Scaling Laws for First-Passage Exponents

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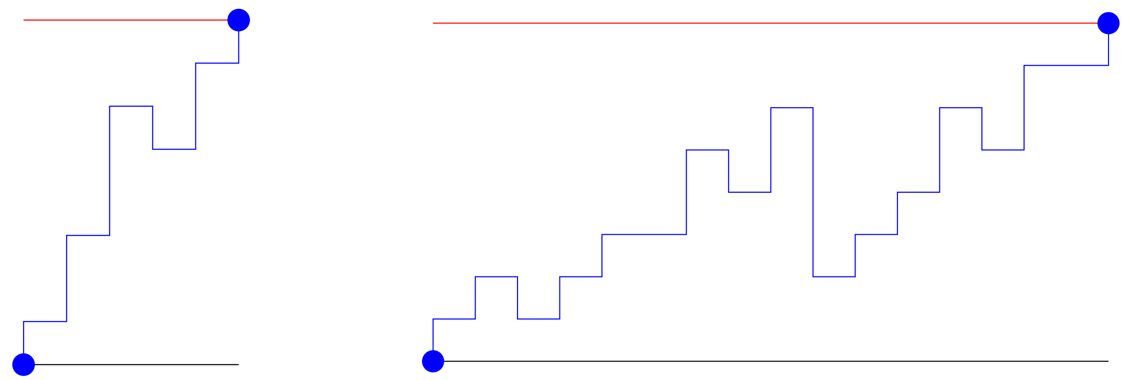
EB PRE 82, 061103 (2010), E. Ben-Naim and P.L. Krapivsky, J. Phys. A **43**, 495007 & 495008 (2010);
arxiv:1306:2990: First passage in conical geometry and ordering of Brownian particles
in *First-Passage Phenomena*, editors: R. Metzler, G. Oshanin, S. Redner

Talk, publications available from: http://cnls.lanl.gov/~ebn

STATPHYS25, Seoul, Korea, July 22, 2013

First-Passage Process



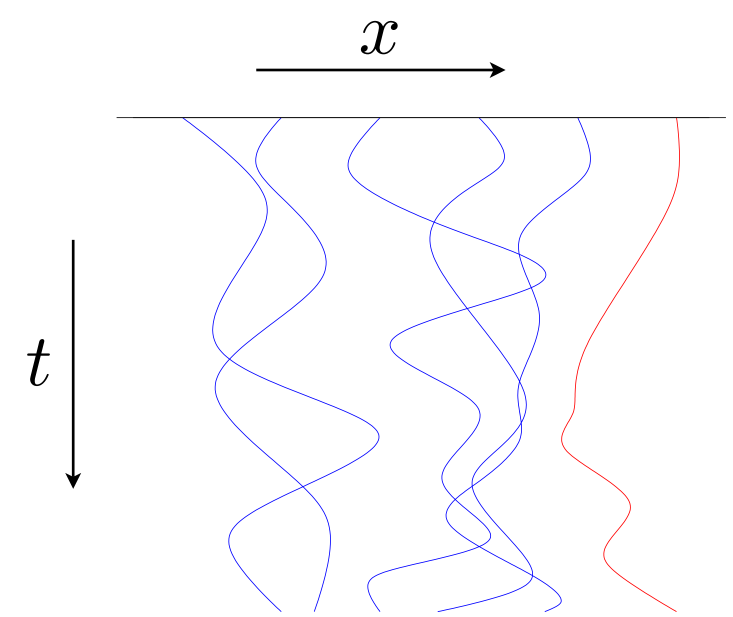
- Process by which a fluctuating quantity reaches a threshold for the <u>first</u> time.
- **First-passage probability:** for the random variable to reach the threshold as a function of time.
- **Total probability**: that threshold is <u>ever</u> reached. May or may not equal 1.
- **First-passage time**: the mean duration of the first-passage process. Can be <u>finite</u> or <u>infinite</u>.

Typically defined by a single threshold

S. Redner, A Guide to First-Passage Processes, 2001

Ordering of Brownian particles

- System: N independent Brownian particles in one dimension
- What is the probability that original leader maintains the lead?
- N Brownian particles



$$\frac{\partial \varphi_i(x,t)}{\partial t} = D\nabla^2 \varphi_i(x,t)$$

- Initial conditions

$$x_N(0) < x_{N-1}(0) < \cdots < x_2(0) < x_1(0)$$

- Survival probability $S(t)$=probability leader remains first until t
- Independent of initial conditions, power-law asymptotic behavior

$$S(t) \sim t^{-\beta} \qquad \text{as} \qquad t \to \infty$$

- Monte Carlo: nontrivial exponents that depend on N

Bramson 91
Redner 96
benAvraham 02
Grassberger 03

N	2	3	4	5	6	10
$\beta(N)$	1/2	3/4	0.913	1.032	1.11	1.37

No analytic expressions for exponents

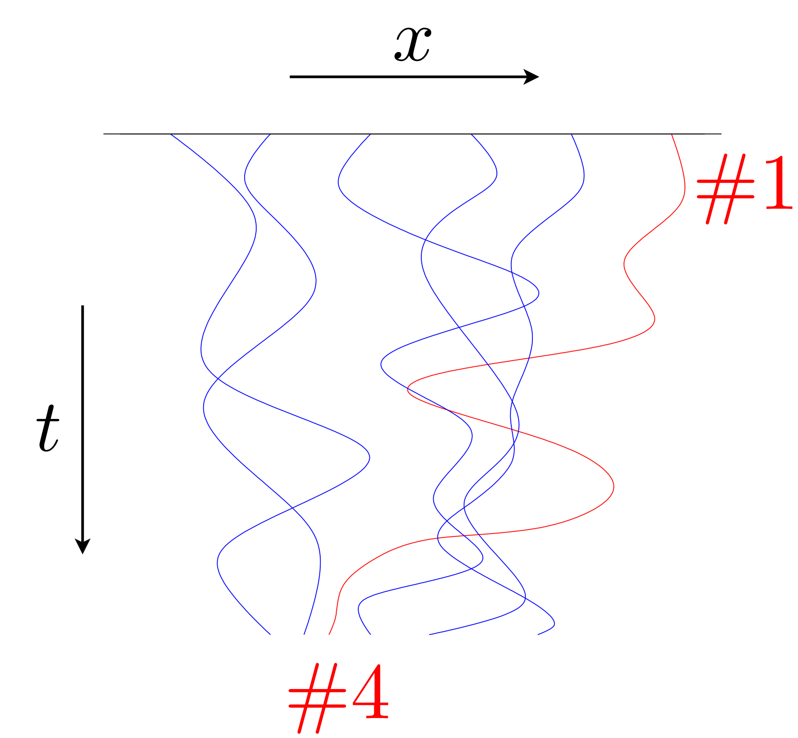
Order statistics

- Generalize the capture problem: $S_m(t)$ is the probability that the leader does not fall below rank m until time t
- $S_1(t)$ is the probability that leader remains first
- $S_{N-1}(t)$ is the probability that leader never becomes last
- Power-law asymptotic behavior is generic

$$S_m(t) \sim t^{-\beta_m(N)}$$

- Spectrum of first-passage exponents

$$\beta_1(N) > \beta_2(N) > \cdots > \beta_{N-1}(N)$$



Can't solve the problem? Make it bigger!

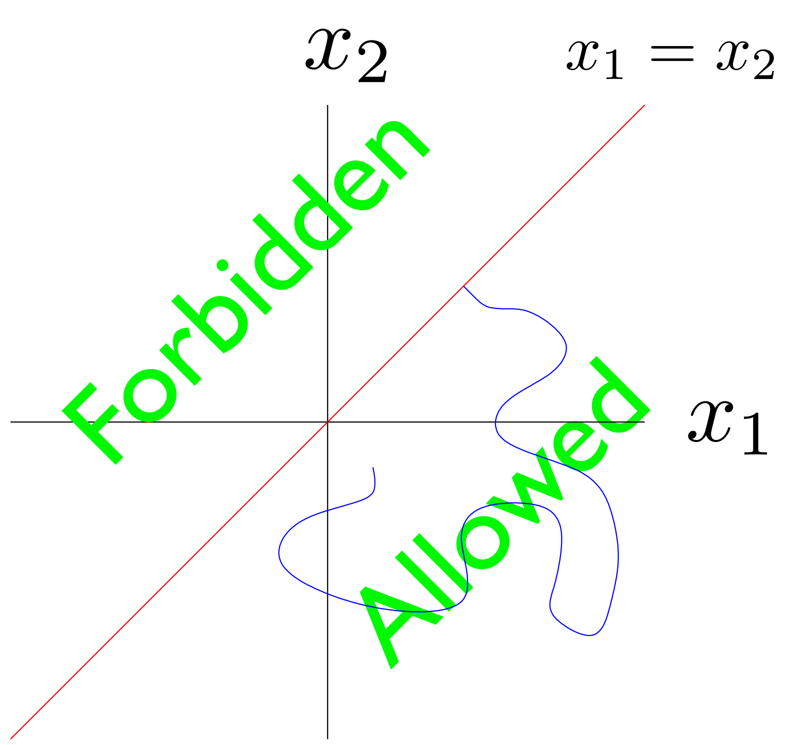
Lindenberg 01

Two particles

- We need the probability that two particles do not meet
- Map <u>two one-dimensional</u> walks onto <u>one two-dimensional</u> walk
- Space is divided into allowed and forbidden regions
- Boundary separating the two regions is absorbing
- Coordinate $x_1 - x_2$ performs one-dimensional random walk
- Survival probability decays as power-law

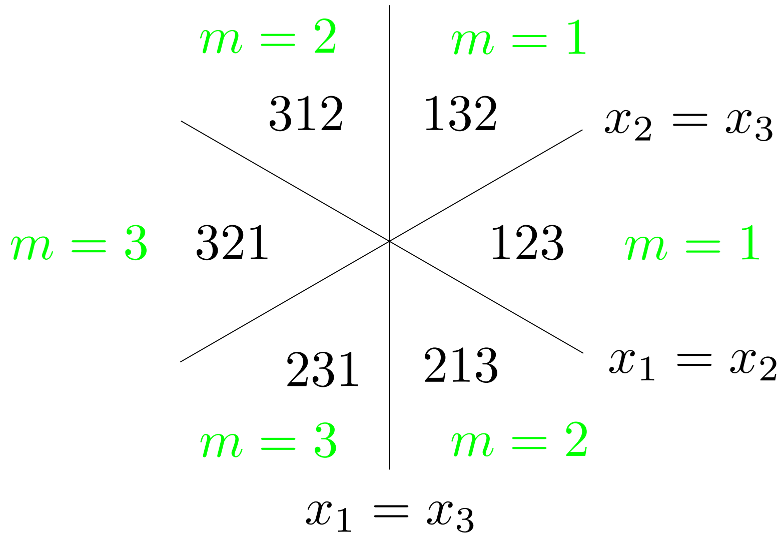
$$S_1(t) \sim t^{-1/2}$$

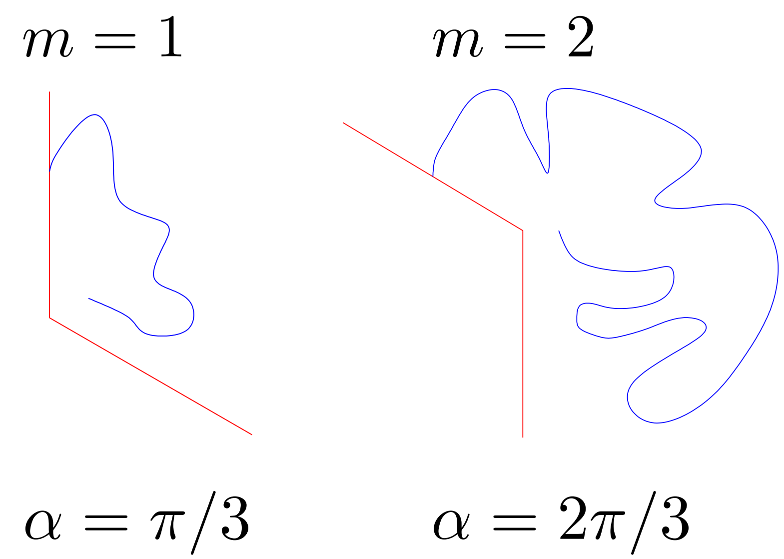
- In general, map N one-dimensional walk onto one walk in N dimension with complex boundary conditions



Three particles

- Diffusion in three dimensions; now, allowed regions are wedges





- Survival probability in wedge with opening angle $0<\alpha<\pi$

$$S(t)\sim t^{-\pi/(4\alpha)}$$

Spitzer 58
Fisher 84

- Survival probabilities decay as power-law with time

$$S_1\sim t^{-3/4}\quad\text{and}\quad S_2\sim t^{-3/8}$$

- Indeed, a family of nontrivial first-passage exponents

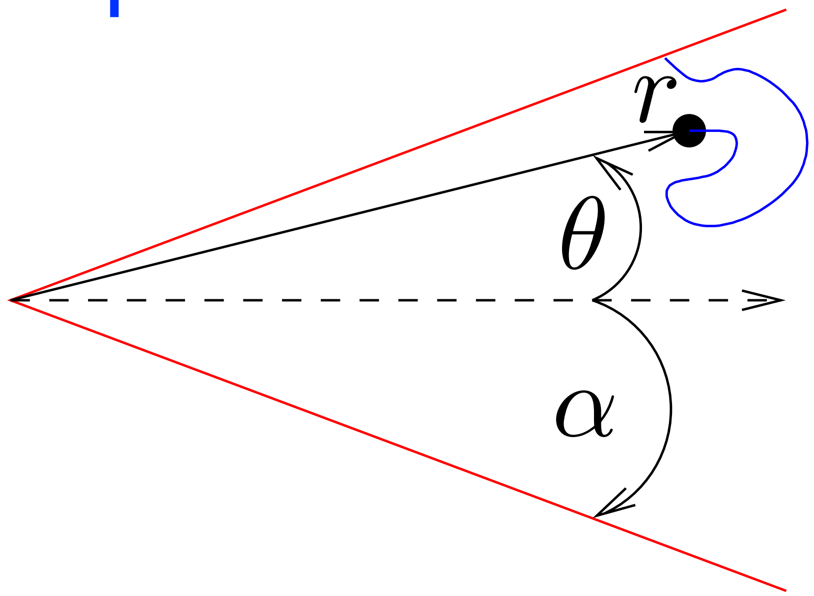
$$S_m\sim t^{-\beta_m}\qquad\text{with}\qquad \beta_1>\beta_2>\cdots>\beta_{N-1}$$

Large spectrum of first-passage exponents

First passage in a wedge

- Survival probability obeys the diffusion equation

$$\frac{\partial S(r,\theta,t)}{\partial t} = D\nabla^2 S(r,\theta,t)$$



- Focus on long-time limit

$$S(r,\theta,t) \simeq \Phi(r,\theta)\, t^{-\beta}$$

- Amplitude obeys Laplace's equation

$$\nabla^2 \Phi(r,\theta) = 0$$

- Use dimensional analysis

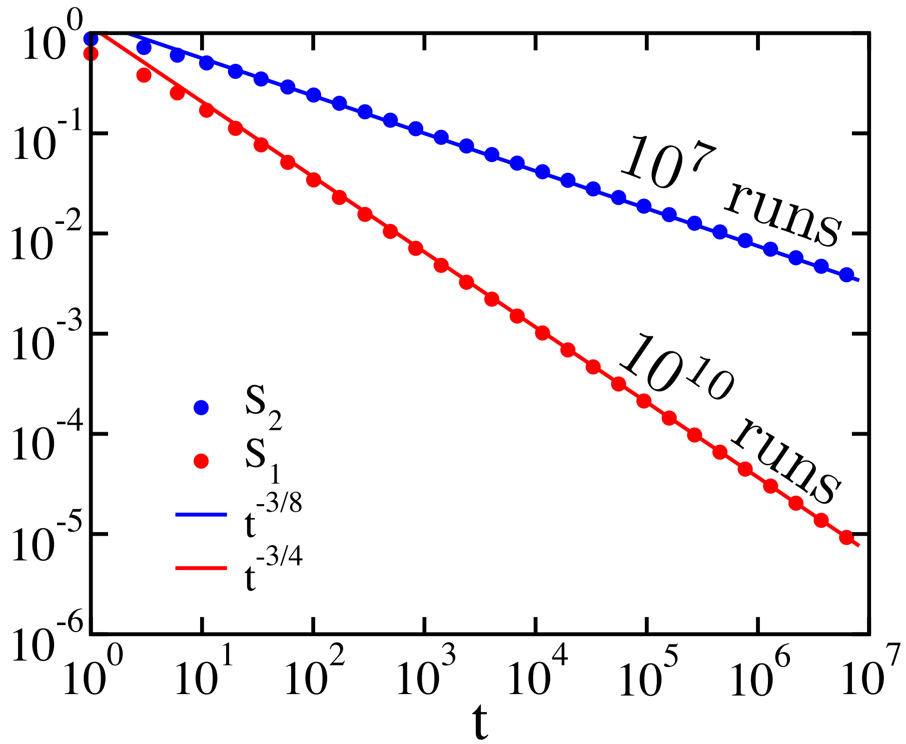
$$\Phi(r,\theta) \sim (r^2/D)^\beta \psi(\theta) \quad \Longrightarrow \quad \psi_{\theta\theta} + (2\beta)^2\psi = 0$$

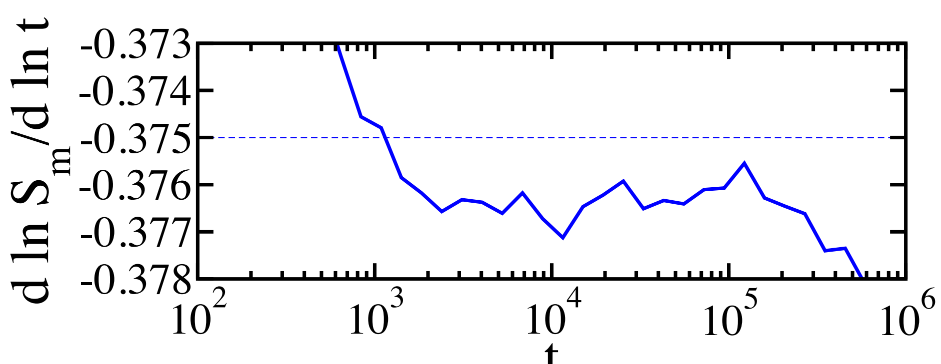
- Enforce boundary condition $S|_{\theta=\alpha} = \Phi|_{\theta=\alpha} = \psi|_{\theta=\alpha}$
- Lowest eigenvalue is the relevant one

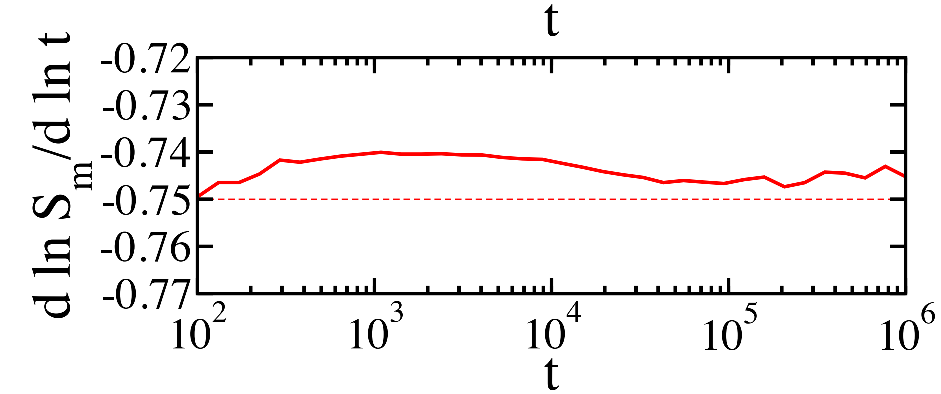
$$\psi_2(\theta) = \cos(2\beta\theta) \quad \Longrightarrow \quad \beta = \frac{\pi}{4\alpha}$$

Monte Carlo simulations

3 particles

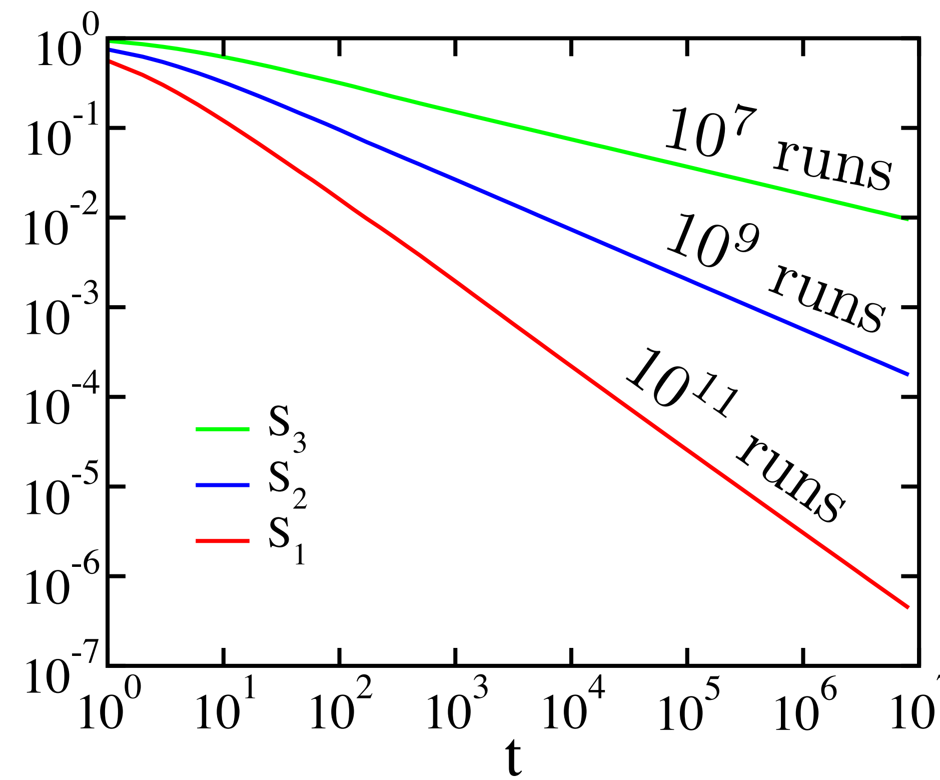






confirm wedge theory results

4 particles



$$\beta_1 = 0.913$$

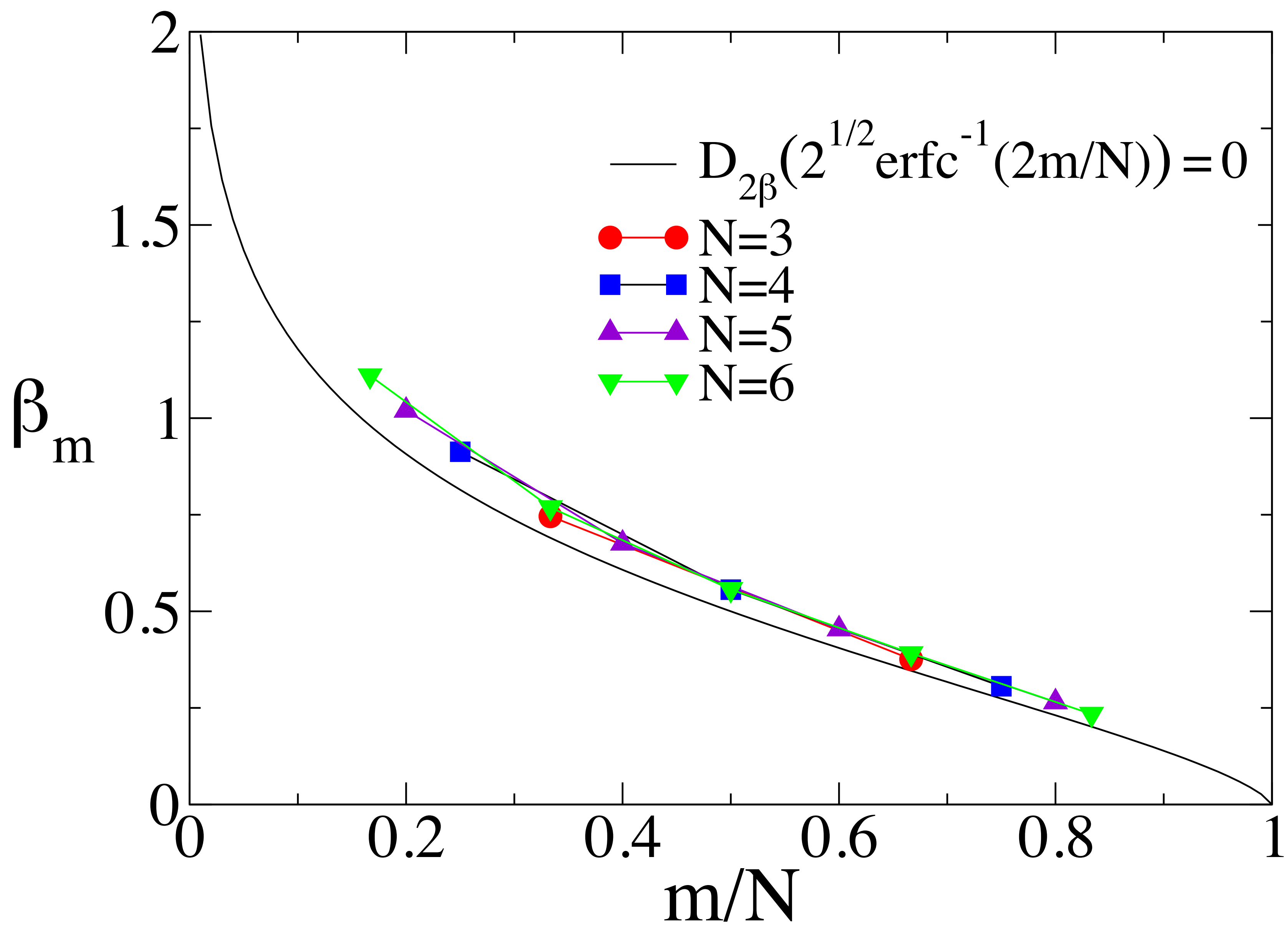
$$\beta_2 = 0.556$$

$$\beta_3 = 0.306$$

as expected, there are 3 nontrivial exponents

Simulations: small number of particles

strongly hints at asymptotic scaling behavior!



$$\beta_m(N) \to F\,(m/N) \quad \text{when} \quad N \to \infty$$

Scaling law for first-passage exponents

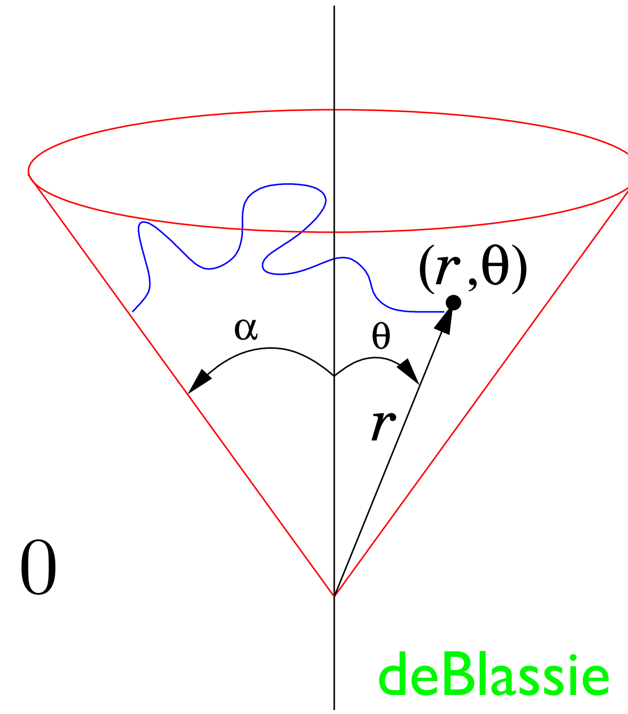
Kinetics of first passage in a cone

- Repeat wedge calculation step by step

$$S(r,\theta,t) \sim \psi(\theta)(Dt/r^2)^{-\beta}$$

- Angular function obeys Poisson-like equation

$$\frac{1}{(\sin\theta)^{d-2}}\frac{d}{d\theta}\left[(\sin\theta)^{d-2}\frac{d\psi}{d\theta}\right] + 2\beta(2\beta+d-2)\psi = 0$$



deBlassie 88

- Solution in terms of associated Legendre functions

$$\psi_d(\theta) = \begin{cases} (\sin\theta)^{-\delta} P^{\delta}_{2\beta+\delta}(\cos\theta) & d \text{ odd}, \\ (\sin\theta)^{-\delta} Q^{\delta}_{2\beta+\delta}(\cos\theta) & d \text{ even} \end{cases} \qquad \delta = \frac{d-3}{2}$$

- Enforce boundary condition, choose <u>lowest</u> eigenvalue

$$\begin{aligned} P^{\delta}_{2\beta+\delta}(\cos\alpha) &= 0 \qquad d \text{ odd}, \\ Q^{\delta}_{2\beta+\delta}(\cos\alpha) &= 0 \qquad d \text{ even}. \end{aligned}$$

Exponent is root of Legendre function

Additional results

- Explicit results in *2d* and *4d*

$$\beta_2(\alpha) = \frac{\pi}{4\alpha} \quad \text{and} \quad \beta_4(\alpha) = \frac{\pi - \alpha}{2\alpha}$$

- Root of ordinary Legendre function in *3d*

$$P_{2\beta}(\cos\alpha) = 0$$

- Flat cone is equivalent to one-dimension

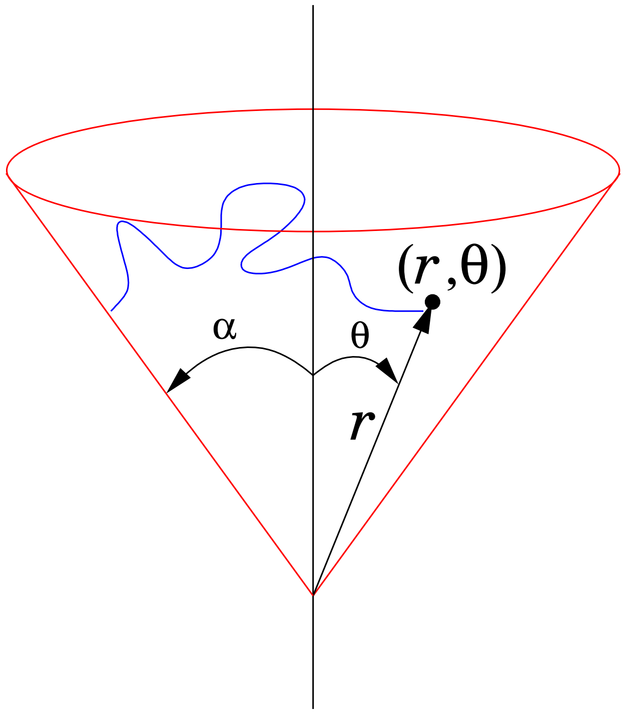
$$\beta_d(\alpha = \pi/2) = 1/2$$

- First-passage time obeys Poisson's equation

$$D\nabla^2 T(r,\theta) = -1$$

- First-passage time (when finite)

$$T(r,\theta) = \frac{r^2}{2D}\,\frac{\cos^2\theta - \cos^2\alpha}{d\cos^2\alpha - 1}$$



$$\alpha < \cos^{-1}(1/\sqrt{d})$$

Asymptotic analysis

- Limiting behavior of scaling function

$$\beta(y) \simeq \begin{cases} \sqrt{y^2/8\pi}\, \exp\left(-y^2/2\right) & y \to -\infty, \\ y^2/8 & y \to \infty. \end{cases}$$

- Thin cones: exponent diverges

$$\beta_d(\alpha) \simeq B_d \alpha^{-1} \quad \text{with} \quad J_\delta(2B_d) = 0$$

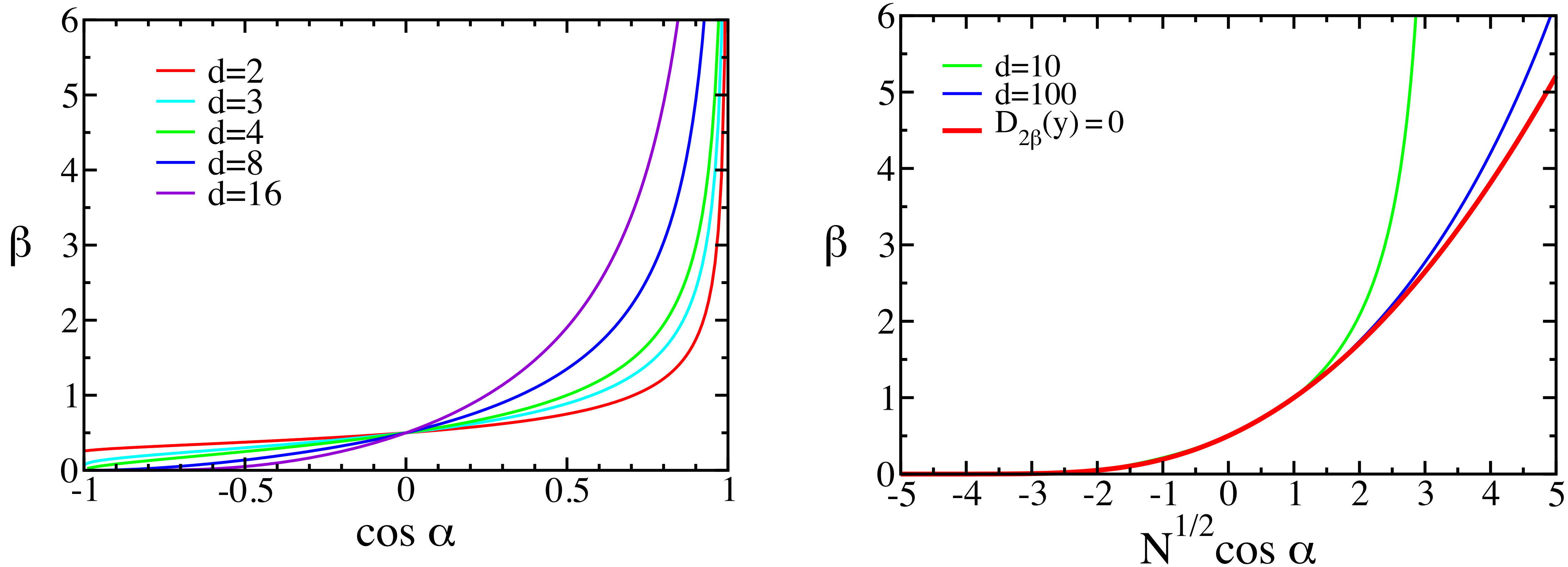
- Wide cones: exponent vanishes when $d \geq 3$

$$\beta_d(\alpha) \simeq A_d\, (\pi - \alpha)^{d-3} \quad \text{with} \quad A_d = \frac{1}{2} B\left(\frac{1}{2}, \frac{d-3}{2}\right)$$

- A needle is reached with certainty only when $d < 3$
- Large dimensions

$$\beta_d(\alpha) \simeq \begin{cases} \frac{d}{4}\left(\frac{1}{\sin\alpha} - 1\right) & \alpha < \pi/2, \\ C(\sin\alpha)^d & \alpha > \pi/2. \end{cases}$$

High dimensions



- Exponent varies sharply for opening angles near $\pi/2$
- Universal behavior in high dimensions

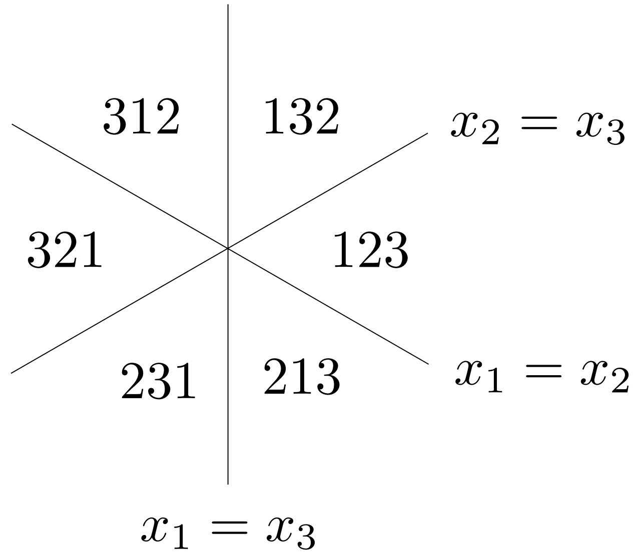
$$\beta_d(\alpha) \to \beta(\sqrt{N}\cos\alpha)$$

- Scaling function is smallest root of parabolic cylinder function

$$D_{2\beta}(y) = 0$$

Exponent is function of one scaling variable, not two

Diffusion in high dimensions

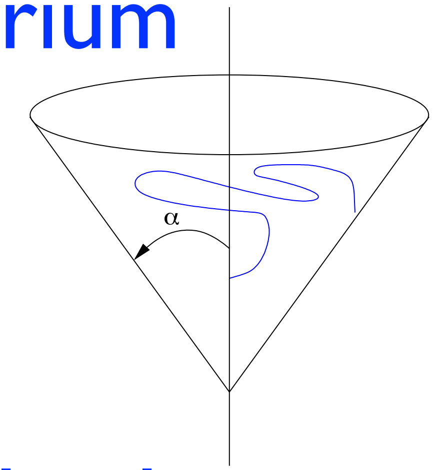


- In general, map N one-dimensional walk onto one walk in N dimension with complex boundary conditions
- There are $\binom{N}{2} = \frac{N(N-1)}{2}$ planes of the type $x_i = x_j$
- These planes divide space into $N!$ "chambers"
- Particle order is unique to each chamber
- The absorbing boundary encloses multiple chambers
- We do not know the shape of the allowed region
- However, we do know the volume of the allowed region
- Equilibrium distribution of particle order

$$V_m = \frac{m}{N}$$

Cone approximation

- Fractional volume of allowed region given by equilibrium distribution of particle order



$$V_m(N) = \frac{m}{N}$$

- Replace allowed region with cone of same fractional volume

$$V(\alpha) = \frac{\int_0^\alpha d\theta\,(\sin\theta)^{N-3}}{\int_0^\pi d\theta\,(\sin\theta)^{N-3}}$$

$$d\Omega \propto \sin^{d-2}\theta\, d\theta$$
$$d = N - 1$$

- Use analytically known exponent for first passage in cone

$$Q^{\gamma}_{2\beta+\gamma}(\cos\alpha) = 0 \qquad N \text{ odd},$$
$$P^{\gamma}_{2\beta+\gamma}(\cos\alpha) = 0 \qquad N \text{ even}.$$

$$\gamma = \frac{N-4}{2}$$

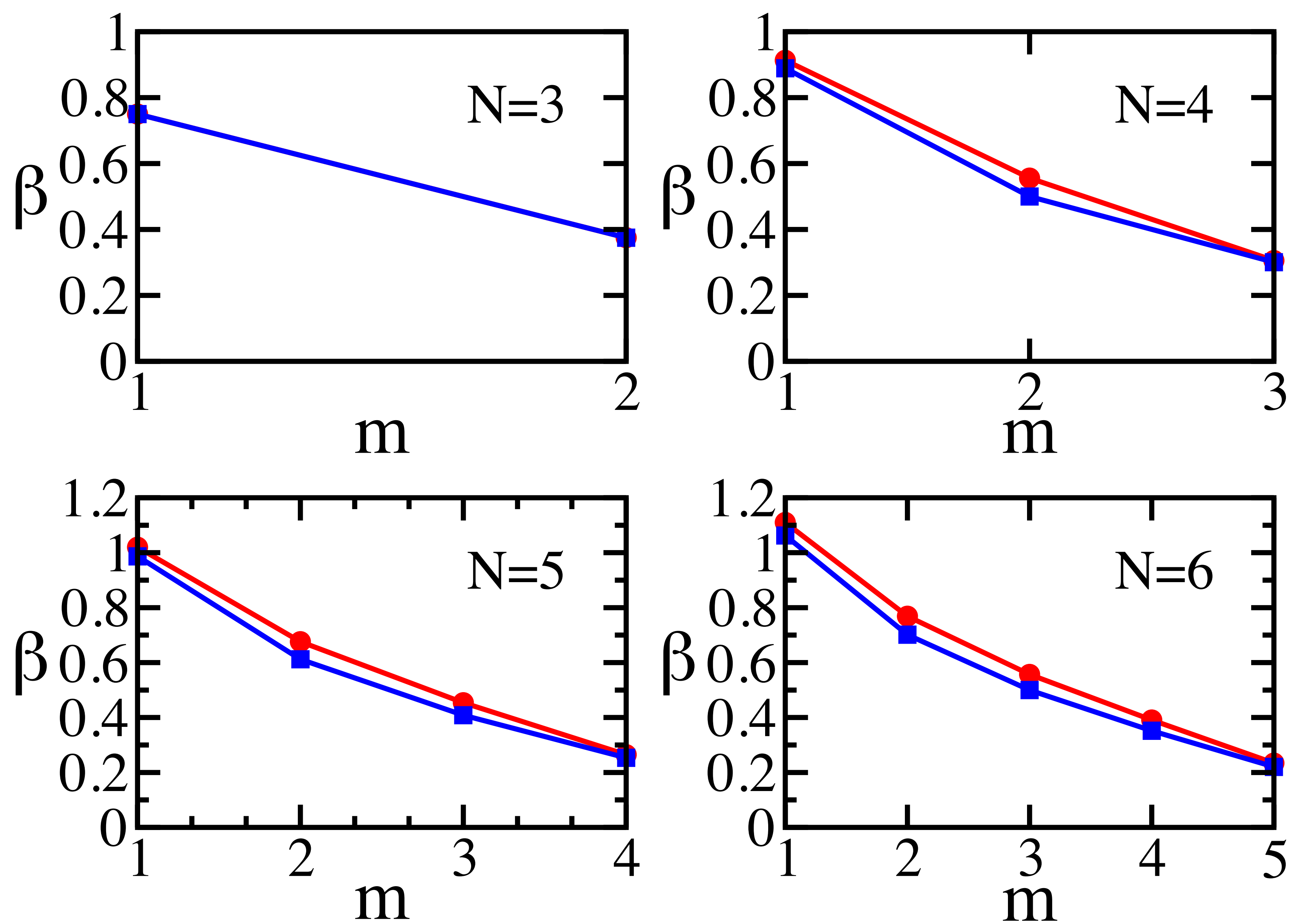
- Good approximation for four particles

m	1	2	3
V_m	$1/4$	$1/2$	$3/4$
β_m^{cone}	0.888644	$1/2$	0.300754
β_m	0.913	0.556	0.306

Small number of particles

- By construction, cone approximation is exact for $N=3$
- Cone approximation gives a formal lower bound

Rayleigh 1877
Faber-Krahn theorem



Excellent, consistent approximation!

Very large number of particles ($N \to \infty$)

- Equilibrium distribution is simple

$$V_m = \frac{m}{N}$$

- Volume of cone is also given by error function

$$V(\alpha, N) \to \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\left(\frac{-y}{\sqrt{2}}\right) \qquad \text{with} \qquad y = (\cos\alpha)\sqrt{N}$$

- First-passage exponent has the scaling form

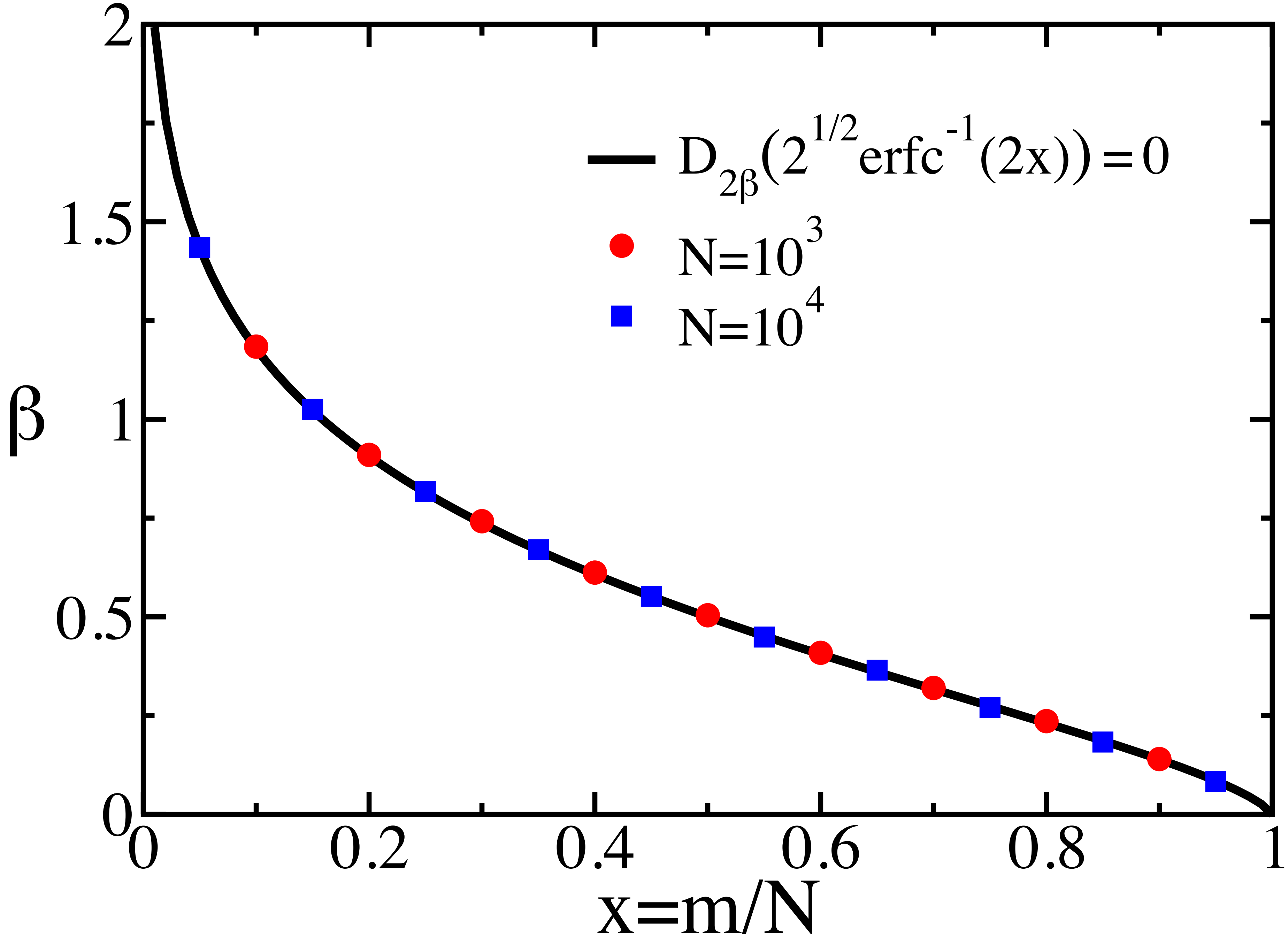
$$\beta_m(N) \to \beta(x) \quad \text{with} \quad x = m/N$$

- Scaling function is root of equation involving parabolic cylinder function

$$D_{2\beta}\left(\sqrt{2}\,\mathrm{erfc}^{-1}(2x)\right) = 0$$

Scaling law for scaling exponents!

Simulation results



Numerical simulation of diffusion in 10,000 dimensions!

Cone approximation is asymptotically exact!

Extreme exponents

- Extremal behavior of first-passage exponents

$$\beta(x) \simeq \begin{cases} \frac{1}{4}\ln\frac{1}{2x} & x \to 0 \\ (1-x)\ln\frac{1}{2(1-x)} & x \to 1 \end{cases}$$

- Probability leader never loses the lead (capture problem)

$$\beta_1 \simeq \frac{1}{4}\ln N$$

- Probability leader never becomes last (laggard problem)

$$\beta_{N-1} \simeq \frac{1}{N}\ln N$$

- Both agree with previous heuristic arguments Krapivsky 02

Extremal exponents can not be measured directly
Indirect measurement via exact scaling function

Small number of particles

N	β_1^{cone}	β_1
3	$3/4$	$3/4$
4	0.888644	0.91
5	0.986694	1.02
6	1.062297	1.11
7	1.123652	1.19
8	1.175189	1.27
9	1.219569	1.33
10	1.258510	1.37

N	$\beta_{N-1}^{\mathrm{cone}}$	β_{N-1}
2	$1/2$	$1/2$
3	$3/8$	$3/8$
4	0.300754	0.306
5	0.253371	0.265
6	0.220490	0.234
7	0.196216	0.212
8	0.177469	0.190
9	0.162496	0.178
10	0.150221	0.165

Decent approximation for the exponents even for small number of particles

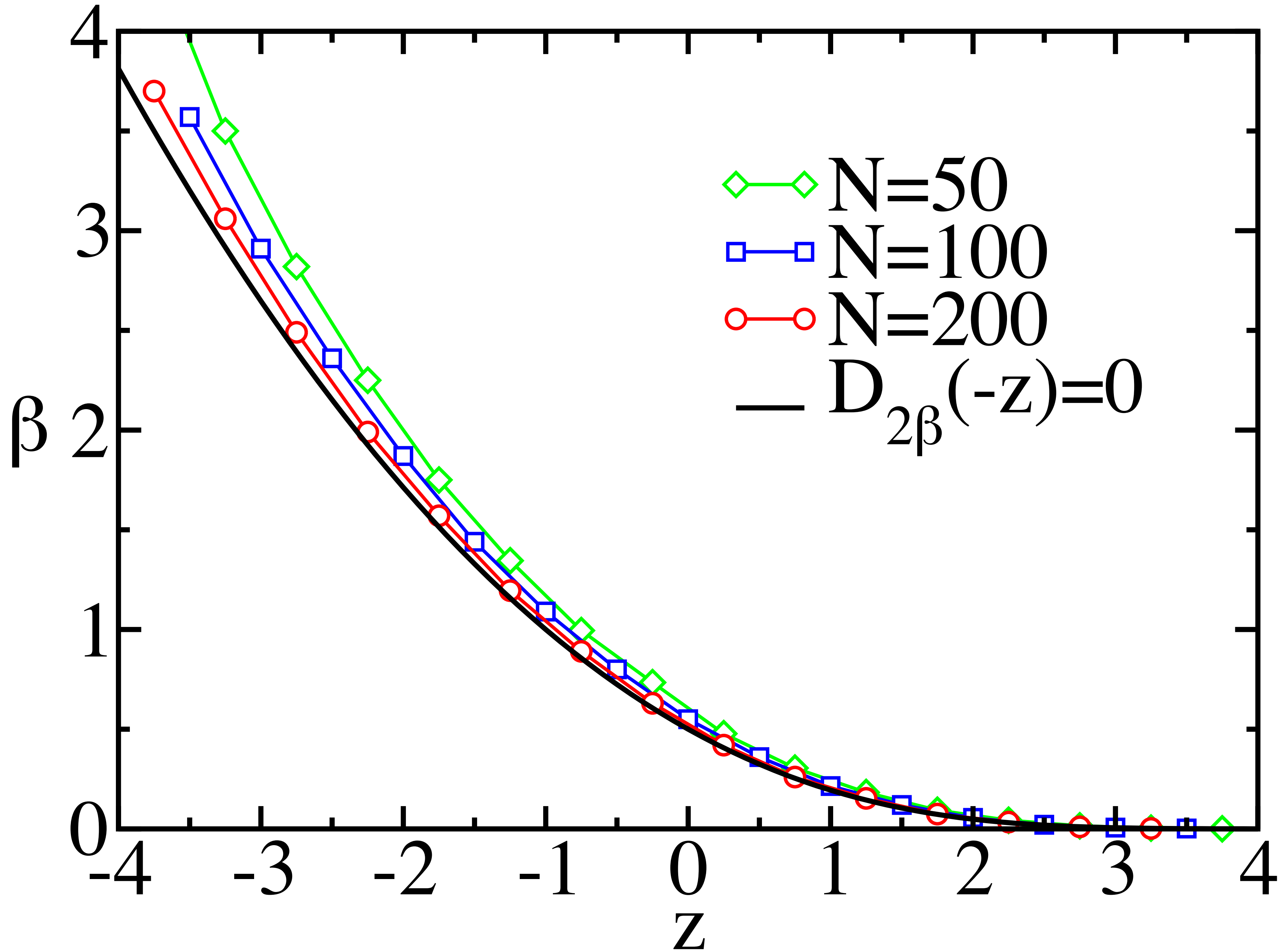
Summary

- First-passage kinetics are rich
- Family of first-passage exponents
- Cone approximation gives good estimates for exponents
- Exponents follow a scaling behavior in high dimensions
- Cone approximation yields the exact scaling function
- Combine equilibrium distribution and geometry to obtain exact or approximate nonequilibrium behavior, namely, first-passage kinetics

Outlook

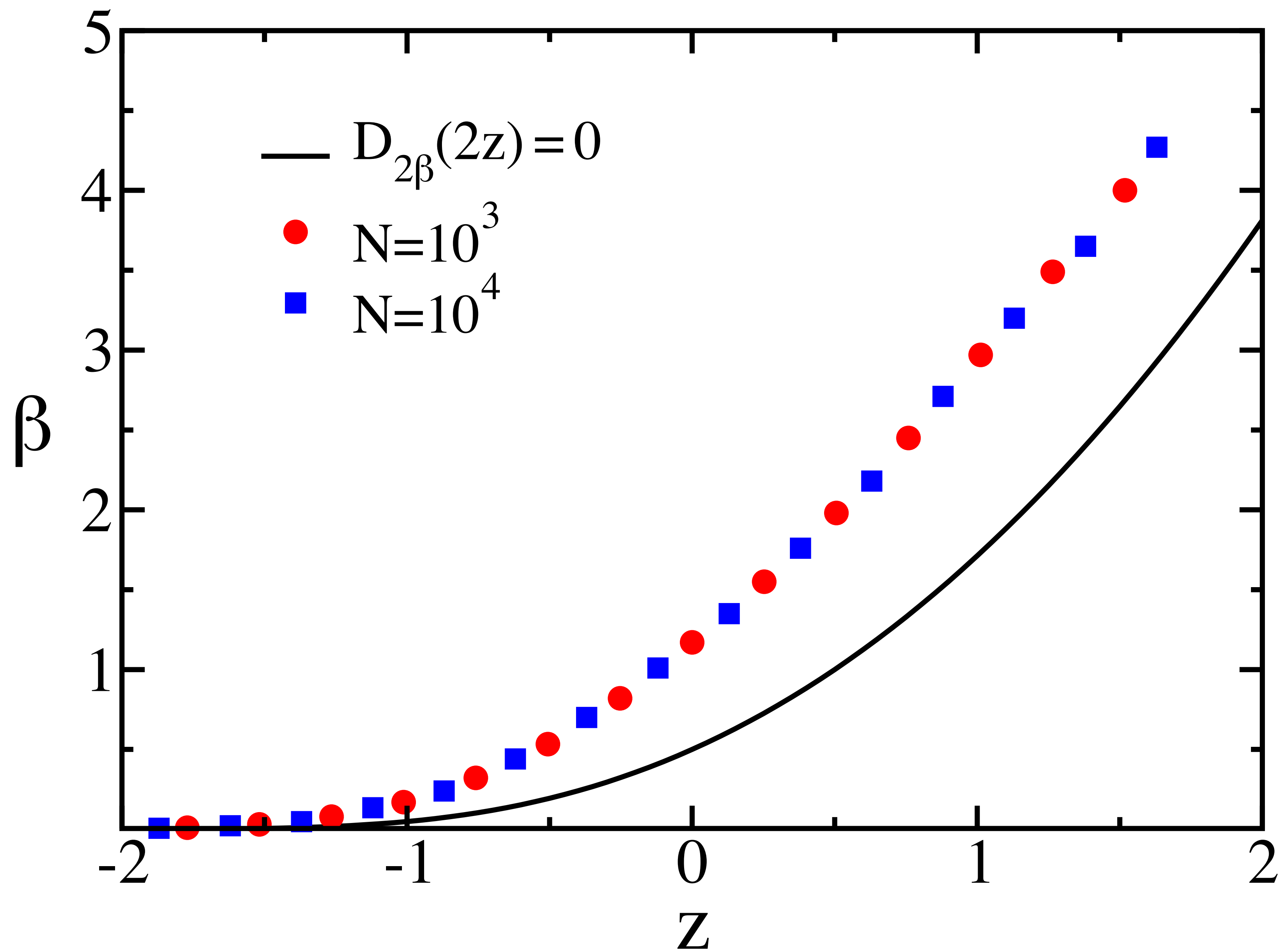
- Heterogeneous Diffusion
- Accelerated Monte Carlo methods
- Scaling occurs in general
- Cone approximation: sometimes exact,
- is not always asymptotically exact
- Geometric proof for exactness
- Limiting shapes in general

Number of pair inversions



Cone approximation is asymptotically exact!

Number of particles avoiding the origin



Counter example: cone is not limiting shape