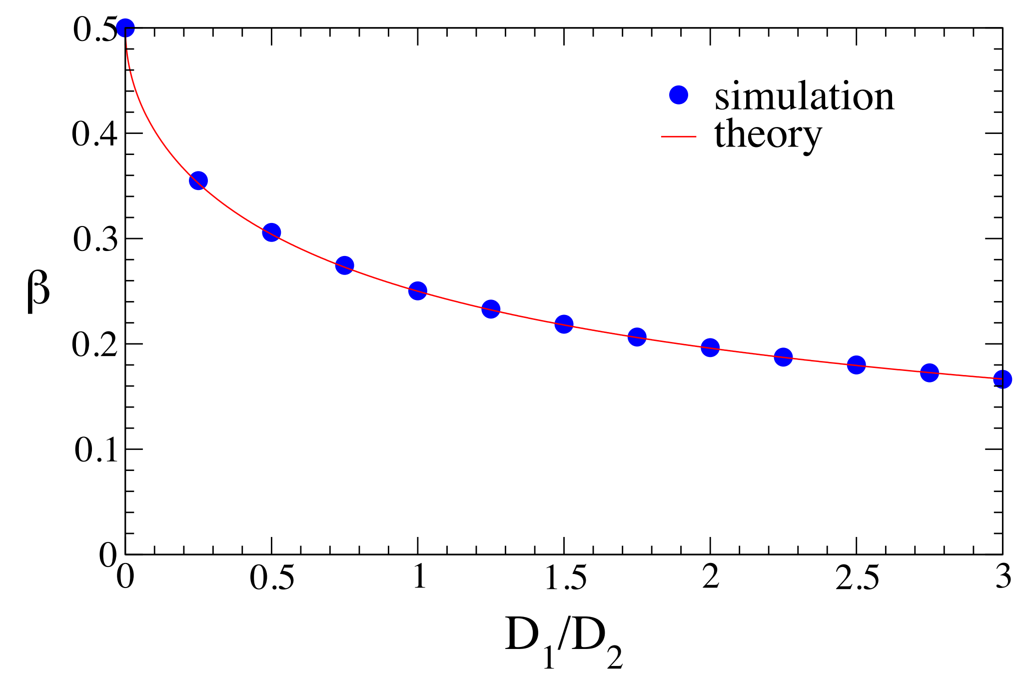
$$(x_1, x_2) \to (\widehat{x}_1, \widehat{x}_2) \quad \text{with} \quad (\widehat{x}_1, \widehat{x}_2) = \left(\frac{x_1}{\sqrt{D_1}}, \frac{x_2}{\sqrt{D_2}} \right)$$

- Condition on maxima involves ratio of mobilities

$$\sqrt{\frac{D_1}{D_2}}\, \widehat{m}_1 > \widehat{m}_2$$

- Analysis straightforward to repeat

$$\sqrt{\frac{D_1}{D_2}} \tan(\beta\pi) = 1$$



- First-passage exponent: nonuniversal, mobility-dependent

$$\beta = \frac{1}{\pi} \arctan \sqrt{\frac{D_2}{D_1}}$$

Duality Relation

- Depends on ratio of diffusion constants

$$\beta(D_1, D_2) \equiv \beta\left(\frac{D_1}{D_2}\right)$$

- Bounds: involve one immobile particle

$$\beta(0) = \frac{1}{2} \qquad \beta(\infty) = 0$$

- Rational for special values of diffusion constants

$$\beta(1/3) = 1/3 \qquad \beta(1) = 1/4 \qquad \beta(3) = 1/6$$

- Duality: between "fast chasing slow" and "slow chasing fast"

$$\beta\left(\frac{D_1}{D_2}\right) + \beta\left(\frac{D_2}{D_1}\right) = \frac{1}{2}$$

Alternating kinetics: slow-fast-slow-fast

Summary

- First-passage kinetics of extremes in Brownian motion
- Problem reduces to diffusion in a two-dimensional corner with mixed boundary conditions
- First-passage exponent obtained analytically
- Exponent is continuously varying function of mobilities
- Relaxation is generally slower compared with positions
- Open questions: multiple particles, higher dimensions
- “Race between maxima” as a data analysis tool