Granular Gases

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Plan

I **Granular gases**

II **Burgers shocks:**
1D,2D freely cooling

III **Scaling, multiscaling, nontrivial exponents:**
the inelastic Maxwell model

“A gas of marbles”

- Granular materials: powders, grains
- Geophysics: sand dunes, volcanic flows
- Astrophysics: large scale formation

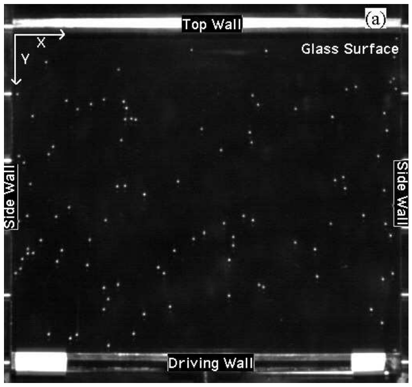
Characteristics

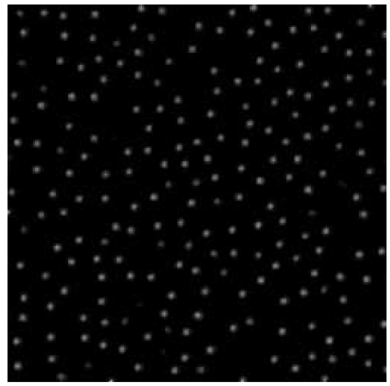
- Hard sphere interactions
- Dissipative collisions

Challenges

- Hydrodynamics: flow equations
- Kinetic Theory: velocity statistics
- Sharp validity criteria are missing

Experiments





- Vibrated horizontal monolayers Urbach 97

$$P(v) \sim \begin{cases} \exp(-v), & g \cong 1; \\ \exp(-v^2), & g \cong 6. \end{cases}$$

- Vibrated vertical monolayers Menon 00
- Electrostatically vibrated layers Aronson 01

$$P(v) \sim \exp\left(-|v|^{3/2}\right)$$

- Kinetic theory (white noise forcing) Ernst 97

$$P(v) \sim \exp\left(-A(r)|v|^{3/2}\right)$$

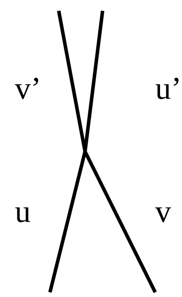
non-Maxwellian statistics

Inelastic collisions

- Relative velocity reduced by $r = 1 - 2\epsilon$

$$\begin{aligned} \Delta v' &= -r\Delta v \\ v' &= v - \epsilon \Delta v \end{aligned}$$

- Energy dissipation $\Delta E \propto -\epsilon(\Delta v)^2$



Freely cooling gases

- N point particles in 1D ring.
  Random velocity distribution.
  Typical velocity v_0. Typical distance x_0.

- **Dimensionless variables** $x \to x/x_0$, $t \to tv_0/x_0$

— “Temperature” $T(t) = \langle v^2(t) \rangle - \langle v(t) \rangle^2$

— Characteristic time/length scales.

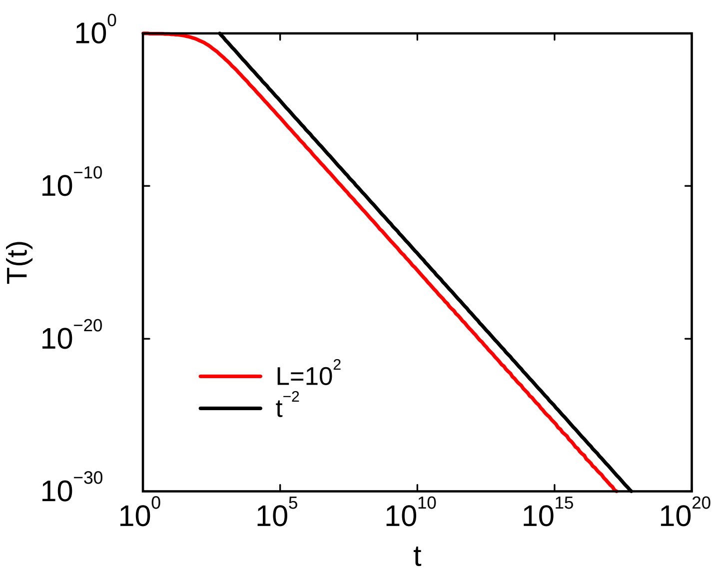
Mean Field Theory

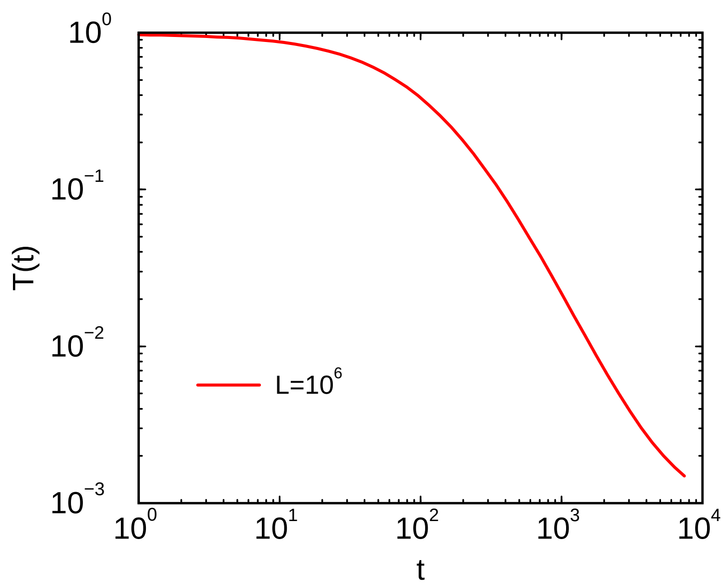
- Energy dissipation $\Delta T \propto -\epsilon(\Delta v)^2$
- Collision frequency $\Delta t \sim \ell/\Delta v \sim (\Delta v)^{-1}$
- Assuming uniform gas $dT/dt \propto -\epsilon\, T^{3/2}$
- Cooling law

Haff 83

$$T(t) \simeq (1 + A\epsilon t)^{-2} \sim \begin{cases} 1 & t \ll \epsilon^{-1} \\ \epsilon^{-2} t^{-2} & t \gg \epsilon^{-1} \end{cases}$$

- Simulation





Valid only in small systems/early time

The Inelastic Collapse: $N = 3$

1D: Bernu 91, Young 91, 2D: Kadanoff 95

- Deterministic collision sequence $12, 23, 12, \ldots$
- Velocities given by linear combination

$$\begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix} = \begin{pmatrix} \epsilon & 1-\epsilon & 0 \\ 1-\epsilon & \epsilon & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

- After a pair of collisions (12,23)

$$\mathbf{v}' = \mathbf{M}\mathbf{v} \qquad \mathbf{M} = \mathbf{M}_{12}\mathbf{M}_{23}$$

- $r < 7 - 4\sqrt{3} \Rightarrow$

$$\Delta x, \Delta t \propto \lambda^n \qquad \lambda < 1$$

- Infinite number of collisions in finite time

Particles clump

The Sticky Gas ($r = 0$)

Carnavale, Pomeau, Young 90

- Multiparticle aggregate of typical mass m
- **Momentum conservation**

$$P_m = \sum_{i=1}^{m} P_i \quad \Rightarrow \quad P \sim m^{1/2}, \quad v \sim m^{-1/2}$$

- **Mass conservation**

$$\rho = cm = \text{const} \quad \Rightarrow \quad c \sim m^{-1}$$

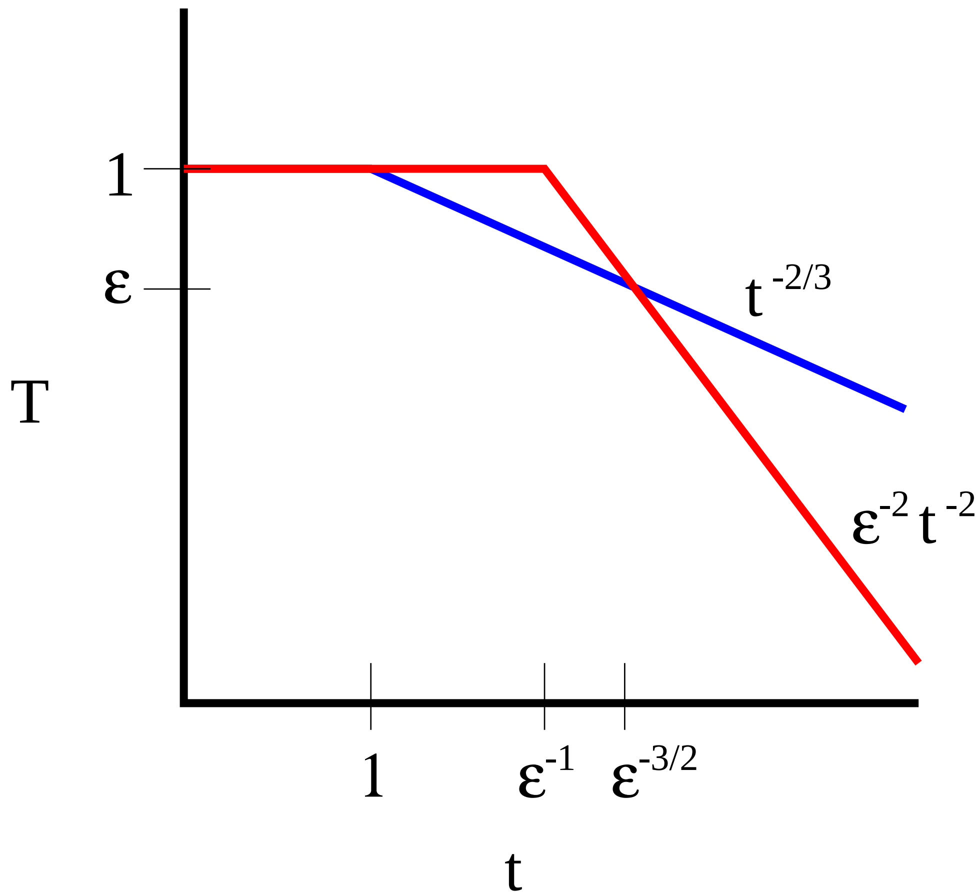
- **Dimensional analysis** $[cv] = [t]^{-1}$

$$m \sim t^{2/3} \quad v \sim t^{-1/3} \quad T \sim t^{-2/3}$$

- **Final state** 1 aggregate with $m = N$

$$T(t) \sim \begin{cases} 1 & t \ll 1; \\ t^{-2/3} & 1 \ll t \ll N^{3/2}; \\ N^{-1} & N^{3/2} \ll t \end{cases}$$

Monotonicity



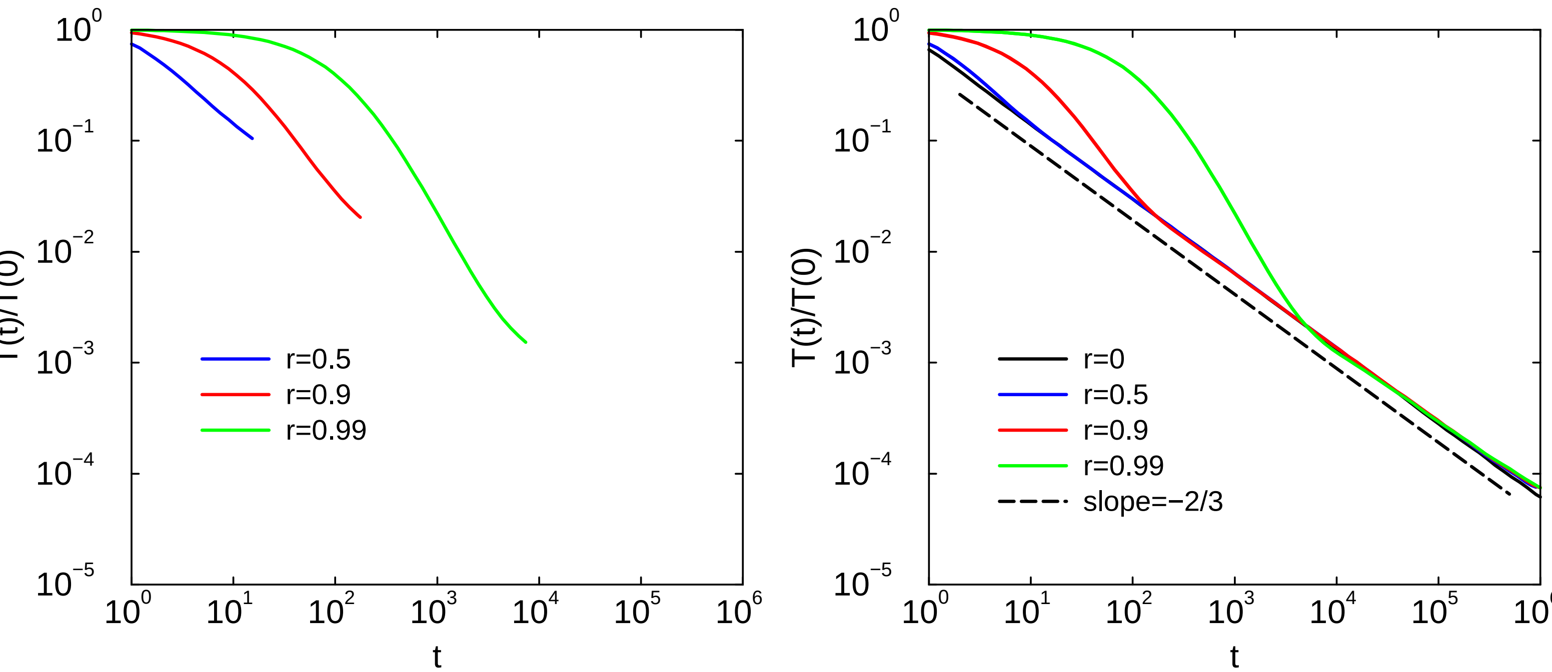
- $T(\epsilon, t)$ decreases monotonically with ϵ, t
- **Sticky gas ($\epsilon = 1/2$) is a lower bound**
- Homogeneous when $N \ll \epsilon^{-1}$ **or** $t \ll \epsilon^{-3/2}$
- Clustering when $N \gg \epsilon^{-1}$ **and** $t \gg \epsilon^{-3/2}$

The crossover picture

- Method: relax dissipation below cutoff (mimic granular particles)

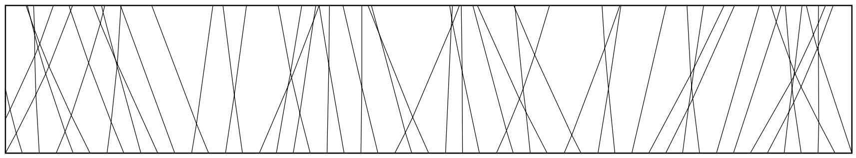
$$r(\Delta v) = \begin{cases} 1 & \Delta v < \delta \\ r & \Delta v > \delta \end{cases}$$

- Simulation results: $N = 10^7$, $\delta = 10^{-3}$
- Universal cooling law $T(t) \sim t^{-2/3}$

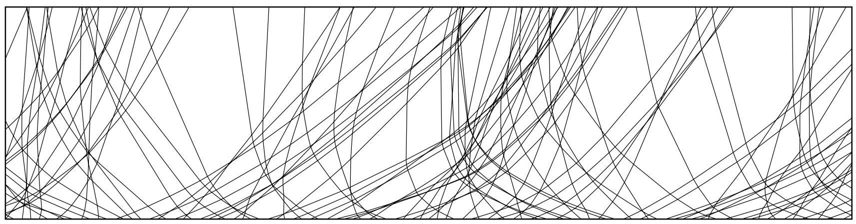


Asymptotic behavior is independent of r

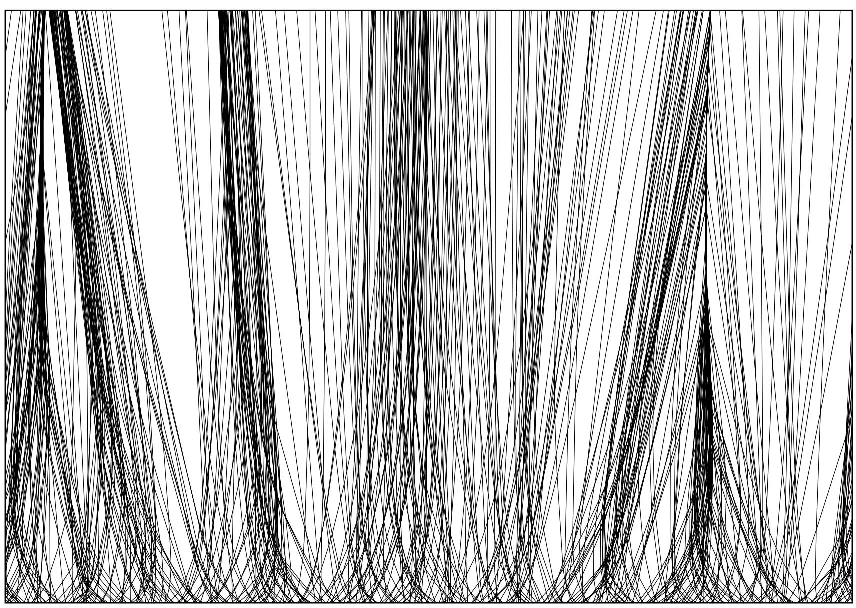
Early = Elastic gas ($r = 1$)



Intermediate = Inelastic gas ($r = 0.9$)



Late = Sticky gas ($r = 0$)



$r = 0$ is fixed point

The Velocity Distribution

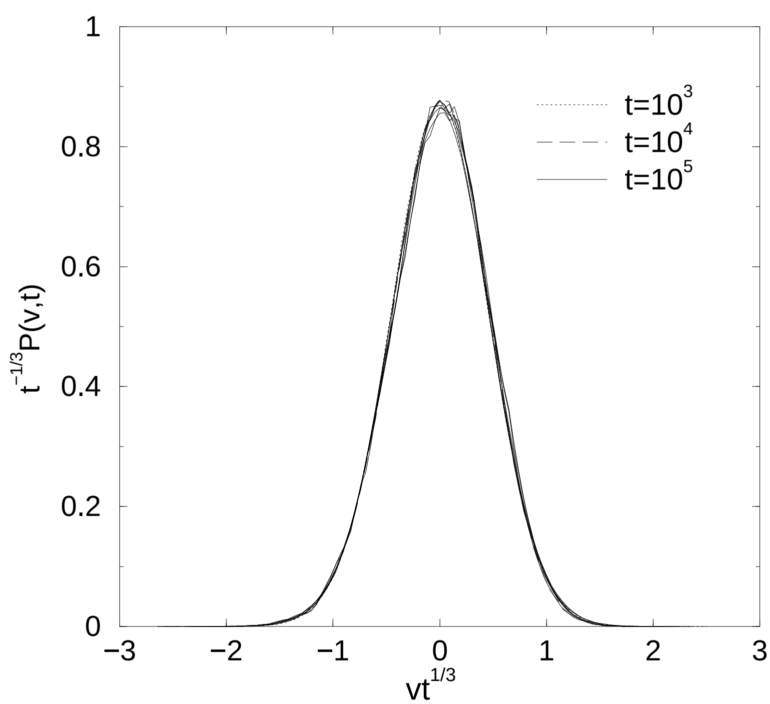
- Self similar distribution

$$P(v,t) \sim t^{1/3}\Phi(vt^{1/3})$$

- Anomalous tail — Frachebourg 99

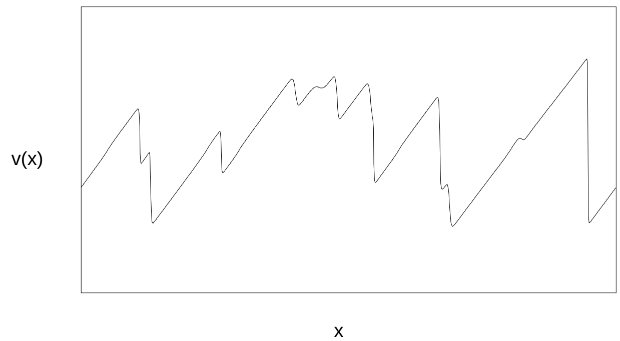
$$\Phi(z) \sim \exp(-\text{const.} \times z^3) \qquad z \gg 1$$

- Simulation results $r = 0,\ 0.5,\ 0.9$



$P(r,v,t)$ is function of $z = vt^{1/3}$ only

The Inviscid Burgers Equation



- Nonlinear diffusion equation

$$v_t + v v_x = \nu v_{xx} \qquad \nu \to 0$$

- Transform to linear diffusion equation

$$u_t = \nu u_{xx} \qquad \Leftarrow \qquad v = -2\nu(\ln u)_x$$

- Sawtooth (shock) velocity profile

$$v(x,t) = \frac{x - q(x,t)}{t}$$

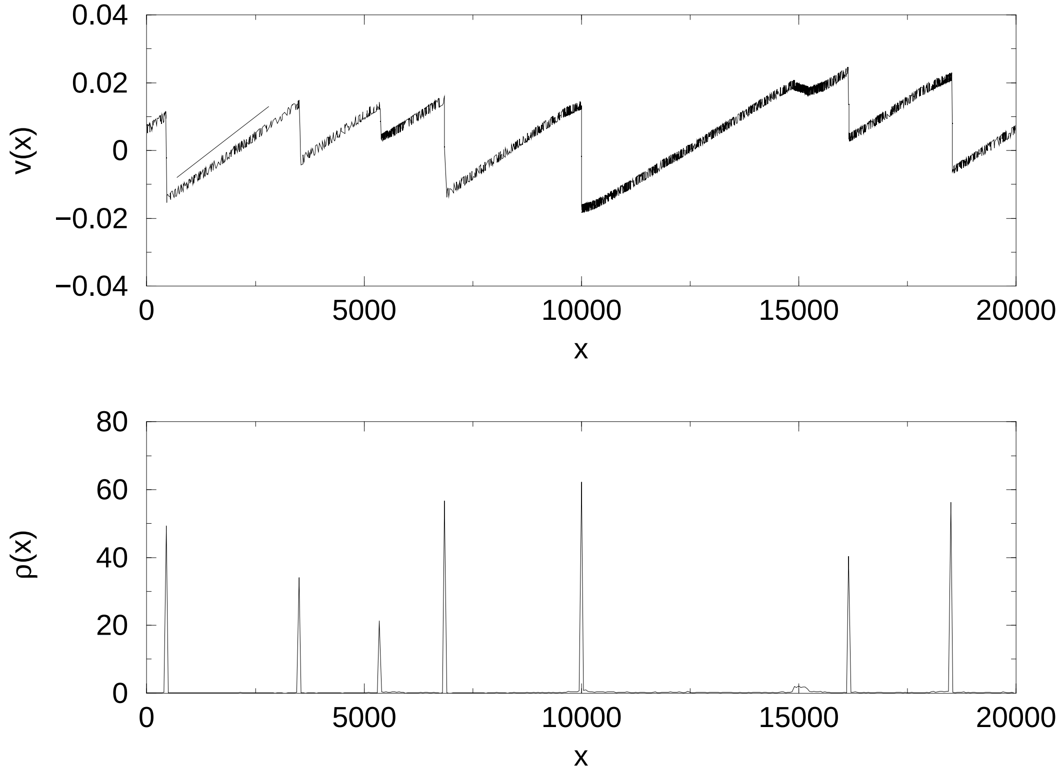
- Shock collisions conserve mass & momentum

- Describes "sticky gas" $r = 0$ Zeldovich RMP 89

Burgers equation $\equiv$ sticky gas $\equiv$ inelastic gas

Burgers' eqn Predictions verified in 1D

- Velocity statistics $v \sim t^{-1/3}$
- Discontinuous (shock) velocity profile
- Slope $= t^{-1}$ (simulation with $r = 0.99$)

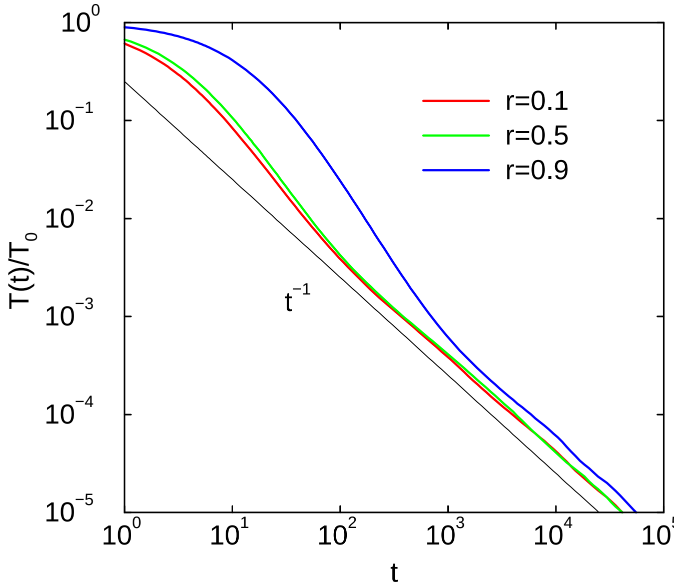


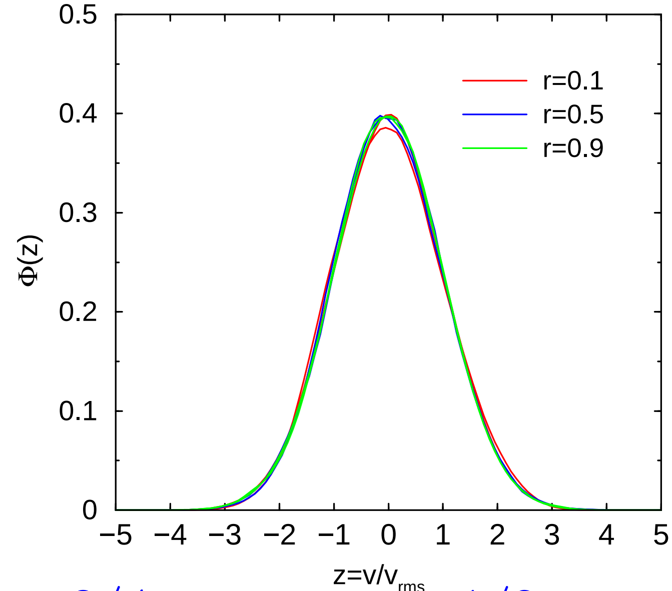
Collapse $\equiv$ shock formation

Two Dimensions

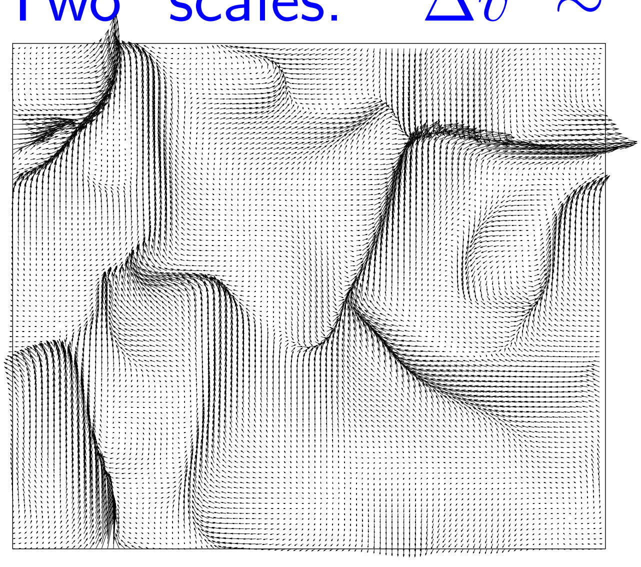
Goldhirsch/Zannetti 93

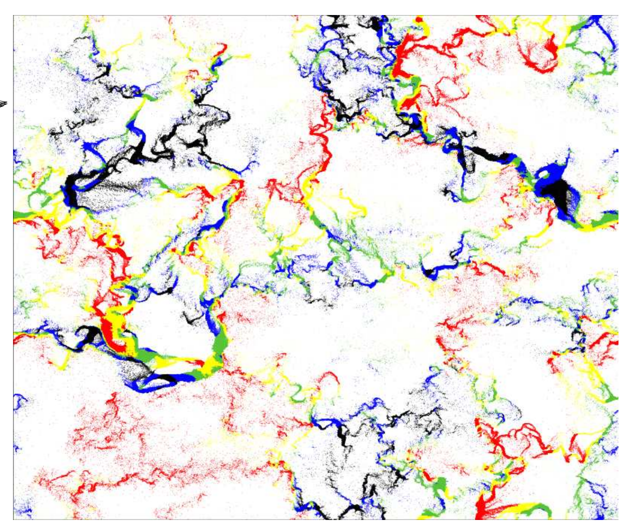
- Simulations: $N = 10^6$, $\delta = 10^{-5}$
- Universal temperature, velocity distributions





- Two scales: $\Delta v \sim t^{-3/4}$, $v \sim t^{-1/2}$





Preliminary evidence: $r = 0$ remains fixed point

Conclusions I

Asymptotic behavior:

- **Governed by cluster-cluster coalescence**
- **Independent of restitution coefficient**
- **Described by inviscid Burgers equation**

Outlook

- **Velocity & spatial correlations**
- **Higher dimensions**

EB, Chen, Doolen, Redner, PRL 83, 4069 (1999)

The Elastic Maxwell Model

J.C. Maxwell, Phil. Tran. Roy. Soc **157**, 49 (1867)

- Infinite particle system
- Binary collisions
- Random collision partners
- Random impact directions $\mathbf{n}$
- Elastic collisions $(\mathbf{g} = \mathbf{v_1} - \mathbf{v_2})$

$$\mathbf{v_1} \to \mathbf{v_1} - \mathbf{g} \cdot \mathbf{n}\,\mathbf{n}$$

- Mean-field collision process is solvable
- Purely Maxwellian velocity distributions

$$P(\mathbf{v}) = \frac{1}{(2\pi T)^{d/2}} \exp\left(-\frac{v^2}{2T}\right)$$

What about inelastic, dissipative collisions?

The Inelastic Maxwell Model

- **Inelastic collisions** $r = 1 - 2\epsilon$

$$\mathbf{v}_{1,2} = \mathbf{u}_{1,2} \mp (1-\epsilon)\,(\mathbf{g}\cdot\mathbf{n})\,\mathbf{n},$$

- **Boltzmann equation** $\mathbf{g}\cdot\mathbf{n} \to \langle g \rangle$

$$\begin{aligned}\frac{\partial P(\mathbf{v},t)}{\partial t} &= \int d\mathbf{n} \int d\mathbf{u}_1 \int d\mathbf{u}_2 \, \langle g \rangle \, P(\mathbf{u}_1,t)\, P(\mathbf{u}_2,t) \\ &\times \Big\{ \delta\left(\mathbf{v}-\mathbf{v}_1\right) - \delta\left(\mathbf{v}-\mathbf{u}_1\right) \Big\}\end{aligned}$$

- **Fourier transform**

$$F(\mathbf{k},t) = \int d\mathbf{v} e^{i\mathbf{k}\cdot\mathbf{v}} P(\mathbf{v},t)$$

- **Closed equations** $\mathbf{q} = (1-\epsilon)\mathbf{k}\cdot\mathbf{n}\,\mathbf{n}$

$$\frac{\partial}{\partial t} F(\mathbf{k},t) + F(\mathbf{k},t) = \int d\mathbf{n}\, F\left[\mathbf{k}-\mathbf{q},t\right] F\left[\mathbf{q},t\right],$$

Theory is analytically tractable

One Dimension

- **Scaling of velocity distribution**

$$P(\mathbf{v},t) \to \frac{1}{T^{d/2}}\Phi\left(\frac{v}{T^{1/2}}\right) \quad \text{or} \quad F(k,t) \to f\left(k^2T\right)$$

- **Governing equation**

$$-2\epsilon(1-\epsilon)f'(x) + f(x) = f(\epsilon^2 x)f\left((1-\epsilon)^2x\right)$$

- **Exact solution** BKW

$$f(x) = \left(1+\sqrt{x}\right)e^{-\sqrt{x}} \cong 1 - \frac{1}{2}x + \frac{1}{3}x^{3/2} + \cdots$$

- **Lorentzian^2 velocity distribution**

$$\Phi(w) = \frac{2}{\pi}\frac{1}{(1+w^2)^2}$$

- **Algebraic tail** Baldassari 01

$$\Phi(w) \sim w^{-4} \qquad w \gg 1$$

Universal scaling function, exponent

Scaling, Nontrivial Exponents

- **Freely cooling case**

$$T = \langle v^2 \rangle \sim t^{-2}$$

- **Scaling equation**

$$-\lambda\, x\, \Phi'(x) + \Phi(x) = \int d\mathbf{n}\, \Phi(x\xi)\, \Phi(x\eta)$$

$$\lambda = 2\epsilon(1-\epsilon)/d,\ \xi = 1-(1-\epsilon^2)\cos^2\theta,\ \eta = (1-\epsilon)^2\cos^2\theta$$

- **Power-law tails**

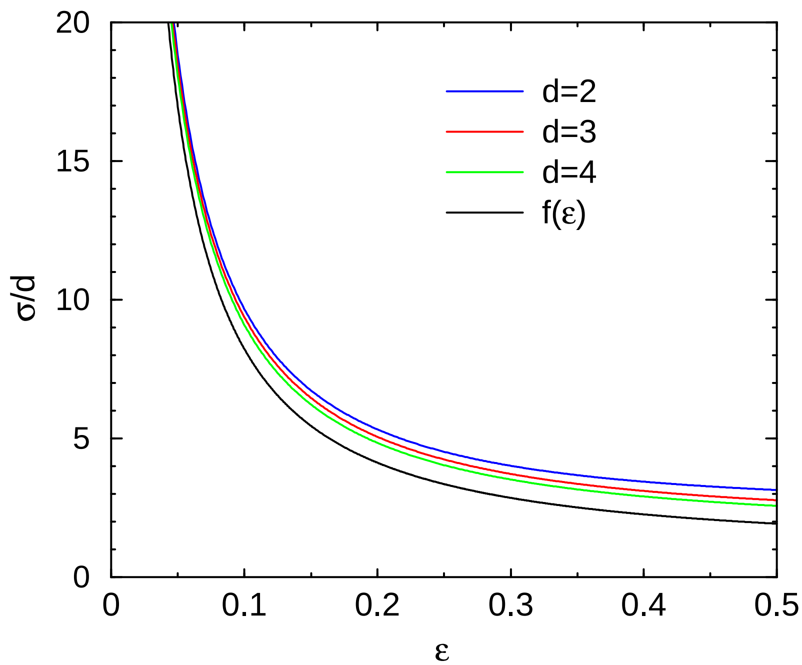
$$\Phi(w) \sim w^{-\sigma}, \qquad w \to \infty.$$

- **Exact solution for the exponent σ**

$$1-\epsilon(1-\epsilon)\,\frac{\sigma-d}{d} = {}_2F_1\left[\tfrac{d-\sigma}{2},\tfrac{1}{2};\tfrac{d}{2};1-\epsilon^2\right]+(1-\epsilon)^{\sigma-d}\,\frac{\Gamma\left(\frac{\sigma-d+1}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{\sigma}{2}\right)\Gamma\left(\frac{1}{2}\right)}$$

Nonuniversal tails, exponents depend on ϵ, d

The exponent σ



- **Maxwellian distributions:** $d = \infty$, $\epsilon = 0$
- **Diverges in high dimensions**

$$\sigma \cong d\, f(\epsilon)$$

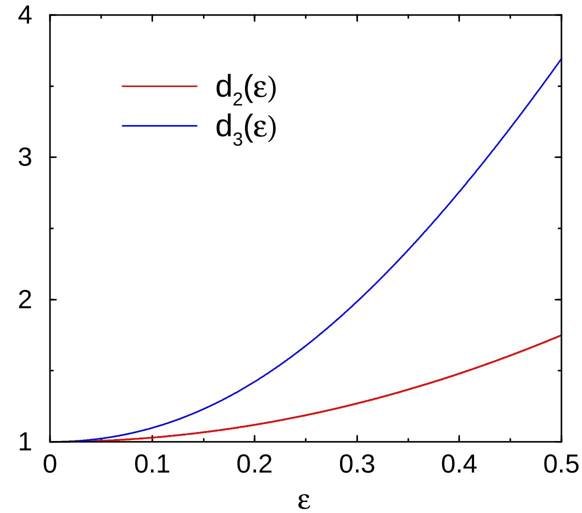
- **Diverges for low dissipation**

$$\sigma \cong d\, \epsilon^{-1}$$

- **In practice, huge**

$$\sigma(d = 3, r = 0.8) \cong 30!$$

Dynamics



- Moments of the velocity distribution

$$M_{2n}(t) = \int d\mathbf{v}|\mathbf{v}|^{2n}P(\mathbf{v},t)$$

- Multiscaling asymptotic behavior

$$M_{2n} \sim M_2^{\xi_n/2} \qquad \xi_n = \begin{cases} n & d < d_n(\epsilon), \\ \alpha_n(\epsilon) & d > d_n(\epsilon). \end{cases}$$

- Crossover dimensions: $d_2(\epsilon) = 1 + 3\epsilon^2$
- Nonlinear multiscaling spectrum (1D):

$$\alpha_n(\epsilon) = \frac{1 - \epsilon^{2n} - (1-\epsilon)^{2n}}{1 - \epsilon^2 - (1-\epsilon)^2}$$

Sufficiently large moments exhibit multiscaling

Velocity Correlations

- Definition (correlation between v_x^2 and v_y^2)

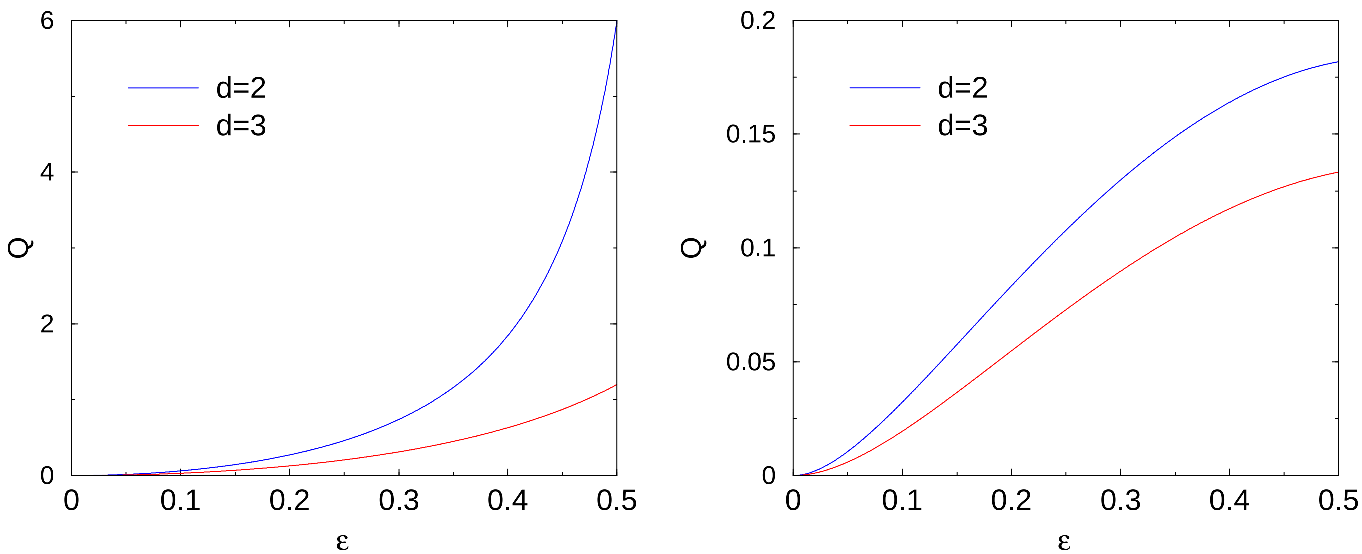
$$Q = \frac{\langle v_x^2 v_y^2 \rangle - \langle v_x^2 \rangle \langle v_y^2 \rangle}{\langle v_x^2 \rangle \langle v_y^2 \rangle}$$

- Unforced case (freely cooling): $P(v) \sim v^{-\sigma}$

$$Q = \frac{6\epsilon^2}{d - (1 + 3\epsilon^2)}$$

- Forced case (white noise): $P(v) \sim e^{-|v|}$

$$Q = \frac{6\epsilon^2(1-\epsilon)}{(d+2)(1+\epsilon) - 3(1-\epsilon)(1+\epsilon^2)}.$$



Correlations diminish with energy input

The "brazil nut" problem

- Fluid background: mass 1, restitution α
- Impurity: mass m, restitution β
- Theory: impurity enslaved to background
- Series of transition masses

$$1 < m_1 < m_2 < \cdots < m_\infty$$

- Ratio of moments diverges asymptotically

$$\frac{\langle v_I^{2n} \rangle}{\langle v_F^{2n} \rangle} \sim \begin{cases} c_n & m < m_n; \\ \infty & m > m_n. \end{cases}$$

- Light impurity: moderate violation of equipartition, impurity mimics the fluid
- Heavy impurity: extreme violation of equipartition, impurity sees a static fluid

series of phase transitions

Conclusions II

- Velocity distribution obeys scaling
- Power-law high energy tails
- Exponent depends on d and ϵ
- Multiscaling of the moments, Temperature insufficient to characterize large moments
- Velocity correlations develop
- Algebraic autocorrelations, aging
- **Inelasticity responsible for "fat" tails**

- E. Ben-Naim and P. L. Krapivsky, *J. Phys. A* **35**, L147 (2002); Phys. Rev. E **66**, 011309 (2002); EPJE, in press.
- M. H. Ernst and R. Britto, Europhys. Lett. **58**, 182 (2002). Phys. Rev. E **65**, 040301 (2002).
- A. Baldassari, U. M. B Marconi, and A Puglisi, Europhys. Lett. **58**, 14 (2002); Phys Rev. E **65**, 011301 (2002).