Jamming and tiling in fragmentation of rectangles

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arXiv://1905.06984
Physical Review E **100**, 032122 (2019)

Talk, publications available from: http://cnls.lanl.gov/~ebn

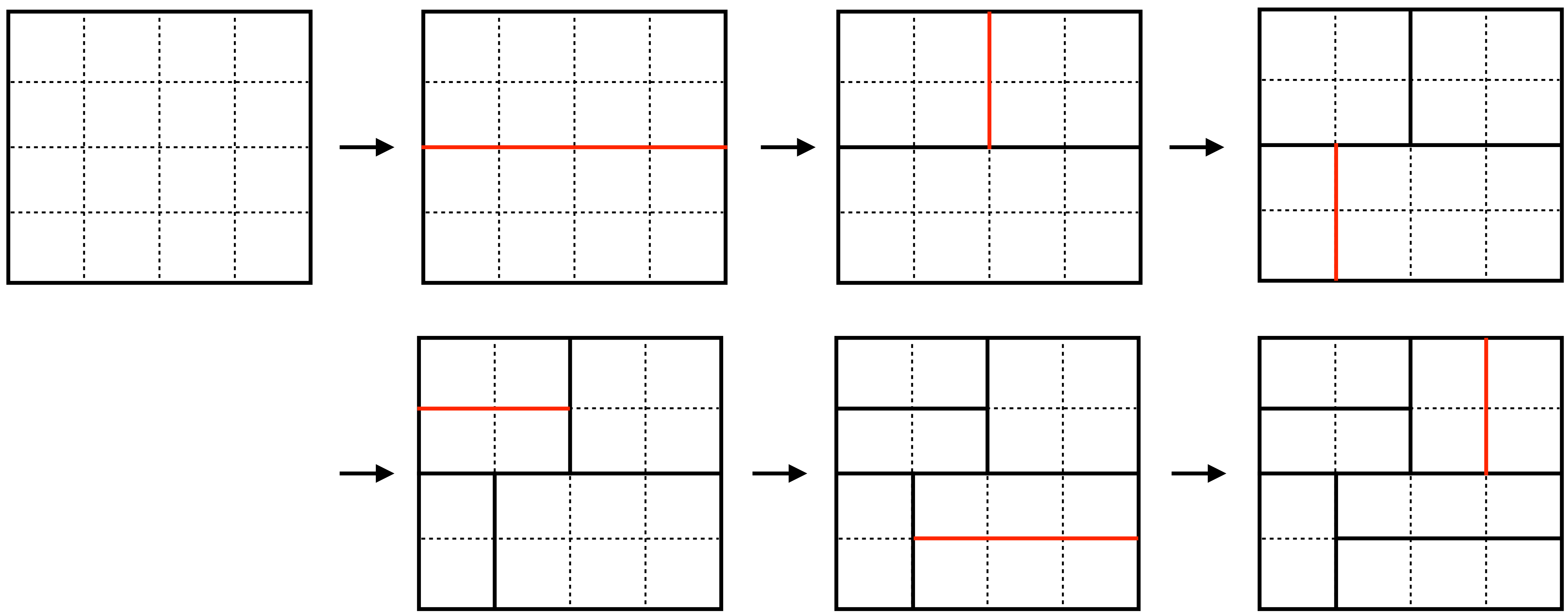
APS March Meeting
Denver, CO, March 4, 2020

Fragmentation of rectangles

Start with a perfect grid

Pick (i) random grid point (ii) random direction

Fragment rectangle into two smaller rectangles



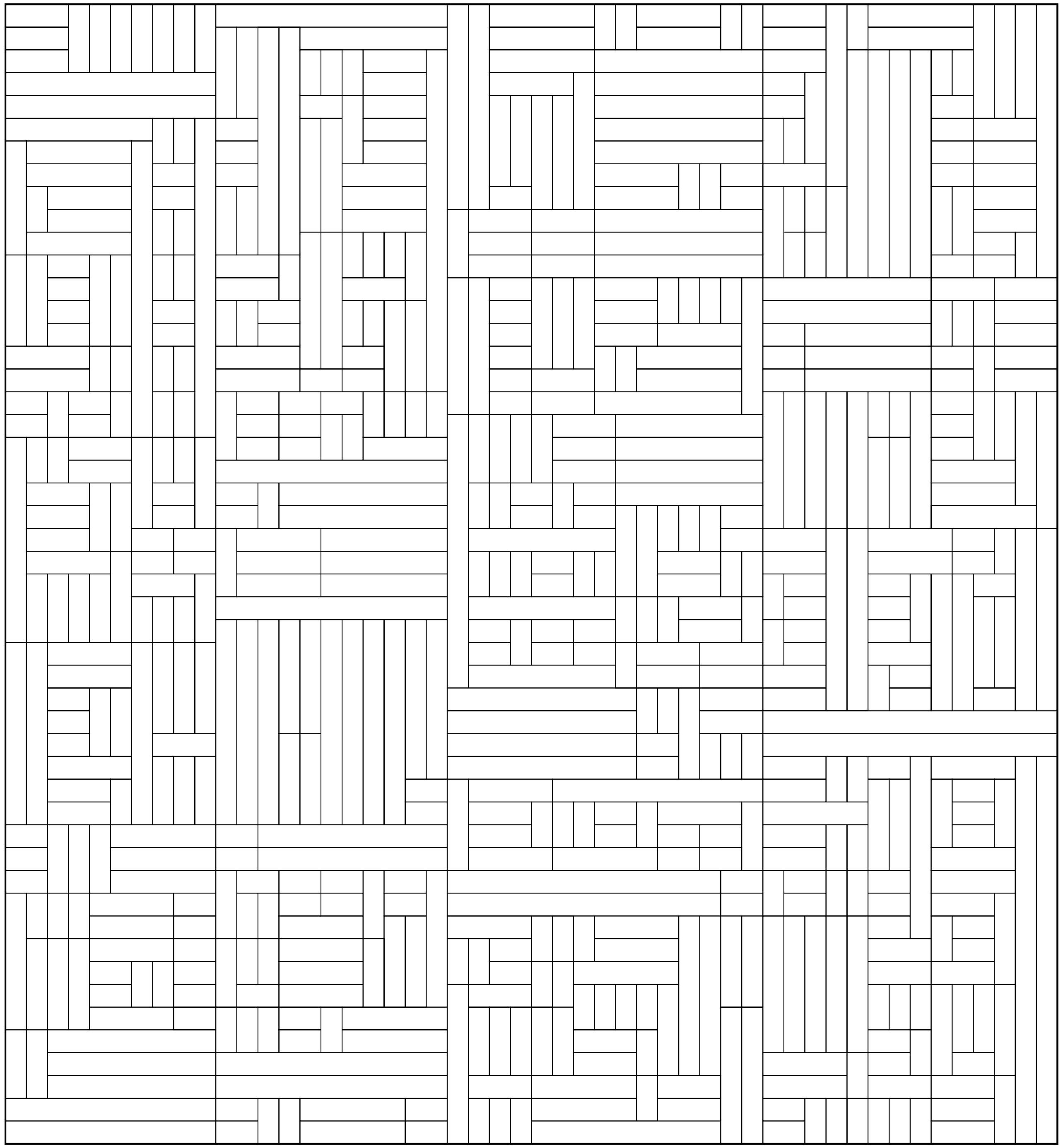
System reaches a jammed state

All rectangles are sticks (1 x k or k x 1)

The jammed state

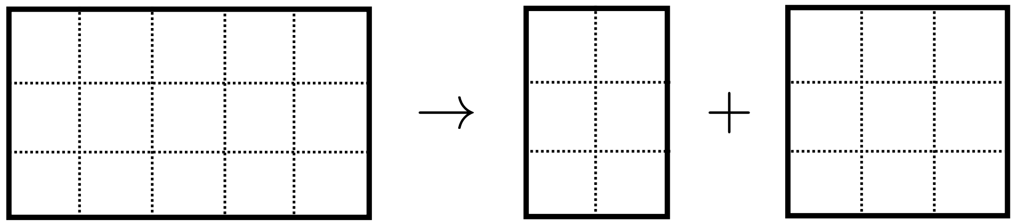
Tiling by sticks

Tiling is:
- Polydisperse
- Dynamical



How many sticks? How long? How many jammed states?

Theoretical approach: recursion equations



1D
Filippov 61
Spouge 84
Ziff, McGrady 85

- Random fragmentation process

$$(m,n) \to \begin{cases} (i,n) + (m-i,n) & \text{with prob. } 1/2 \\ (m,j) + (m,n-j) & \text{with prob. } 1/2 \end{cases}$$

- Average number of sticks $S(m,n)$ in an $m \times n$ rectangle

- Recursion: sum over all possible (i) grid points (ii) directions

$$S(m,n) = \frac{1}{2} \times \frac{1}{m-1} \sum_{i=1}^{m-1} \Big[S(i,n) + S(m-i,n)\Big] + \frac{1}{2} \times \frac{1}{n-1} \sum_{j=1}^{n-1} \Big[S(m,j) + S(m,n-j)\Big]$$

- Linear recursion equations for number of jammed sticks

$$S(m,n) = \frac{1}{m-1} \sum_{i=1}^{m-1} S(i,n) + \frac{1}{n-1} \sum_{j=1}^{n-1} S(m,j)$$

2D
Torrents, Illa,
Vives, Planes
PRE 2017

Theory: (i) linear (ii) bypasses dynamics (iii) 2d

Asymptotic analysis

1. Continuum limit (very large rectangles)

$$S(m,n) = \frac{1}{m}\int_1^m di\, S(i,n) + \frac{1}{n}\int_1^n dj\, S(m,j)$$

2. Convert integral equation into partial differential equation

$$\partial_\mu \partial_\nu S(\mu,\nu) = S(\mu,\nu) \qquad \begin{aligned} \mu &= \ln m \\ \nu &= \ln n \end{aligned}$$

3. Introduce double Laplace transform

$$\widehat{S}(p,q) = \int_0^\infty d\mu\, e^{-p\mu} \int_0^\infty d\nu\, e^{-q\nu}\, S(\mu,\nu)$$

4. Obtain Laplace transform in compact form

$$\widehat{S}(p,q) = \frac{1}{pq-1}$$

5. Invert double Laplace transform (saddle point analysis)

$$S(\mu,\nu) = \int_{-i\infty}^{i\infty} \frac{dp}{2\pi i} \int_{-i\infty}^{i\infty} \frac{dq}{2\pi i}\, \frac{e^{p\mu+q\nu}}{pq-1} \quad \to \quad S(\mu,\nu) \simeq \frac{e^{2\sqrt{\mu\nu}}}{\sqrt{4\pi\sqrt{\mu\nu}}}$$

Average number of jammed sticks

- Asymptotic behavior

$$S(m,n) \simeq \frac{e^{2\sqrt{(\ln m)(\ln n)}}}{\sqrt{4\pi\sqrt{(\ln m)(\ln n)}}}$$

- Focus on very large rectangles with finite aspect ratio

$$m \to \infty \quad \text{and} \quad n \to \infty \quad \text{with} \quad m/n = \text{constant}$$

- Universal behavior for all rectangles with same area

$$S(A) \simeq \frac{A}{\sqrt{2\pi \ln A}} \qquad\qquad A = mn$$

- Average stick length $\langle k \rangle = A/S$ grows slowly with area

$$\langle k \rangle \simeq \sqrt{2\pi \ln A}$$

Behavior is independent of aspect ratio

Distribution of stick length

- Number of sticks of given length obeys same recursion

$$S_k(m,n) = \frac{1}{m-1}\sum_{i=1}^{m-1} S_k(i,n) + \frac{1}{n-1}\sum_{j=1}^{n-1} S_k(m,j)$$

- Leading asymptotic behavior

$$P_k \simeq 2k^{-2}\,\exp\left[-\frac{(\ln k)^2}{2\ln A}\right]$$

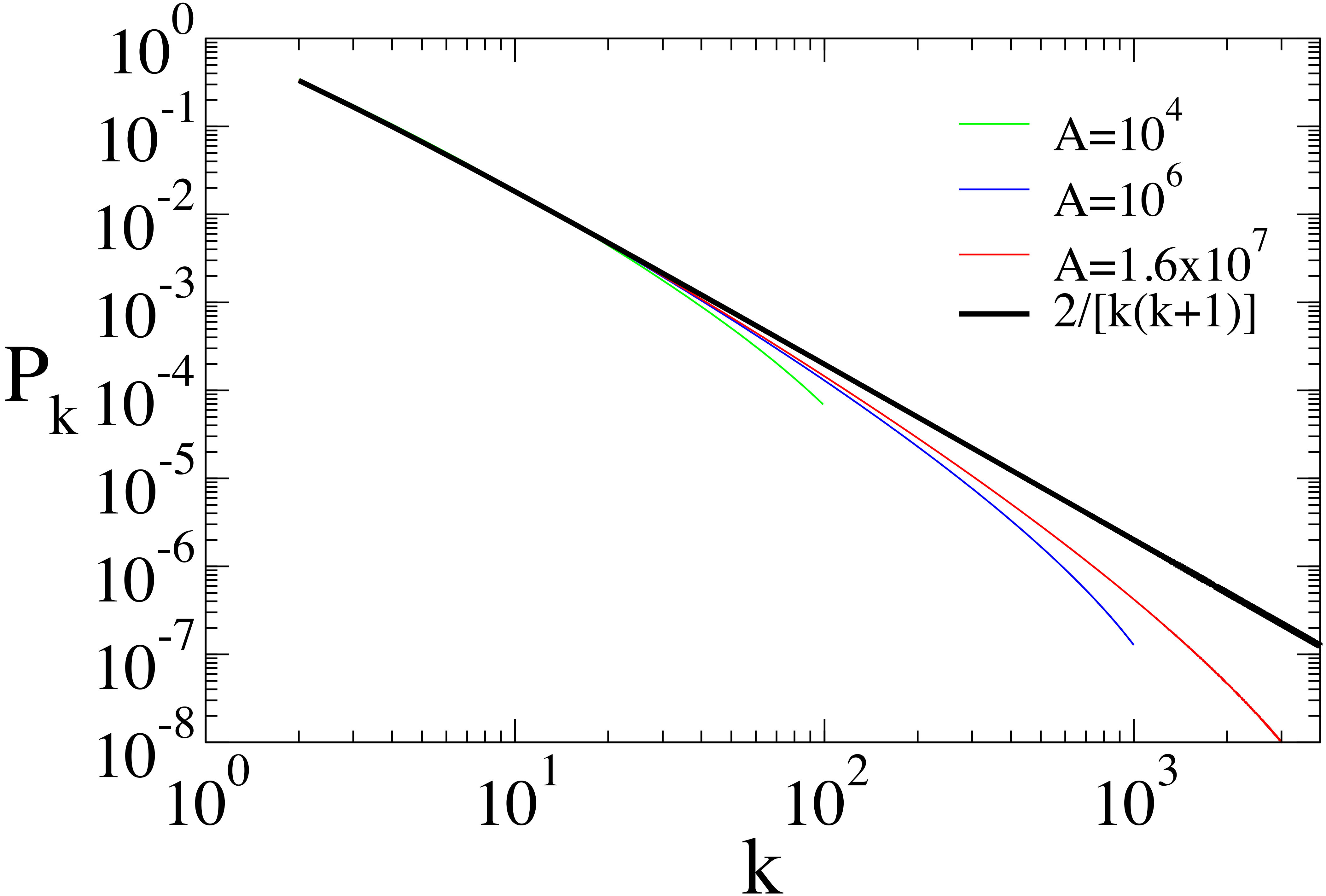
- Infinite-area limit: exact result

$$P_k = \frac{2}{k(k+1)}$$

Below average length: power law tail

Above average length: log-normal decay

Numerical validation



perfect agreement for small length (within 0.1%)
convergence is very slow

Moments of length distribution

- Normalized moments

$$M_h = \frac{\langle k^h \rangle}{\langle k \rangle} \qquad \langle k^h \rangle = \sum_{k \geq 2} k^h P_k$$

- Multiscaling asymptotic behavior

$$M_h \sim A^{\mu(h)} \quad \text{with} \quad \mu(h) = \frac{(h-1)^2}{h}$$

- Different spectrum than continuum version

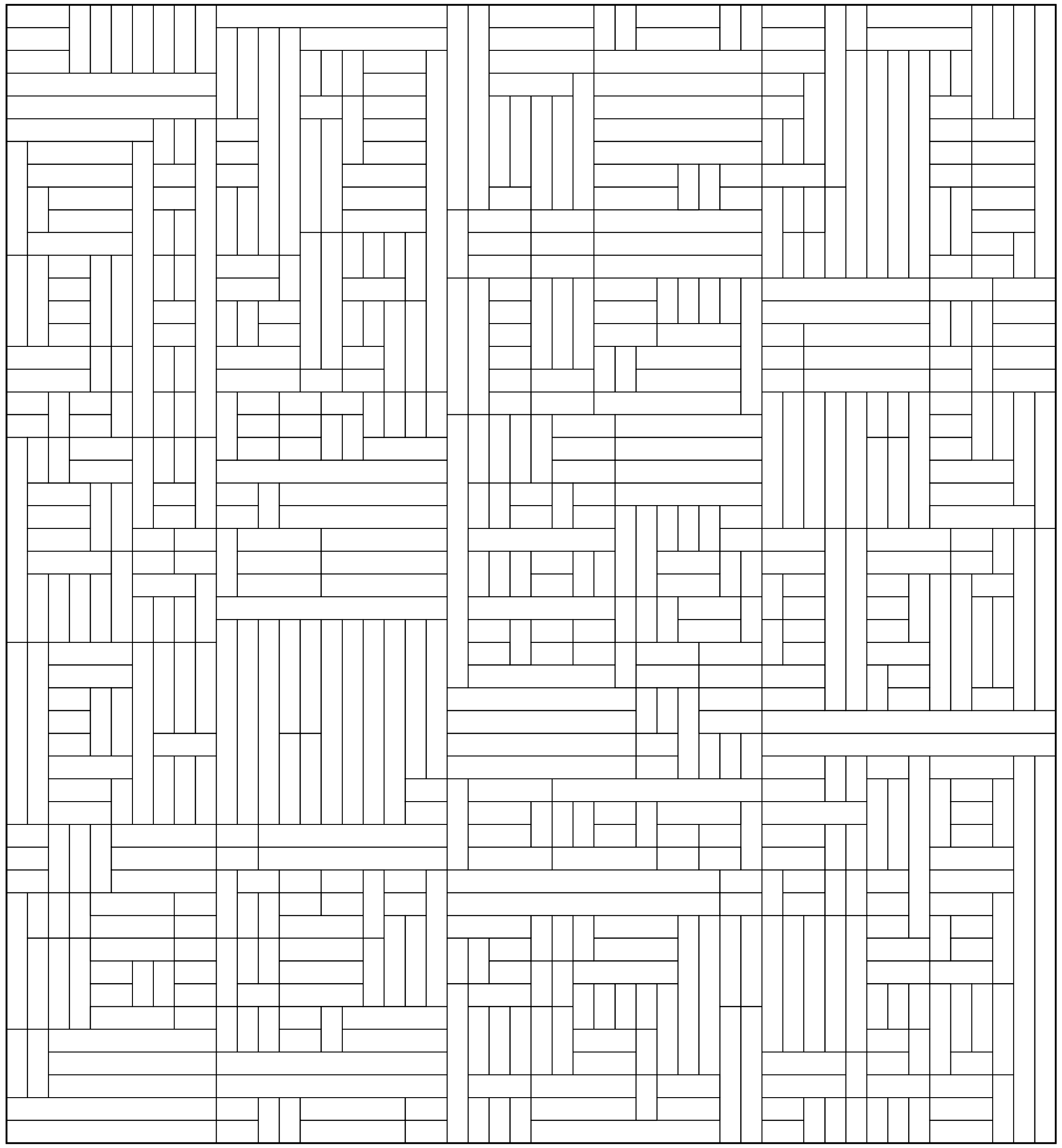
EB, Krapivsky 96

$$M_h \sim A^{\mu_{\text{nojam}}(h)} \quad \text{with} \quad \mu_{\text{nojam}}(h) = \sqrt{h^2+1} - \sqrt{2}$$

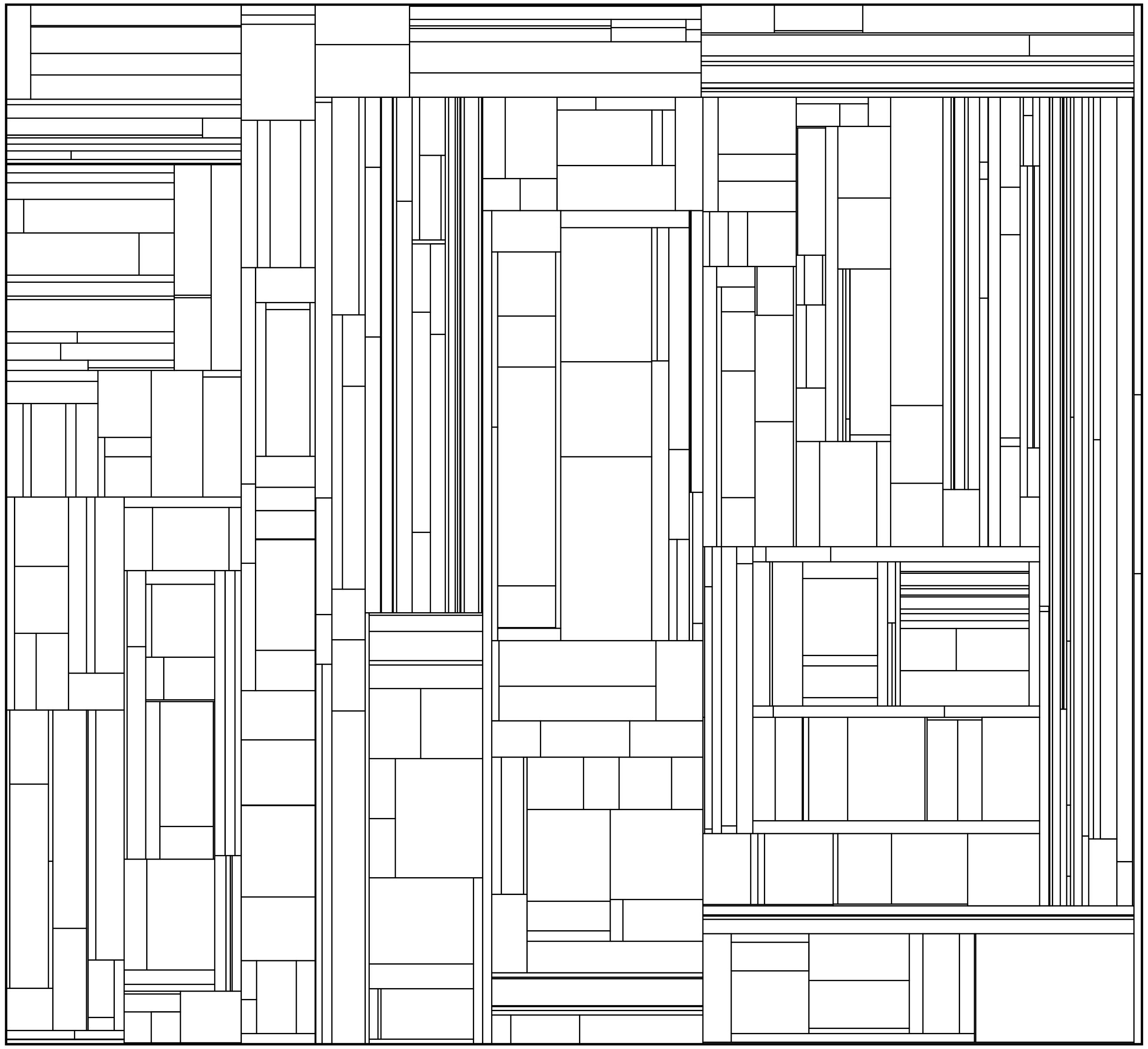
Nonlinear spectrum of scaling exponents
Discrete and continuous versions differ!!!

Discrete versus continuous fragmentation

discrete version
process stops



continuous version
process never stops



Asymmetric fragmentation

- Two fragmentation events realized with different probabilities

$$(m,n) \to \begin{cases} (i,n) + (m-i,n) & \text{with prob.} \quad (1-\alpha)/2 \\ (m,j) + (m,n-j) & \text{with prob.} \quad (1+\alpha)/2 \end{cases}$$

- Discrepancy between two extreme cases

$$S = \sqrt{A} \qquad \alpha = 1 \quad \text{(perfectly asymmetric)}$$

$$S \simeq A/\sqrt{2\pi \ln A} \qquad \alpha = 0 \quad \text{(perfectly symmetric)}$$

- Strongly asymmetric phase: purely power law

$$S \sim A^{\sqrt{1-\alpha^2}} \qquad \alpha > \frac{1}{\sqrt{2}}$$

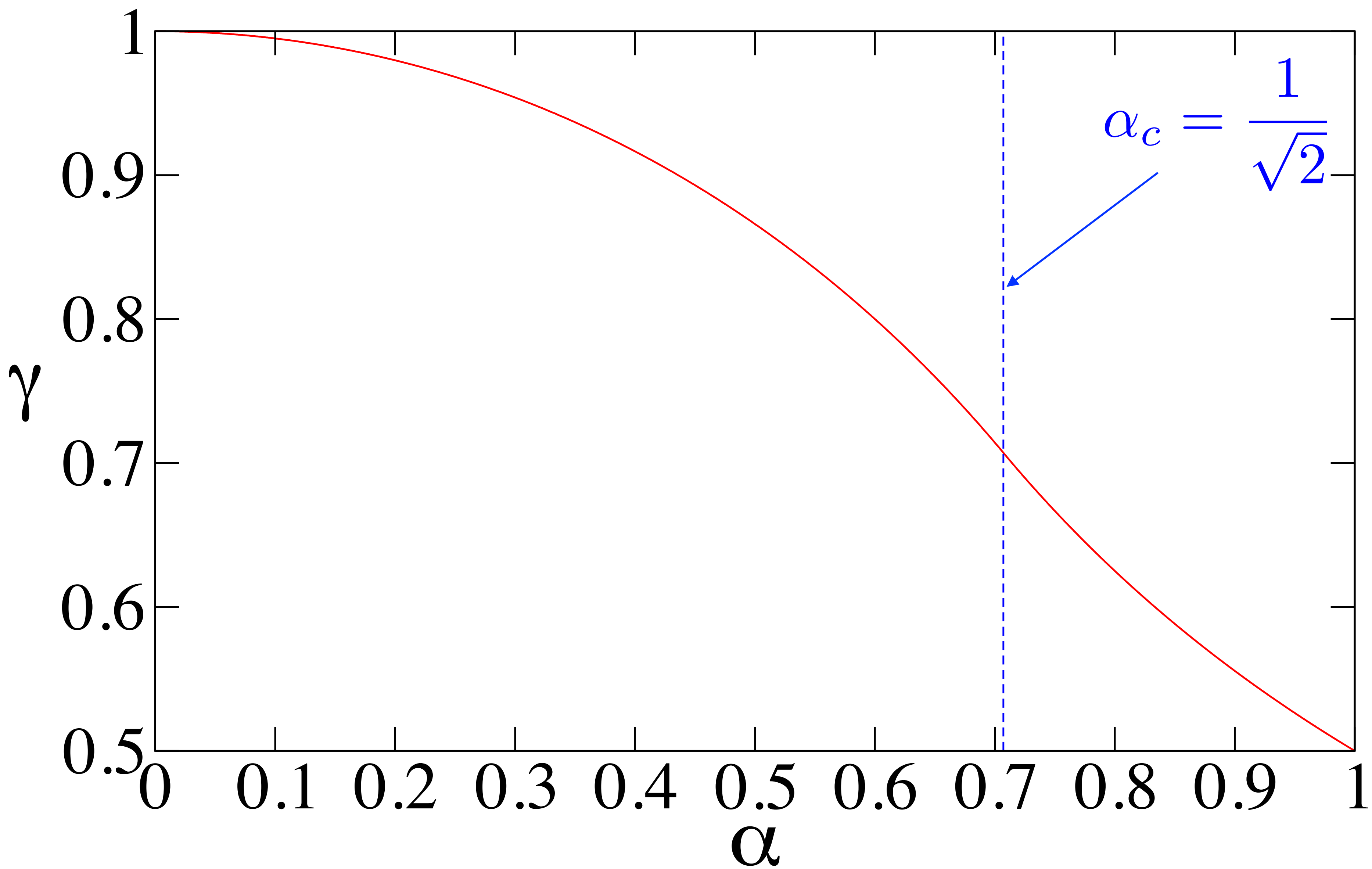
- Weakly asymmetric phase: power law + logarithmic correction

$$S \sim (\ln A)^{-1/2} A^{1/(2\alpha)} \qquad \alpha < \frac{1}{\sqrt{2}}$$

Phase transition at finite asymmetry strength

The growth exponent

$$S \sim A^{\gamma} \quad \text{with} \quad \gamma = \begin{cases} \sqrt{1-\alpha^2} & \alpha \leq \alpha_c \\ 1/(2\alpha) & \alpha \geq \alpha_c \end{cases}$$

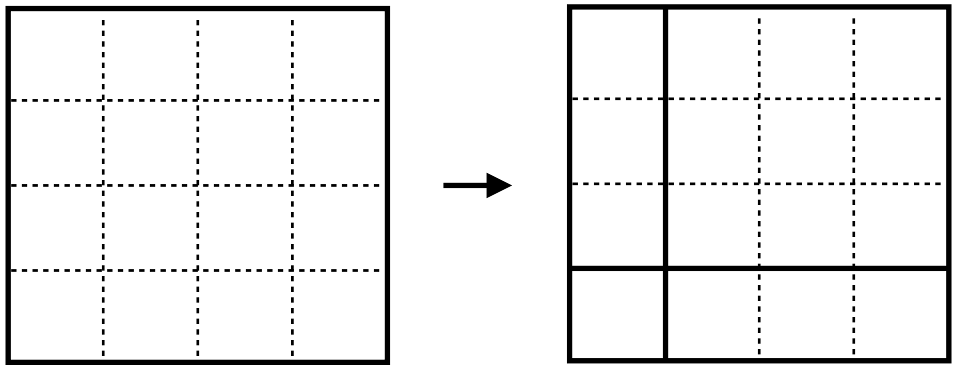


Sub-linear growth with area

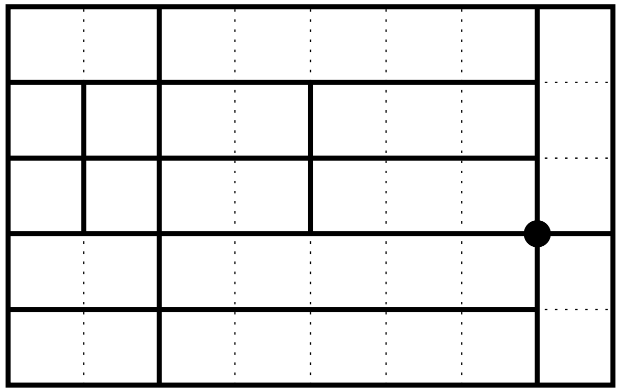
Growth exponent has two distinct forms

Number of jammed configurations

"deterministic" fragmentation into four rectangles



first fragmentation point can be uniquely identified



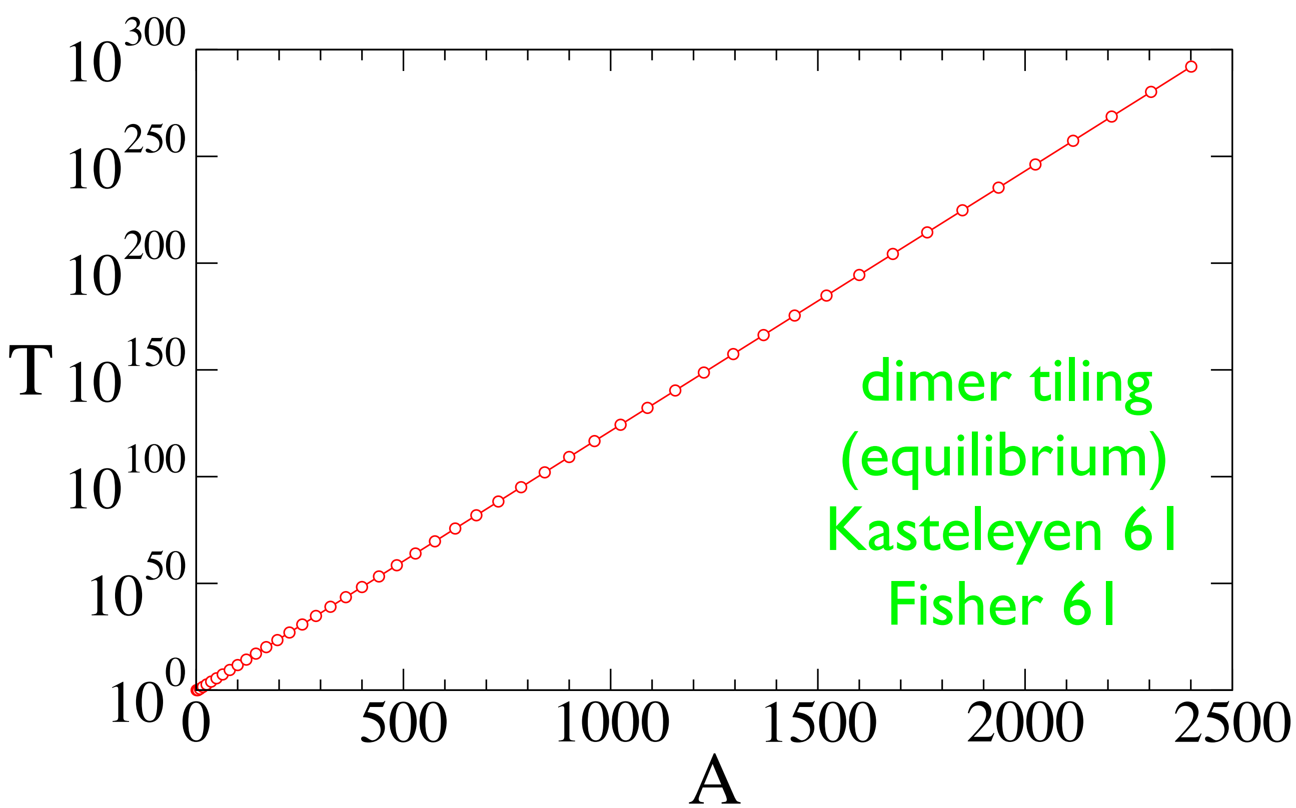
recursion equation for the total number of jammed states

$$T(m,n) = \sum_{\substack{1\le i\le m-1\\ 1\le j\le n-1}} T(i,j)T(m-i,j)T(i,n-j)T(m-i,n-j)$$

exponential growth with area

$$T \sim e^{\lambda A}$$

$$\lambda = 0.2805$$



Aspect ratio dependence?

Conclusions

- Random fragmentation of rectangles
- Process reaches a jammed state where all rectangles are sticks
- Recursion equations give statistical property of jammed state
- Number of jammed sticks is independent of aspect ratio
- Distribution of stick length decays as a power law
- Multiscaling: nonlinear spectrum of exponents for moments
- Asymmetric fragmentation: phase transition for growth exponent
- Generally, number of sticks grows sub-linearly with area
- Number of jammed states grows exponentially with area
- Abundance of exact analytic results