Front Propagation in the Simplex Algorithm

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T. Antal, D. Ben-Avraham, E. Ben-Naim, and P.L. Krapivsky, J. Phys. A **41**, 465002 (2008)

Talk, paper available from: http://cnls.lanl.gov/~ebn

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A peculiar spin-flip process

t ↓

...01001101...
...01001010...
...00110101...
...00001010...

- **System:** unbounded one-dimensional lattice of spins

$$\sigma_i = 0 \text{ or } 1$$

- **Dynamics:**
  - Each 1-spin flips independently
  - A flip causes all spins to the right to flip as well

$$\sigma_i \to 1 - \sigma_i, \quad \text{for all} \quad i \geq j$$

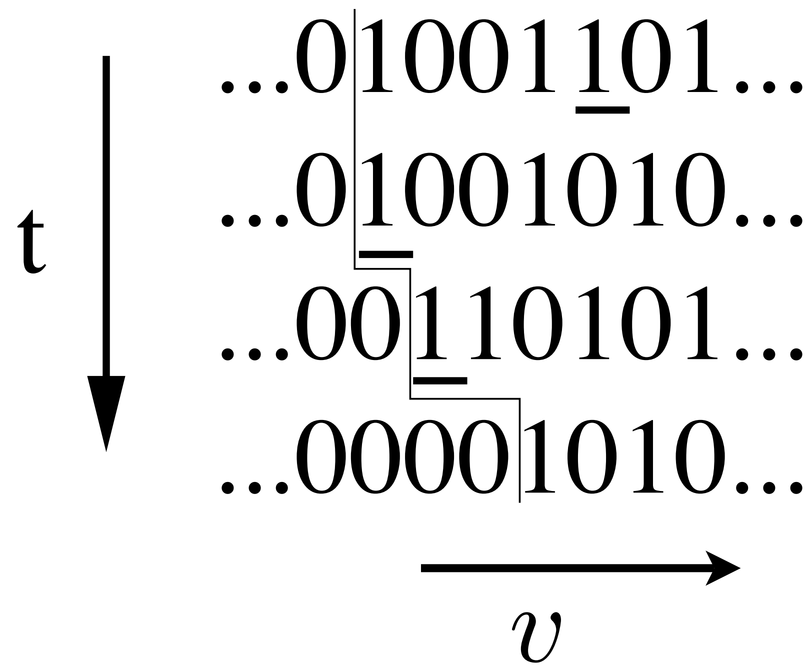
Racz 85

- Every 1-spin affects an infinite number of spins!

How good is the simplex algorithm?

Klee & Minty 72
Pemantle 07

Front propagation



- Stable phase = no 1-spins (...000000...)
- Unstable phase = some 1-spins (...101101...)
- Stable phase propagates into unstable phase
- Front = position of leftmost 1

1. A problem with no parameters!
2. Universal state regardless of initial state

Questions

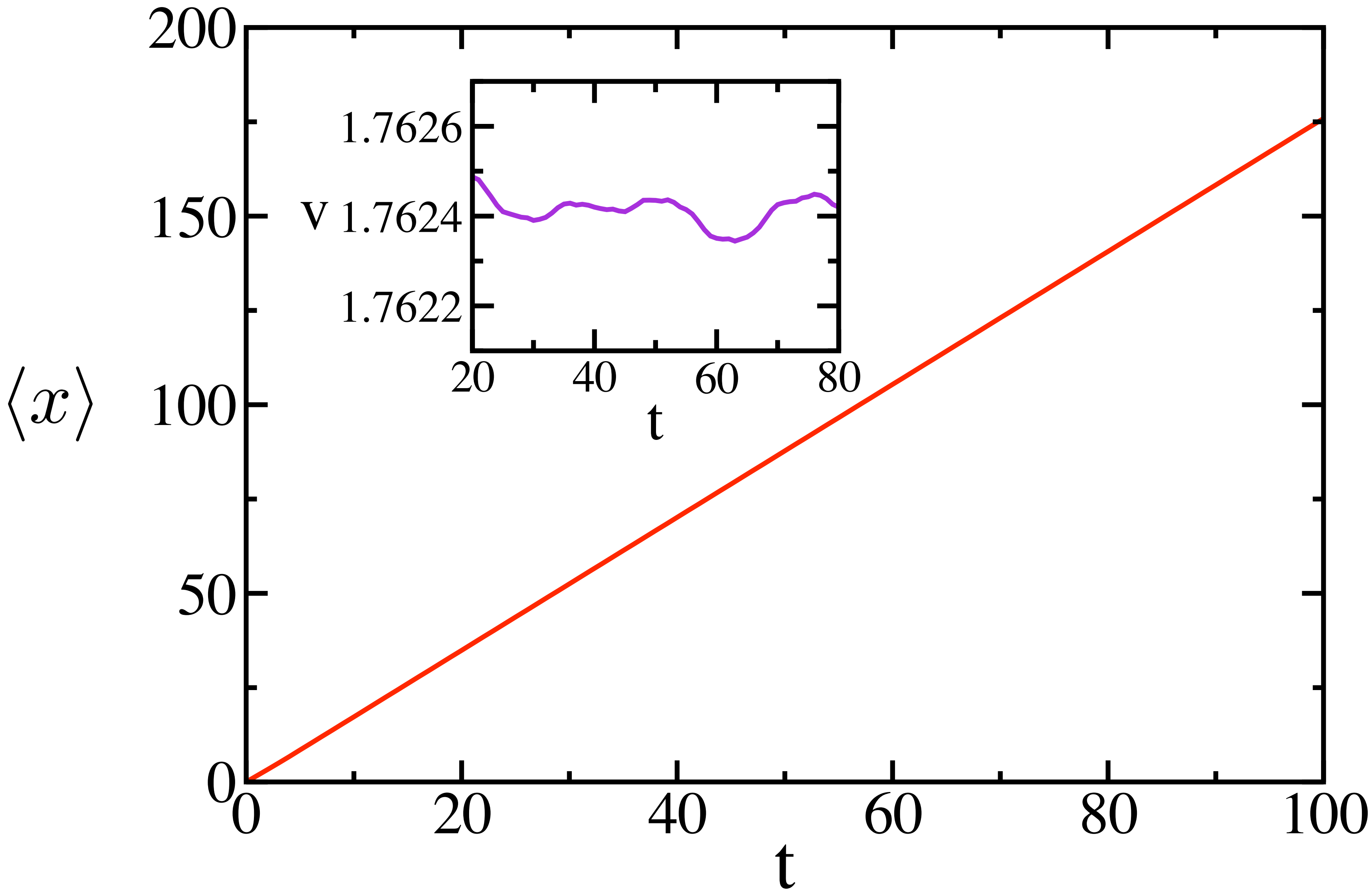
- What is the speed of the front?
- What is the shape of the front?
- What is the spatial structure of the front?
- What is the time evolution of the front?
- What is the speed of the front?

$$v = \,?$$

Monte Carlo simulations

On average, front propagates ballistically

$$v_{\rm MC} = 1.7624 \pm 0.0001$$



A simple observation

- Take the first neighbor to the front
- Lifetime of 0 is double that of 1

$$|1\underline{0} \to X\underline{1} \quad \text{with rate} = 1$$
$$|1\underline{1} \to X\underline{0} \quad \text{with rate} = 2$$

- Twice more likely that first spin is 0

$$\rho_1 \quad = \quad \frac{1}{3} \qquad\qquad \rho_k \equiv \langle \sigma_k \rangle$$

- Similarly, we expect for all k>0

$$\rho_k \quad < \quad \frac{1}{2}$$

Depletion of 1 spins, nonuniform density

Depletion

- Two assumptions
  1. Quasi-static: no evolution in front reference frame $\frac{d\rho_k}{dt} = 0$
  2. Mean-field: no correlations between spins $\langle \sigma_j \sigma_k \rangle \to \langle \sigma_j \rangle \langle \sigma_k \rangle$
- Generalize argument for k=1

$$\frac{d\rho_k}{dt} = \underbrace{(\rho_0 + \rho_1 + \cdots + \rho_{k-1})(1 - \rho_k)}_{0 \to 1} - \underbrace{(1 + \rho_0 + \rho_1 + \cdots + \rho_{k-1})\rho_k}_{1 \to 0}$$

- Recursion relation for "density"

$$\rho_k = \frac{\rho_0 + \rho_1 + \cdots + \rho_{k-1}}{2(\rho_0 + \rho_1 + \cdots + \rho_{k-1}) + 1}$$

- Indeed, there is a depletion of 1 spins

$$\rho_k = 1, \frac{1}{3}, \frac{4}{11}, \frac{56}{145}, \qquad k = 0, 1, 2, 3, \cdots .$$

Depletion of 1 spins, nonuniform density

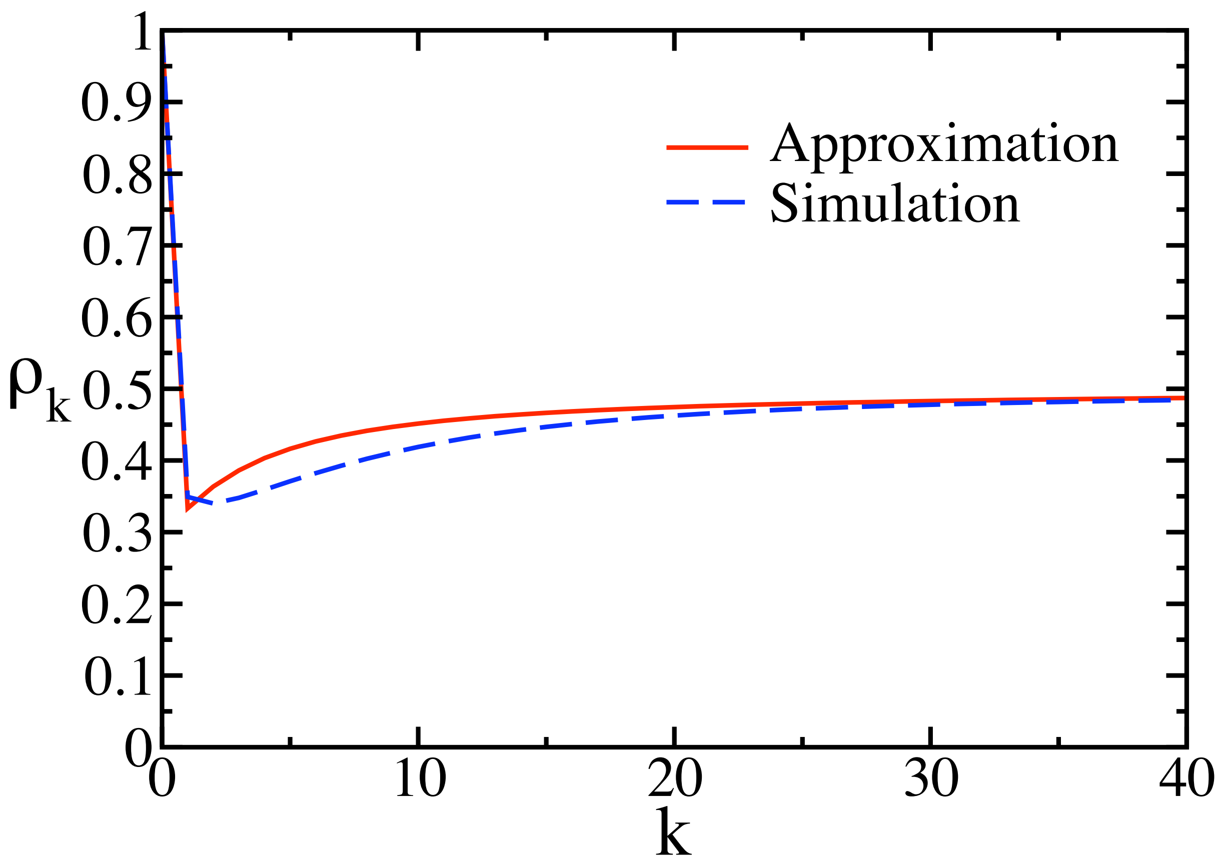
Depletion

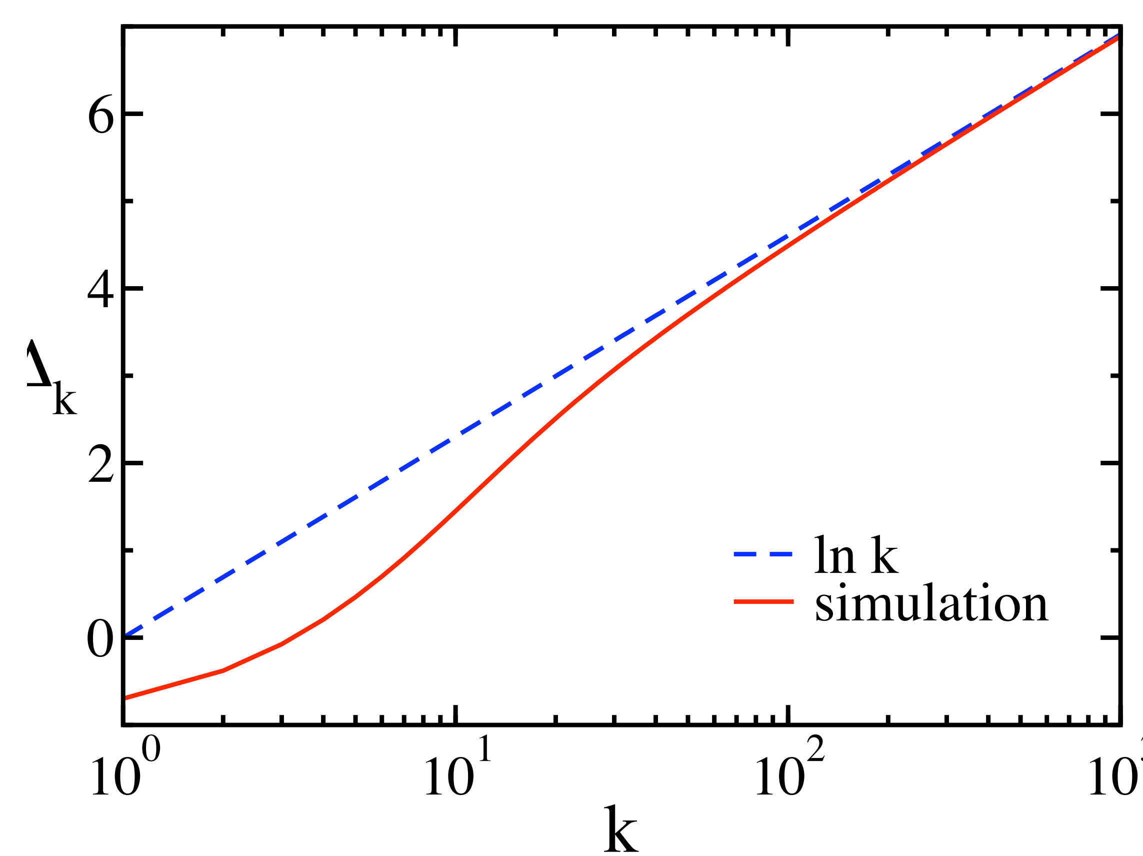
- Depletion penetrates deep into the front

$$\rho_k \simeq \frac{1}{2} - \frac{1}{2k} \qquad k \gg 1$$ asymptotically exact

- Depletion of 0-spins grows logarithmically

$$\Delta_k \simeq \ln k$$ asymptotically exact





Approximation seems to give accurate picture

Velocity and strings

$$\cdots 0000\,|\,\underbrace{11111}_{n}\,0100\cdots \to \cdots 0000\,\underbrace{00000}_{n}\,|\,1011\cdots$$

- Velocity equals the average size of 1 strings $v = \sum_n n(S_n - S_{n-1})$

$$v = \langle n \rangle = \sum_n S_n \qquad S_n \equiv \mathrm{Prob}(\underbrace{11111}_{n})$$

- Bounds for velocity

$$1 \leq v \leq 2$$

- Mean-field: string probability given by product

$$S_n^{\mathrm{MF}} = \rho_1 \rho_2 \cdots \rho_{n-1}$$

- Quasi-static approximation: poor estimate for velocity

$$v_{\mathrm{QSA}} = 1.534070$$

Strong spatial correlations

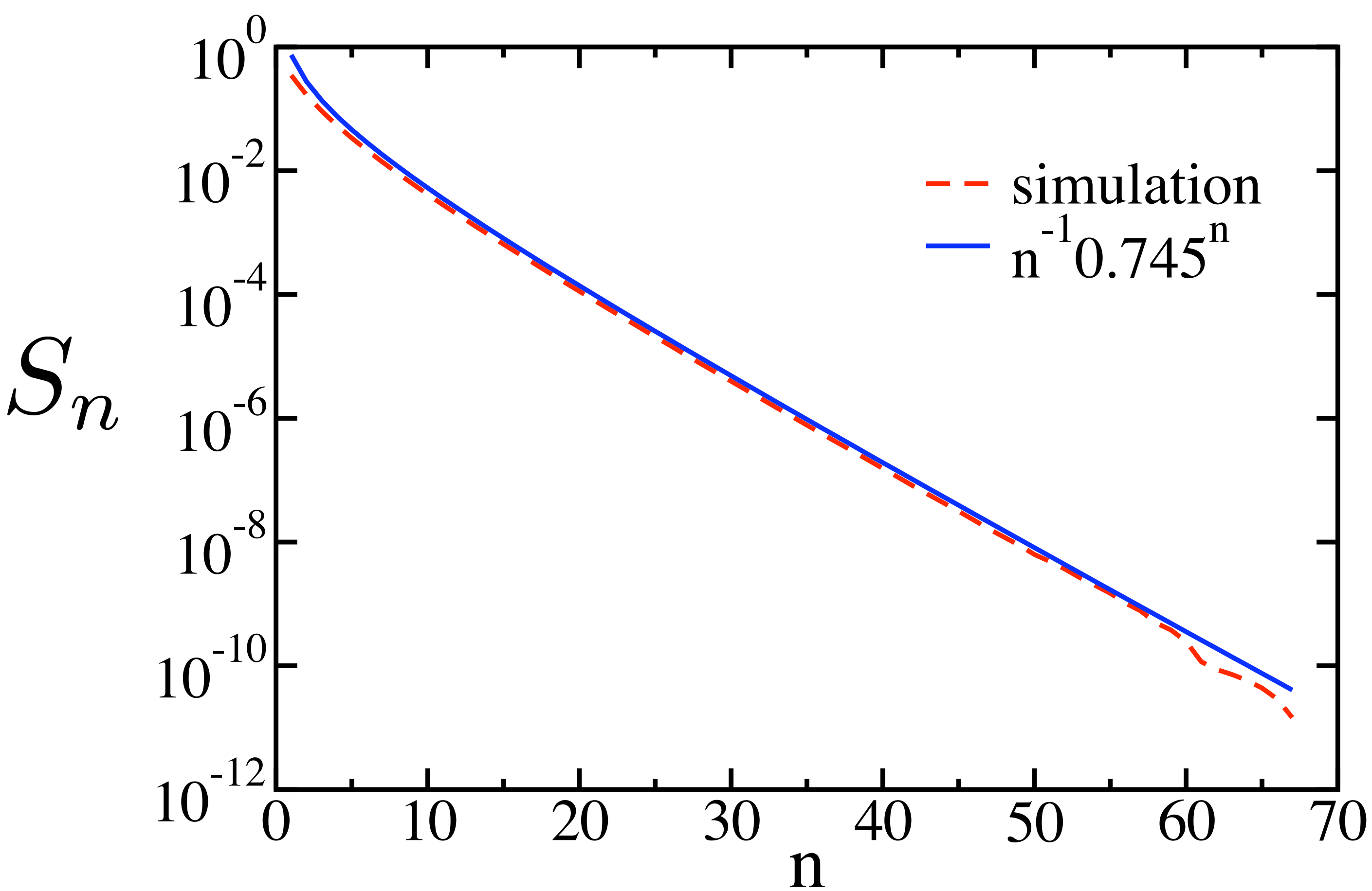
Spatial correlations

- Properly characterized by strings

$$S_n \equiv \mathrm{Prob}(\underbrace{11111}_{n}) \sim n^{-\nu}\lambda^n$$

- Much more likely than Mean-Field suggests

$$\lambda_{\mathrm{MC}} = 0.745 \qquad \lambda_{\mathrm{QSA}} = 1/2 \qquad \nu = 1$$



Temporal correlations

- For a renewal process, if n and n' are successive jumps

$$\langle n n' \rangle = \langle n \rangle^2 = v^2$$

- For flipping process, successive jumps anti-correlated

$$\langle n\, n' \rangle < v^2$$

- Define "age"= time since last jump
- Velocity is age-dependent

$$v = \int_0^\infty d\tau\, u(\tau)\, e^{-\tau}$$

- Density is age-dependent

$$c_k(\tau) = \langle \sigma_k(\tau) \rangle \qquad \rho_k = \int_0^\infty d\tau\, c_k(\tau)\, e^{-\tau}$$

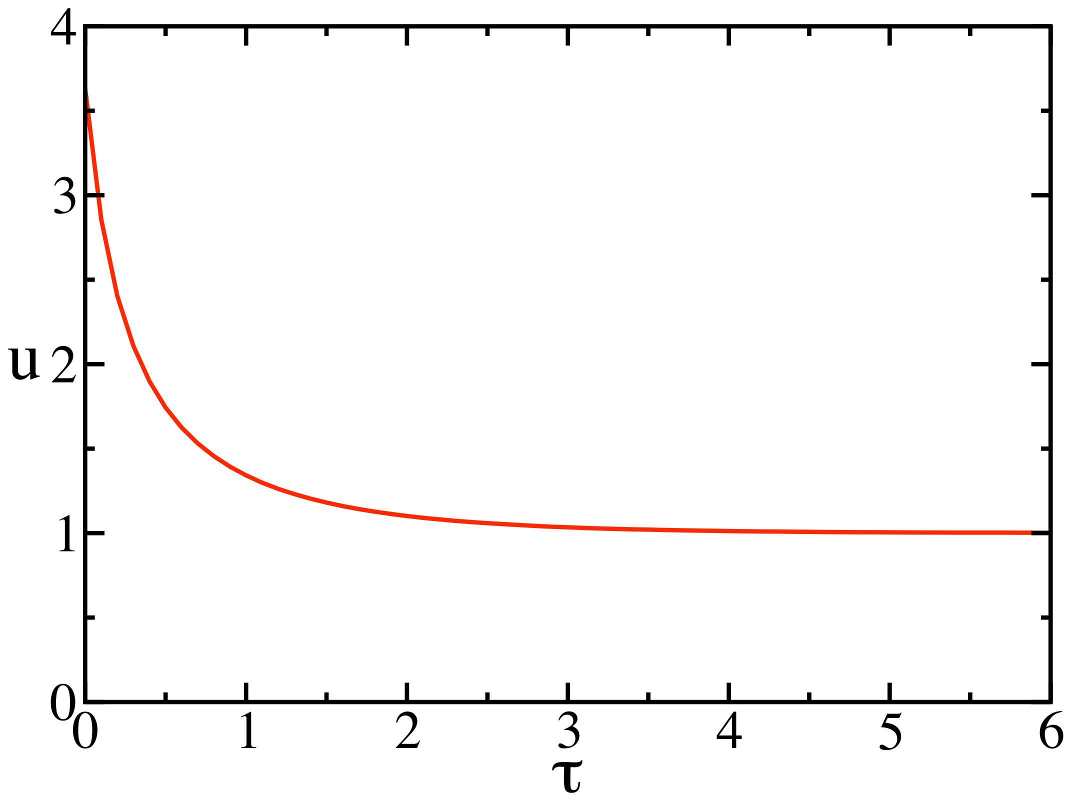
Strong temporal correlations

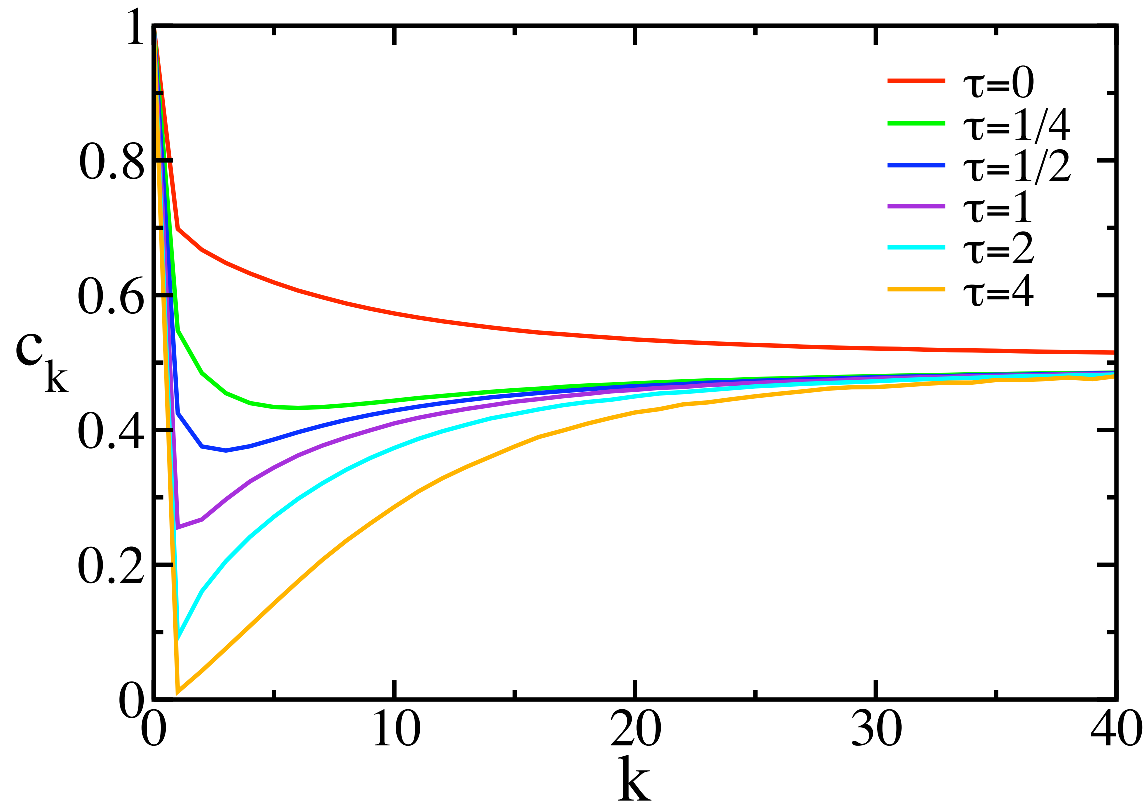
Aging & rejuvenation

- Young fronts are fast, old fronts are slow!
- Rejuvenation: flip re-invigorates slow fronts

$$\cdots\cdots 0|100000\cdots \to \cdots 00|11111\ldots$$

- Shape inversion: new is mirror image of old one





A perpetually repeating life-cycle

Small segments

- *Problem:* infinite hierarchy of equations
- *Solution:* consider small segments of size L
- *Assumption:* complete randomness outside segment
- *Technically:* Approach is exact as $L \to \infty$
- Evolution equation for all possible 2^{L-1} states

$$
\begin{aligned}
\frac{dP_{100}}{dt} &= -P_{100} + \frac{3}{2}P_{101} + \frac{1}{4}P_{110} + \frac{5}{4}P_{111} \\
\frac{dP_{101}}{dt} &= -\frac{3}{2}P_{101} + \frac{5}{4}P_{110} + \frac{1}{4}P_{111} \\
\frac{dP_{110}}{dt} &= \frac{1}{2}P_{100} - \frac{7}{4}P_{110} + \frac{5}{4}P_{111} \\
\frac{dP_{111}}{dt} &= \frac{1}{2}P_{100} + \frac{1}{4}P_{110} - \frac{11}{4}P_{111}.
\end{aligned}
$$

Is this brute-force approach useful?

Shanks transformation

- Obtain velocities from steady-state
- Shanks transformation extrapolates to infinity

$$v_k^{(m+1)} = \frac{v_{k-1}^{(m)} v_{k+1}^{(m)} - v_k^{(m)} v_k^{(m)}}{v_{k-1}^{(m)} + v_{k+1}^{(m)} - 2v_k^{(m)}}$$

- Fast convergence for exponential corrections

k	$v_k^{(0)}$	$v_k^{(1)}$	$v_k^{(2)}$	$v_k^{(3)}$	$v_k^{(4)}$
2	1.500000				
3	1.535714	1.418947			
4	1.587165	1.826205	1.779225		
5	1.629503	1.773099	1.765862	1.764458	
6	1.662201	1.766730	1.764592	1.758245	1.762322
7	1.687108	1.765129	1.763533	1.770104	1.765175
8	1.705987	1.764330	1.762272	1.761669	
9	1.720251	1.763754	1.761864		
10	1.730993	1.763313			
11	1.739055				

- Good estimate for the velocity

$$v_{\text{shanks}} = 1.76 \pm 0.01$$

Pinned fronts

$$\begin{array}{c|l} 0 & 10\underline{1}001 \\ 0 & \underline{1}00110 \\ 0 & 11100\underline{1} \\ 0 & 111000 \end{array}$$

- Small modification: fixed one spin to 1

$$\sigma_0 = 1$$

- Front does not move, but we can still calculate v!
- Provides excellent approximation: velocity within 1%

$$v_{\text{pinned}} = 1.7753 \pm 0.0001$$

- Quasi-static description becomes exact
- Small segment approach exact for all segment lengths
- Some exact results for correlation functions

$$\langle \sigma_k \sigma_{k+1} \rangle = \frac{1}{2} \langle \sigma_{k+1} \rangle$$

- Correlations decay slowly

$$\langle \sigma_k \sigma_{k+1} \rangle - \langle \sigma_k \rangle \langle \sigma_{k+1} \rangle \simeq (4k)^{-1}$$

Small segments

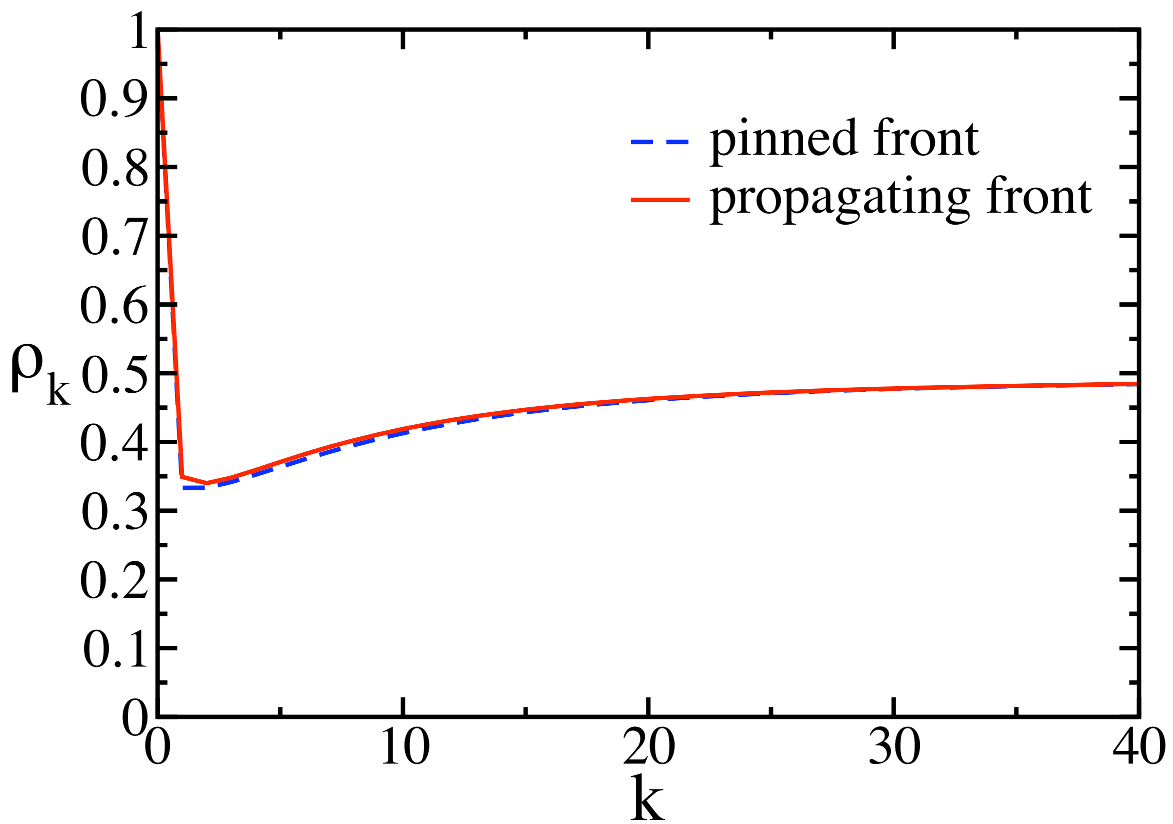
- Now, small segment results are exact
- Shanks transformation converges rapidly and gives impressive estimates

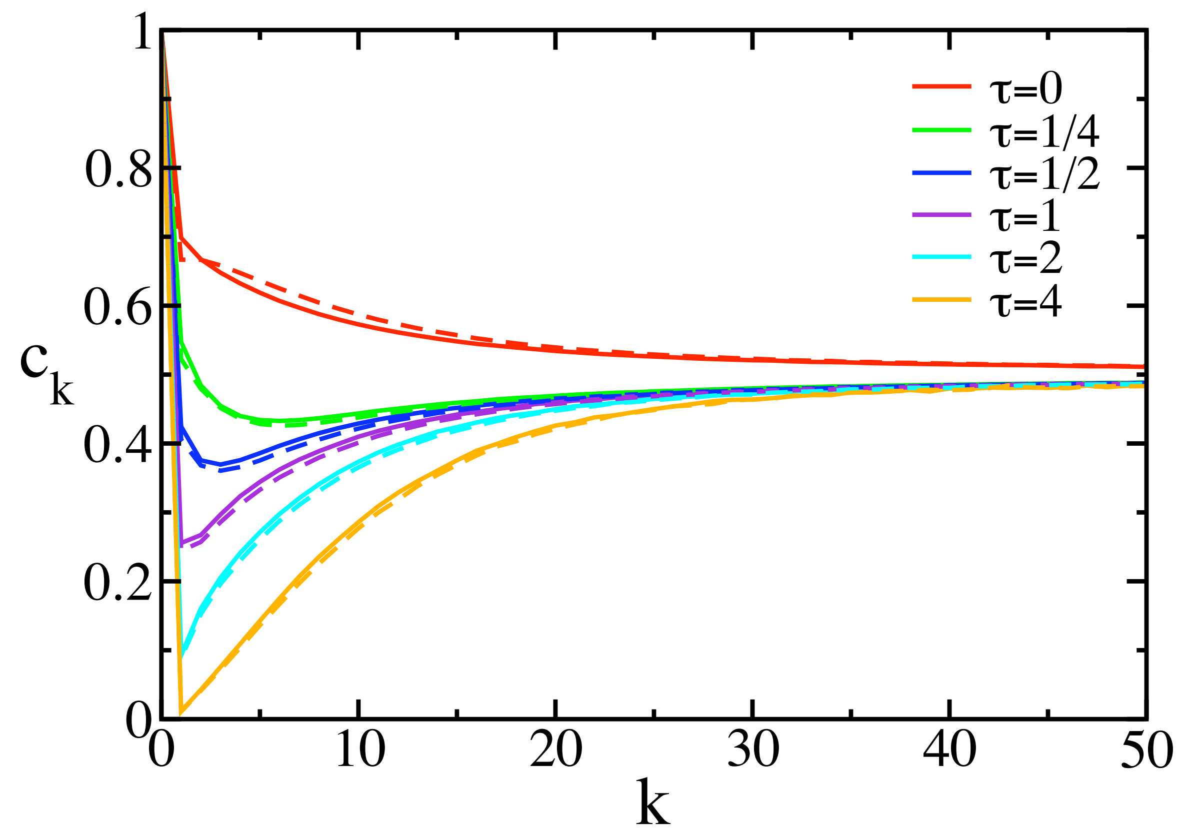
k	$v_k^{(0)}$	$v_k^{(1)}$	$v_k^{(2)}$	$v_k^{(3)}$	$v_k^{(4)}$
1	1.				
2	1.333333	1.666666			
3	1.5	1.72549	1.769737		
4	1.595833	1.750742	1.773156	1.775020	
5	1.655039	1.762616	1.774362	1.775178	1.775278
6	1.693228	1.768521	1.774849	1.775239	1.775289
7	1.718565	1.771576	1.775065	1.775267	1.775293
8	1.735709	1.773205	1.775170	1.775280	1.775293
9	1.747473	1.774095	1.775223	1.775287	
10	1.755632	1.774593	1.775252		
11	1.761337	1.774876			
12	1.765350				

- Perfect estimate for the velocity

$$\begin{aligned} v_{\text{shanks}} &= 1.7753 \pm 0.0001 \\ v_{\text{MC}} &= 1.7753 \pm 0.0001 \end{aligned}$$

Depletion & aging





Age-dependent densities can now be calculated

$$
\begin{aligned}
c_1(\tau) &= \frac{2}{3}e^{-\tau} \\
c_2(\tau) &= \frac{1}{3}(2\tau - 1)e^{-\tau} + e^{-2\tau}
\end{aligned}
$$

Pinned fronts capture all the physics, provide excellent approximation

Summary

- Analysis in a reference frame with the front is useful
- All the hallmarks of nonequilirbium physics
    - Depletion
    - Strong spatial and temporal correlations
    - Aging and rejuvenation
- Mean-field theory explains depletion
- Small segment analysis + extrapolation provides good estimate for velocity
- Pinning the fronts provides excellent approximation and reproduces all qualitative features

Outlook

Exact analytical solution for the velocity remains an open question, requires exact closure