Granular Gases: Stationary Solutions and Driven Steady States

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talk, papers available: http://cnls.lanl.gov/~ebn, CECAM

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Plan

1. The inelastic Boltzmann equation, collision rules, collision rates,
2. Extreme statistics, linear Boltzmann equation
3. Stationary solutions
4. Driven steady states
5. Time dependent solutions

The Inelastic Boltzmann equation (1D)

- **Collision rule (linear)** $r = 1 - 2p, \quad p + q = 1$

$$(u_1, u_2) \to (pu_1 + qu_2, pu_2 + qu_1)$$

- **General collision rate**

$$K(v_1, v_2) = |v_1 - v_2|^\lambda = \begin{cases} 0 & \text{Maxwell molecules} \\ 1 & \text{Hard spheres} \end{cases}$$

- **Boltzmann equation (nonlinear and nonlocal)**

$$\frac{\partial f(v)}{\partial t} = \iint du_1 du_2 f(u_1) f(u_2) |u_1 - u_2|^\lambda [\delta(v - pu_1 - qu_2) - \delta(v - u_2)]$$

collision rate gain loss

Theory: non-linear, non-local, dissipative

The Inelastic Boltzmann equation

Spatially homogeneous systems

$$\frac{\partial f(v)}{\partial t} = \iint du_1 du_2 f(u_1) f(u_2) |u_1 - u_2|^\lambda [\delta(v - pu_1 - qu_2) - \delta(v - u_2)]$$

What is the solution of this equation?
What is the nature of the velocity distribution?

Inelastic Collisions (1D)

- **Relative velocity reduced by** $0 < r < 1$

$$v_1 - v_2 = -r(u_1 - u_2)$$

- **Momentum is conserved**

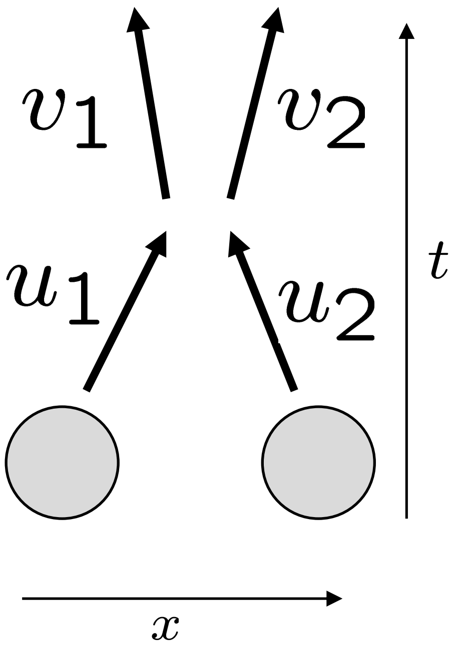
$$v_1 + v_2 = u_1 + u_2$$

- **Energy is dissipated**

$$\Delta E = \frac{1 - r^2}{4}(u_1 - u_2)^2$$

- **Limiting cases**

$$r = \begin{cases} 0 & \text{completely inelastic } (\Delta E = \max) \\ 1 & \text{elastic } (\Delta E = 0) \end{cases}$$



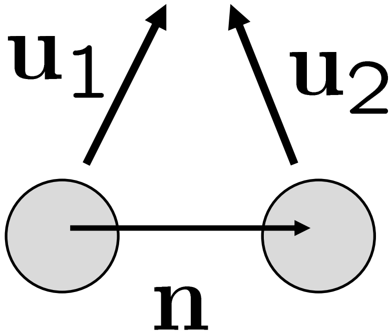
Inelastic Collisions (any D)

- **Normal relative velocity reduced by** $0 < r < 1$

$$(\mathbf{v}_1 - \mathbf{v}_2) \cdot \mathbf{n} = -r(\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{n}$$

- **Momentum conservation**

$$\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{u}_1 + \mathbf{u}_2$$



- **Energy loss**

$$\Delta E = \frac{1 - r^2}{4}[(\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{n}]^2$$

- **Limiting cases**

$$r = \begin{cases} 0 & \text{completely inelastic } (\Delta E = \text{max}) \\ 1 & \text{elastic } (\Delta E = 0) \end{cases}$$

The collision rate

- **Collision rate**

$$K(\mathbf{u}_1, \mathbf{u}_2) = |(\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{n}|^\lambda$$

- **Collision rate related to interaction potential (elastic)**

$$U(r) \sim r^{-\gamma} \qquad \lambda = 1 - 2\frac{d-1}{\gamma} = \begin{cases} 0 & \text{Maxwell molecules} \\ 1 & \text{Hard spheres} \end{cases}$$

- **Balance kinetic and potential energy**

$$v^2 \sim r^{-\gamma} \quad \Rightarrow \quad r \sim v^{-2/\gamma}$$

- **Collisional cross-section**

$$\sigma \sim v r^{d-1} \quad \Rightarrow \quad \sigma \sim v^{1-\frac{2}{\gamma}(d-1)}$$

The Inelastic Boltzmann equation

Spatially homogeneous systems

$$\frac{\partial f(v)}{\partial t} = \iint du_1 du_2 f(u_1) f(u_2) |u_1 - u_2|^\lambda [\delta(v - pu_1 - qu_2) - \delta(v - u_2)]$$

What is the solution of this equation?
What is the nature of the velocity distribution?

Homogeneous cooling state: temperature decay

Haff, JFM 1982

- **Energy loss** $\Delta T \sim (\Delta v)^2$
- **Collision rate** $\Delta t \sim 1/(\Delta v)^\lambda$
- **Energy balance equation**

$$\frac{\Delta T}{\Delta t} \sim -(\Delta v)^{2+\lambda} \quad \Rightarrow \quad \frac{dT}{dt} \sim -T^{1+\lambda/2}$$

- **Temperature decays, system comes to rest**

$$T \sim t^{-2/\lambda} \quad \Rightarrow \quad f(v) \to \delta(v)$$

Trivial stationary solution

Homogeneous cooling states: similarity solutions

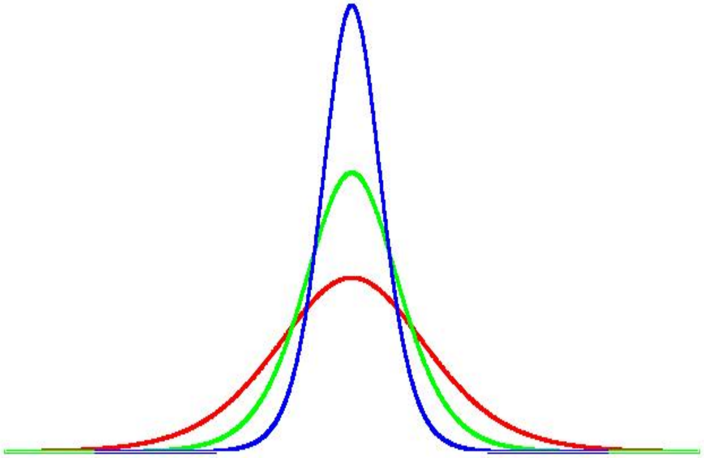
Esipov, Poeschel 97

- **Similarity solution**

$$f(v,t) = t^{1/\lambda}\Phi(vt^{1/\lambda})$$

- **Stretched exponentials (overpopulation)**

$$\Phi(z) \sim \exp\left(-|z|^{\lambda}\right)$$



Are there nontrivial stationary solutions?

- **Stationary Boltzmann equation**

$$0 = \iint du_1 du_2 f(u_1) f(u_2) |u_1 - u_2|^\lambda [\delta(v - pu_1 - qu_2) - \delta(v - u_2)]$$

collision rate gain loss

Naive answer: NO!

- **According to the energy balance equation**

$$\frac{dT}{dt} = -\Gamma$$

- **Dissipation rate is positive**

$$\Gamma > 0$$

An exact solution (1D, λ=0)

Lamboitte & Brenig, unpub

- **One-dimensional Maxwell molecules**
- **Fourier transform obeys a closed equation** $F(k) = \int dv\, e^{ikv} f(v)$

$$F(k) = F(pk)F(qk)$$

- **Exponential solution**

$$F(k) = \exp(-v_0|k|)$$

- **Lorentzian velocity distribution**

$$f(v) = \frac{1}{\pi}\frac{1}{1+v^2}$$

A nontrivial stationary solution does exist!

Properties of stationary solution

- Perfect balance between collisional loss and gain
- Purely collisional dynamics (no source term)
- Family of solutions: scale invariance $v \rightarrow v/v_0$

$$f(v) = \frac{1}{\pi v_0} \frac{1}{1 + (v/v_0)^2}$$

- Power-law high-energy tail

$$f(v) \sim v^{-2}$$

- Infinite energy, infinite dissipation rate!

Are these stationary solutions physical?

Extreme Statistics (1D)

Ernst, Goldhirsh

- **Collision rule: arbitrary velocities**

$$(u_1, u_2) \to (pu_1 + qu_2, pu_2 + qu_1)$$

- **Large velocities: linear but nonlocal process**

$$v \xrightarrow{v^\lambda} (pv, qv)$$

- **High-energies: linear equation**

$$\frac{\partial f(v)}{\partial t} = v^\lambda \left[\frac{1}{p^{1+\lambda}} f\left(\frac{v}{p}\right) + \frac{1}{q^{1+\lambda}} f\left(\frac{v}{q}\right) - f(v)\right]$$

gain gain loss

Linear, nonlocal evolution equation

Stationary solution (1D)

- **High-energies: linear equation**

$$f(v) = \frac{1}{p^{1+\lambda}} f\left(\frac{v}{p}\right) + \frac{1}{q^{1+\lambda}} f\left(\frac{v}{q}\right)$$

loss gain gain

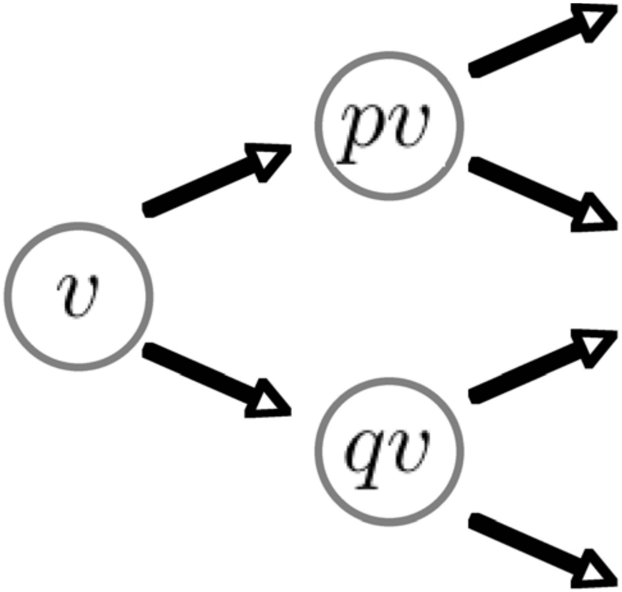
- **Power-law tail**

$$f(v) \sim v^{-2-\lambda}$$

Energy Cascades (1D)

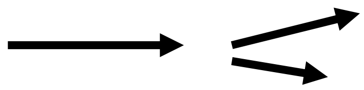
Energetic particles "see" a static medium

$$v \longrightarrow (pv, qv)$$

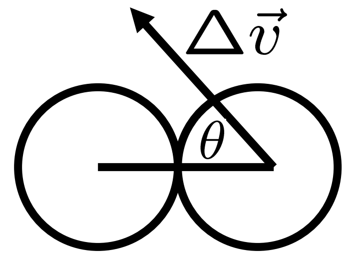


Extreme Statistics (any D)

- **Collision process: large velocities**



$$v \xrightarrow{|v\cos\theta|^{\lambda}} (\alpha v, \beta v)$$



- **Stretching parameters related to impact angle**

$$\alpha = (1-p)\cos\theta \qquad \beta = \sqrt{1-(1-p^2)\cos^2\theta}$$

- **Energy decreases, velocity magnitude increases**

$$\alpha^2 + \beta^2 \le 1 \qquad \alpha + \beta \ge 1$$

- **Linear equation**

$$\frac{\partial f(v)}{\partial t} = \left\langle (v\cos\theta)^{\lambda} \left[\frac{1}{\alpha^{d+\lambda}} f\left(\frac{v}{\alpha}\right) + \frac{1}{\beta^{d+\lambda}} f\left(\frac{v}{\beta}\right) - f(v) \right] \right\rangle$$

Power-laws are generic

- **Velocity distribution always has power-law tail**

$$f(v) \sim v^{-\sigma}$$

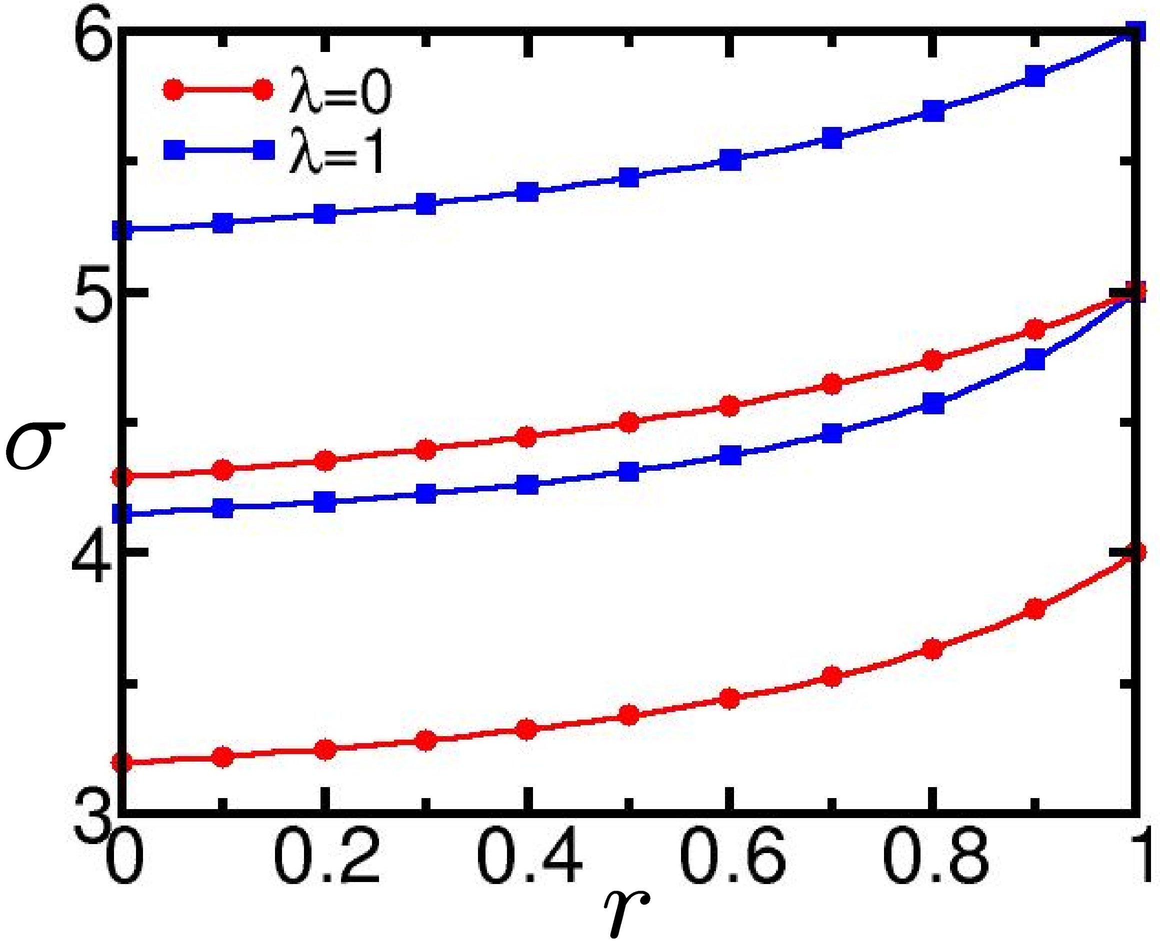
- **Characteristic exponent varies with parameters**

$$\frac{1 - {}_2F_1\left(\frac{d+\lambda-\sigma}{2}, \frac{\lambda+1}{2}, \frac{d+\lambda}{2}, 1-p^2\right)}{(1-p)^{\sigma-d-\lambda}} = \frac{\Gamma(\frac{\sigma-d+1}{2})\Gamma(\frac{d+\lambda}{2})}{\Gamma(\frac{\sigma}{2})\Gamma(\frac{\lambda+1}{2})}$$

- **Tight bounds** $1 \leq \sigma - d - \lambda \leq 2$
- **Elastic limit is singular** $\sigma \to d + 2 + \lambda$

Dissipation rate always divergent
Energy finite or infinite

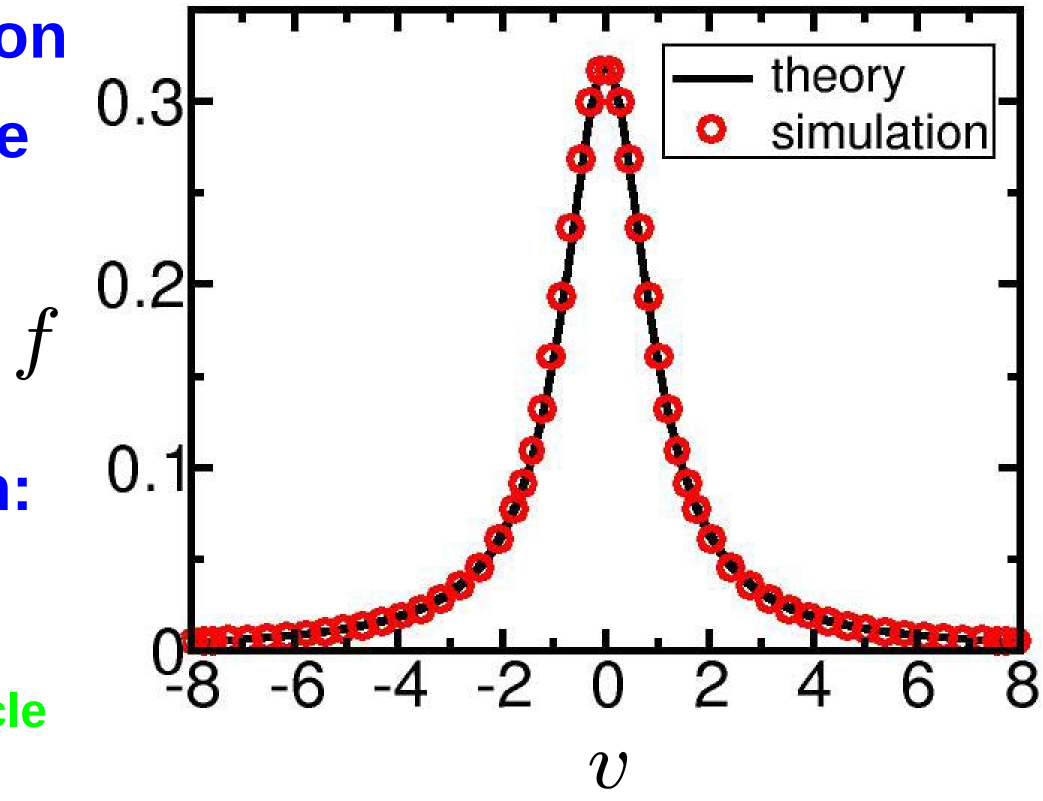
The characteristic exponent σ (d=2,3)



σ varies with spatial dimension, collision rules

Monte Carlo Simulations: Driven Steady States

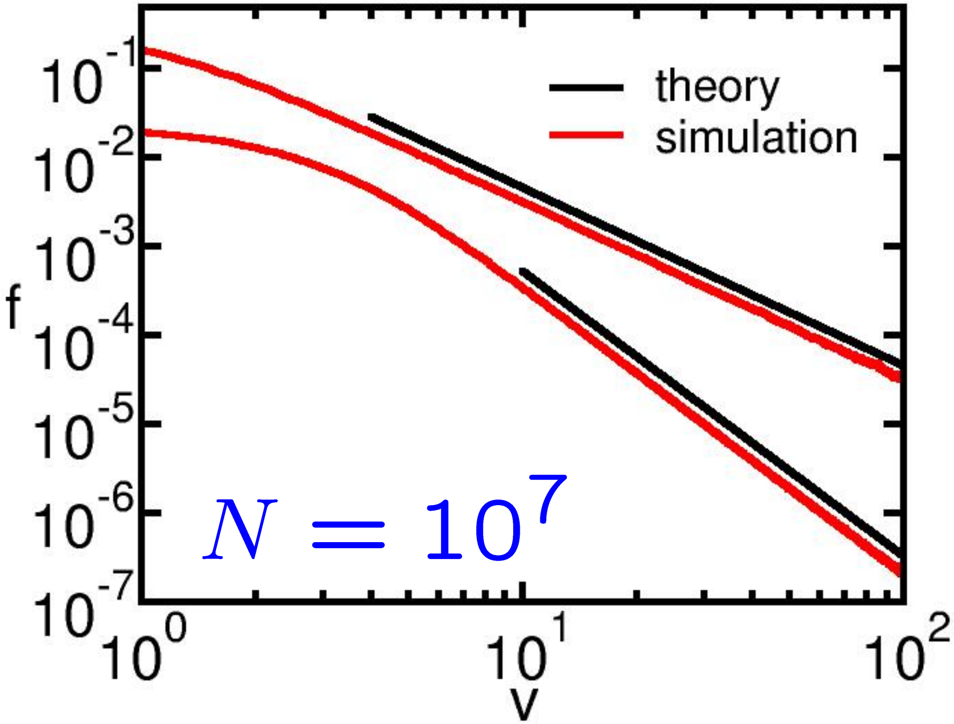
- Compact initial distribution
- Inject energy at very large velocity scales only
- Maintain constant total energy
- "Lottery" implementation:
  - Keep track of total energy dissipated, E_T
  - With small rate, boost a particle by E_T



Excellent agreement between theory and simulation

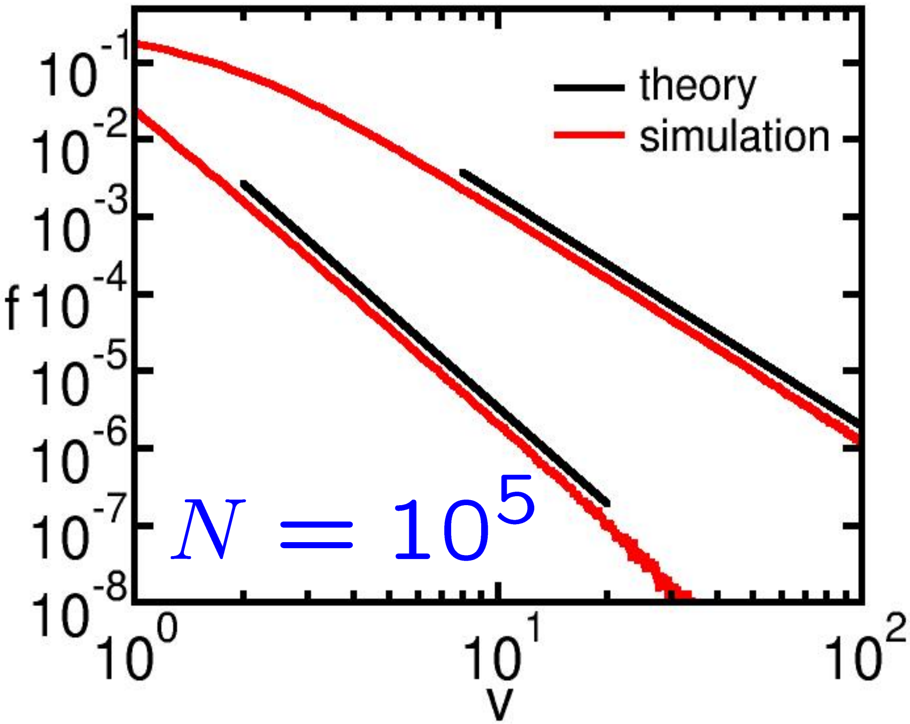
Further confirmation: extremal statistics

Maxwell molecules (1D, 2D)



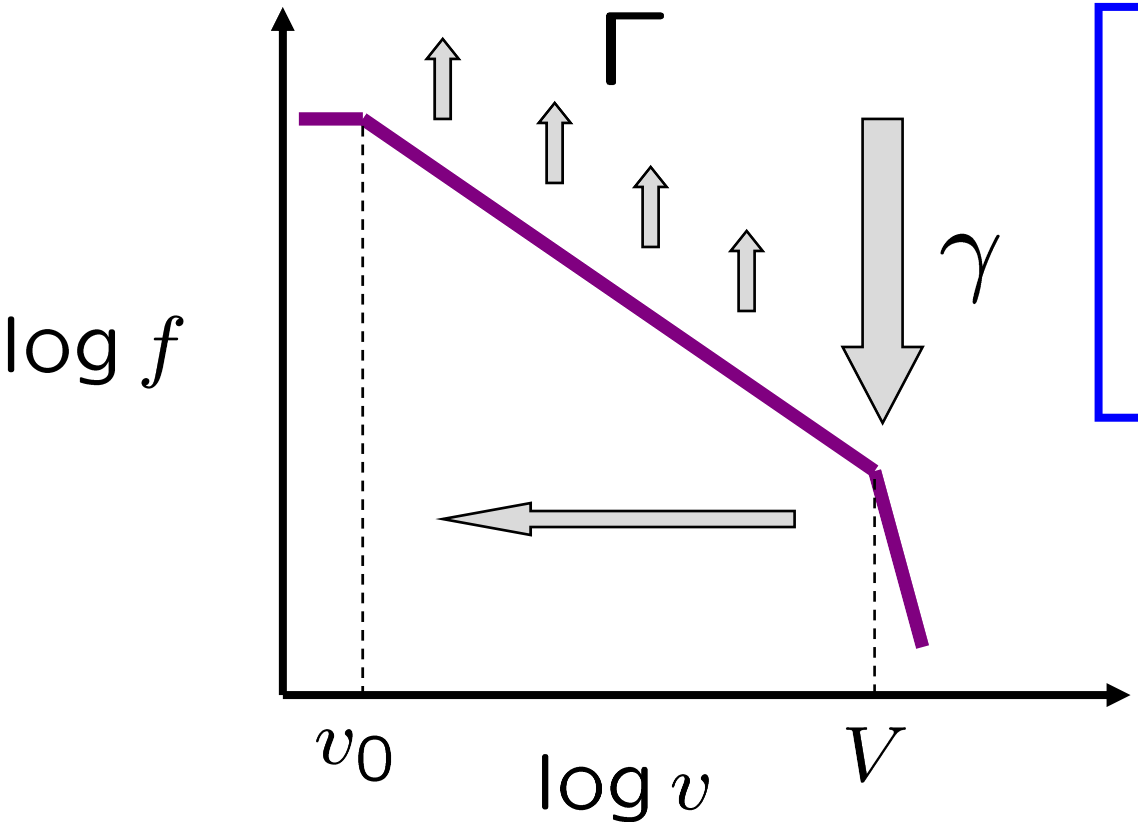
d	theory	simulation
1	2	1.995
2	3.19520	3.19

Hard spheres (1D, 2D)



d	theory	simulation
1	3	2.994
2	4.14922	4.15

Injection, cascade, dissipation



Experimental realization?
Energetic particle "shot" into static medium

Energy balance

$$\Gamma \sim \gamma V^2$$

- ❖ **Energy is injected ONLY AT LARGE VELOCITY SCALES!**
- ❖ **Energy cascades from large velocities to small velocities**
- ❖ **Energy dissipated at small velocity scales**

Conventional forced steady states

T van Noije, M Ernst 97

- **Energy injection: thermal forcing (at all scales)**

$$dv/dt = \eta$$

- **Energy dissipation: inelastic collision**

$$v \longrightarrow (pv, qv)$$

- **Steady state equation**

$$0 = D\frac{d^2 f(v)}{d^2 v} + v^\lambda \left[\frac{1}{p^{1+\lambda}} f\left(\frac{v}{p}\right) + \frac{1}{q^{1+\lambda}} f\left(\frac{v}{q}\right) - f(v)\right]$$

- **Stretched exponentials**

$$f(v) \sim \exp\left(-v^{1+\lambda/2}\right)$$

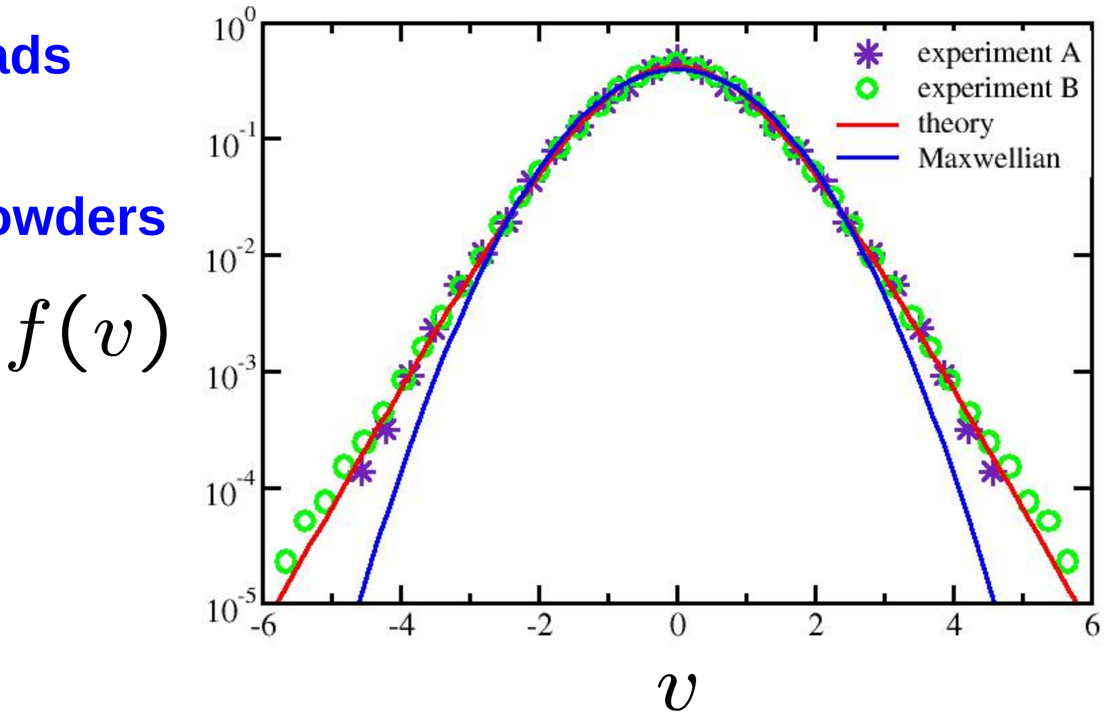
Nonequilibrium velocity distributions

A **Mechanically vibrated beads**

F Rouyer & N Menon 00

B **Electrostatically driven powders**

I Aronson, J Olafsen, EB PRL 05



- <u>**Gaussian core**</u>
- <u>**Overpopulated tail**</u>

$$f(v) \sim \exp\left(-|v|^{\delta}\right)$$

$$1 \le \delta \le 3/2$$

- **Kurtosis**

$$\kappa = \begin{cases} 3.55 & \text{theory} \\ 3.6 & \text{experiment} \end{cases}$$

Excellent agreement between theory and experiment

balance between collisional dissipation, energy injection from walls

Energy balance

- Energy injection rate γ
- Energy injection scale V
- Typical velocity scale v_0
- Balance between energy injection and dissipation

$$\gamma \sim V^{\lambda}(V/v_0)^{d-\sigma}$$

- For "lottery" injection: injection scale diverges with injection rate

$$V \sim \begin{cases} \gamma^{-1/(2-\lambda)} & \sigma < d+2 \\ \gamma^{-1/(\sigma-d-\lambda)} & \sigma > d+2 \end{cases}$$

Energy injection selects stationary solution

with Ben Machta (Brown)

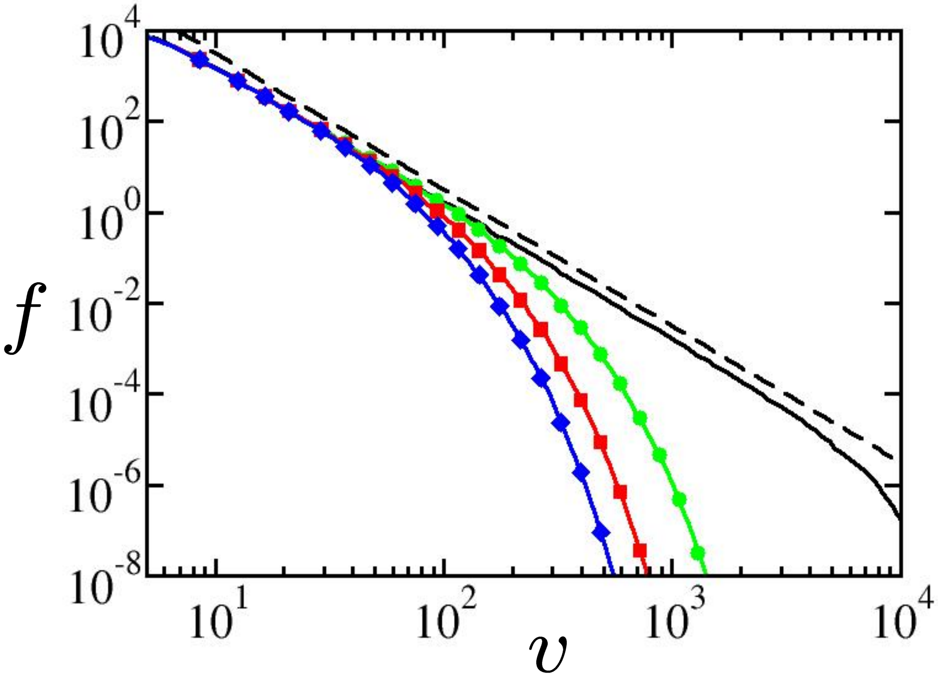
Time dependent solutions (1D, λ>0)

- **Self-similar distribution**

$$f(v,t) \simeq v^{-\sigma}\Phi\left(\frac{v}{V(t)}\right)$$

- **Cutoff velocity decays**

$$V(t) \sim t^{-1/\lambda}$$



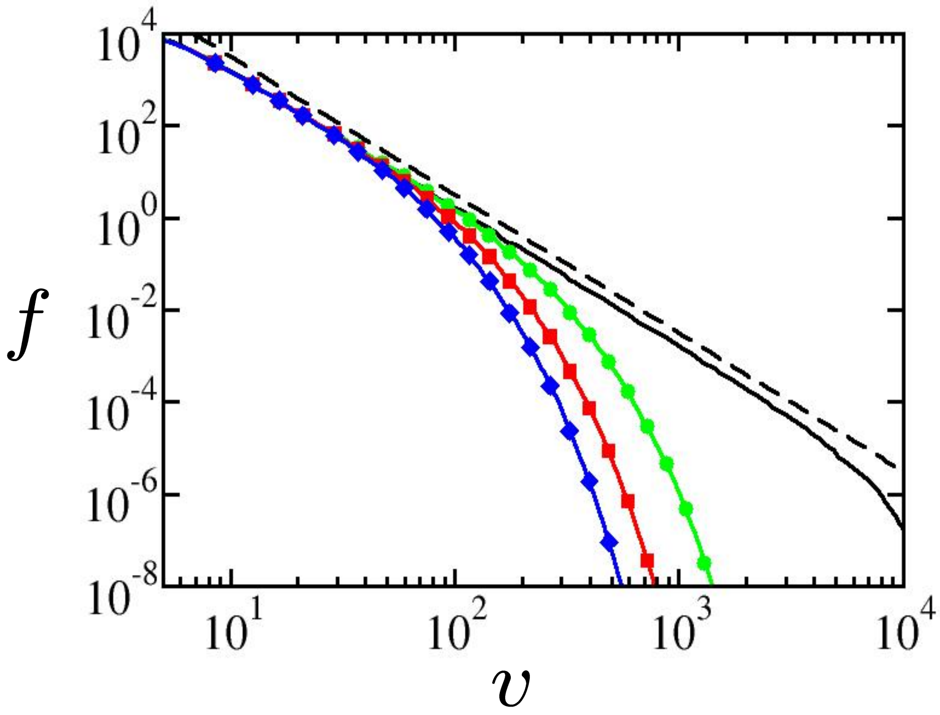
- **Scaling function**

$$\Phi(x) = \sum_{n=1}^{\infty} A_n \exp\left[-(2^n x)^\lambda\right] \qquad A_n = \prod_{\substack{k=1 \\ k\neq n}}^{\infty} \frac{1}{1-2^{\lambda(n-k)}}$$

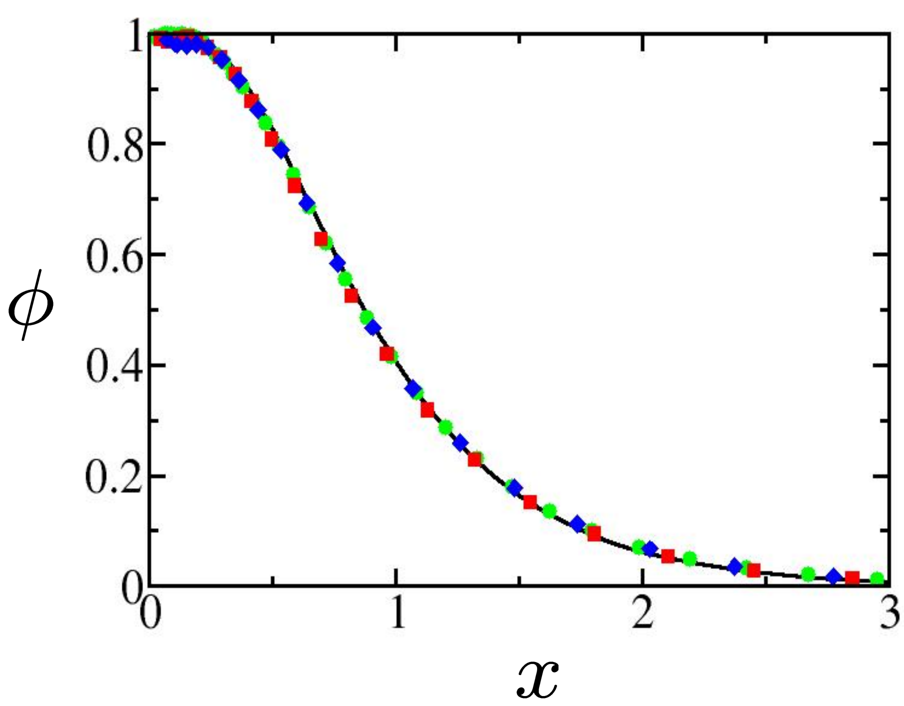
Hybrid between steady-state and time dependent state

Numerical confirmation

Velocity distribution



Scaling function



A third family of solutions exists

Extreme statistics

- **Scaling function**

$$\Phi(x) = \sum_{n=1}^{\infty} A_n \exp\left[-(2^n x)^\lambda\right] \qquad A_n = \prod_{\substack{k=1 \\ k\neq n}}^{\infty} \frac{1}{1-2^{\lambda(n-k)}}$$

- **Large velocities: as in free cooling**

$$\Phi(x) \sim \exp(-x^\lambda) \qquad x \to \infty$$

- **Small velocities: non-analytic behavior**

$$1 - \Phi(x) \sim \exp\left[-(\ln x)^2\right] \qquad x \to 0$$

Hybrid between steady-state and time dependent state

Maxwell Model (λ=0) only unsolved case!

Summary

- **Time dependent solution**

$$f(v,t) = t^{1/\lambda}\Psi(vt^{1/\lambda})$$

- **Time independent solution**

$$f_s(v) \sim v^{-\sigma}$$

- **Hybrid solution**

$$f(v,t) = f_s(v)\Phi(vt^{1/\lambda})$$

Are there other types of solutions?

Conclusions

- New class of nonequilibrium steady states
- Energy cascades from large to small velocities
- Power-law high-energy tail
- Energy input at large scales balances dissipation
- Associated similarity solutions exist as well
- Temperature insufficient to characterize velocities
- Experimental realization: requires a different driving mechanism

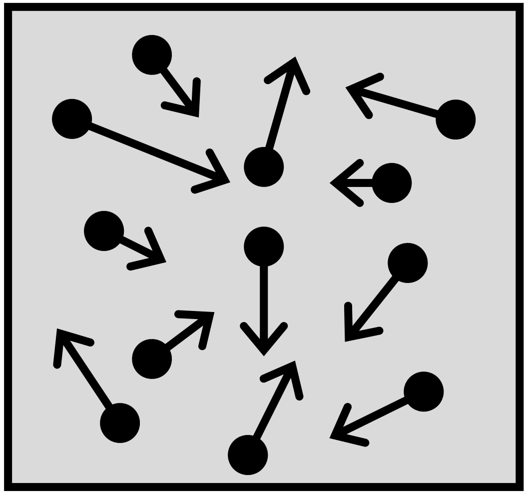
Outlook

- Spatially extended systems
- Spatial structures
- Polydisperses granular media
- Experimental realization

E. Ben-Naim and J. Machta, Phys. Rev. Lett. **94**, 138001 (2005)

E. Ben-Naim, B. Machta, and J. Machta, cond-mat/0504187

Driven Granular gas



- Vigorous driving
- Spatially uniform system
- Particles undergo binary collisions
- Velocities change due to
  1. Collisions: lose energy
  2. Forcing: gain energy
- What is the typical velocity (granular "temperature")?

$$T = \langle v^2 \rangle$$

- What is the velocity distribution?

$$f(v)$$

Comparing kinetic theories