Extinction and Survival in Two-Species Annihilation

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Talk, publications available from: http://cnls.lanl.gov/~ebn

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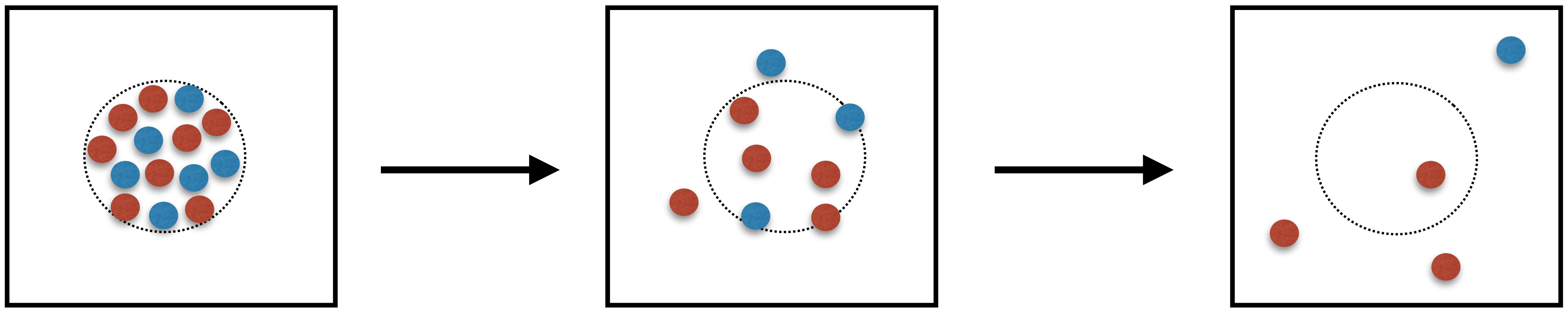
Plan

Diffusion-controlled two-species annihilation

with finite number of particles

1. Equal populations
2. Fixed number difference
3. Equal concentrations

Diffusion-controlled two-species annihilation with finite number of particles



Zeldovich 88
Wilczek 83
Bramson 91
Leyvraz 92

- Initial condition: uniform density in compact domain
- Number of majority & minority particles is N_+ & N_-
- Total number of particles

$$N = N_+ + N_-$$

- Number difference is a conserved quantity

$$\Delta = N_+ - N_-$$

Main result (three dimensions)

- Average number of surviving majority particles is M_+
- Average number of surviving minority particles is M_-
- Conservation law implies majority never goes extinct

$$M_+ - M_- = \Delta$$

- Equal populations

$$M_+ \sim M_- \sim N^{1/3}$$

- Equal concentrations

$$M_+ \sim N^{1/2} \quad \text{and} \quad M_- \sim N^{1/6}$$

Sufficiently small dimensions: extinction

- Probability a random walk returns to origin

$$P = 1 \quad \text{when} \quad d \leq 2$$

- Separation between two random walks itself performs a random walk

- Two diffusing particles are guaranteed to meet

All minority particles eventually disappear

Above critical dimension: survival feasible

- Probability a random walk at distance r returns to origin

$$P \sim r^{-(d-2)} \quad \text{when} \quad d > 2$$

- Two diffusing particles may or may not meet

Uniform-density approximation

- Concentrations obey reaction-diffusion equation

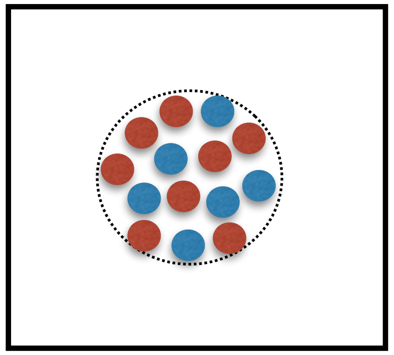
$$\frac{\partial c_-(\mathbf{r},t)}{\partial t} = D\nabla^2 c_-(\mathbf{r},t) - Kc_-(\mathbf{r},t)c_+(\mathbf{r},t)$$

- Dimensionless form $D = K = a = 1$

- Total number of particles obeys rate equation

$$n_-(t) = \int d\mathbf{r}\, c_-(\mathbf{r},t) \quad \Longrightarrow \frac{dn_-}{dt} = -\int d\mathbf{r}\, c_-(\mathbf{r},t)\, c_+(\mathbf{r},t)$$

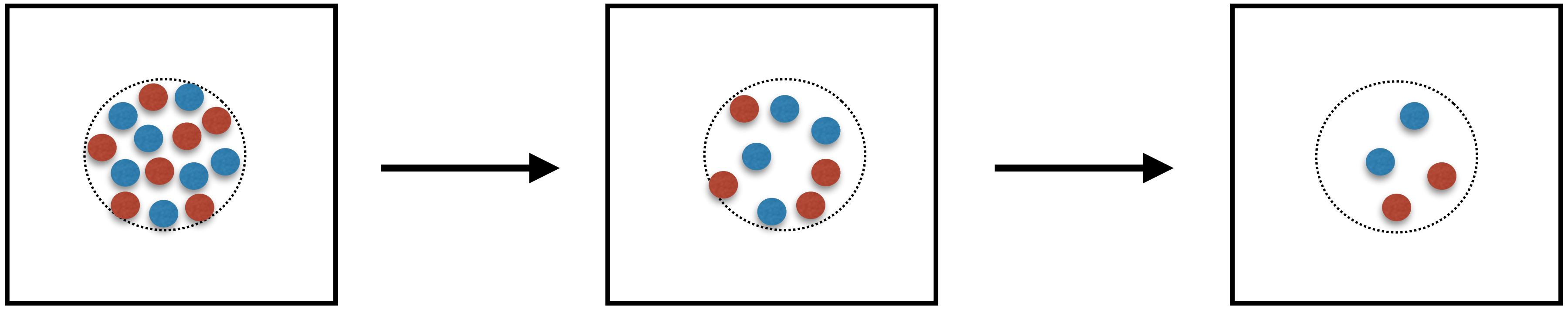
- Two major simplifying assumptions
  1. Particles confined to volume V
  2. Spatial distribution remains uniform



- Closed equation for number of remaining particles

$$\frac{dn_-}{dt} = -\frac{n_- n_+}{V}$$

Equal populations ($\Delta = 0$)



- Particles still inside initial-occupied domain

$$n_+ = n_- = n/2 \quad \& \quad V \sim N \qquad \Longrightarrow \qquad \frac{dn}{dt} = -\frac{n^2}{N}$$

- Mean-field like decay

$$n(t) \sim N\, t^{-1}$$

- Valid until particles exit initially-occupied domain

$$\ell^{3/2} \sim t^{3/2} \sim N \qquad \Longrightarrow \qquad T \sim N^{2/3}$$

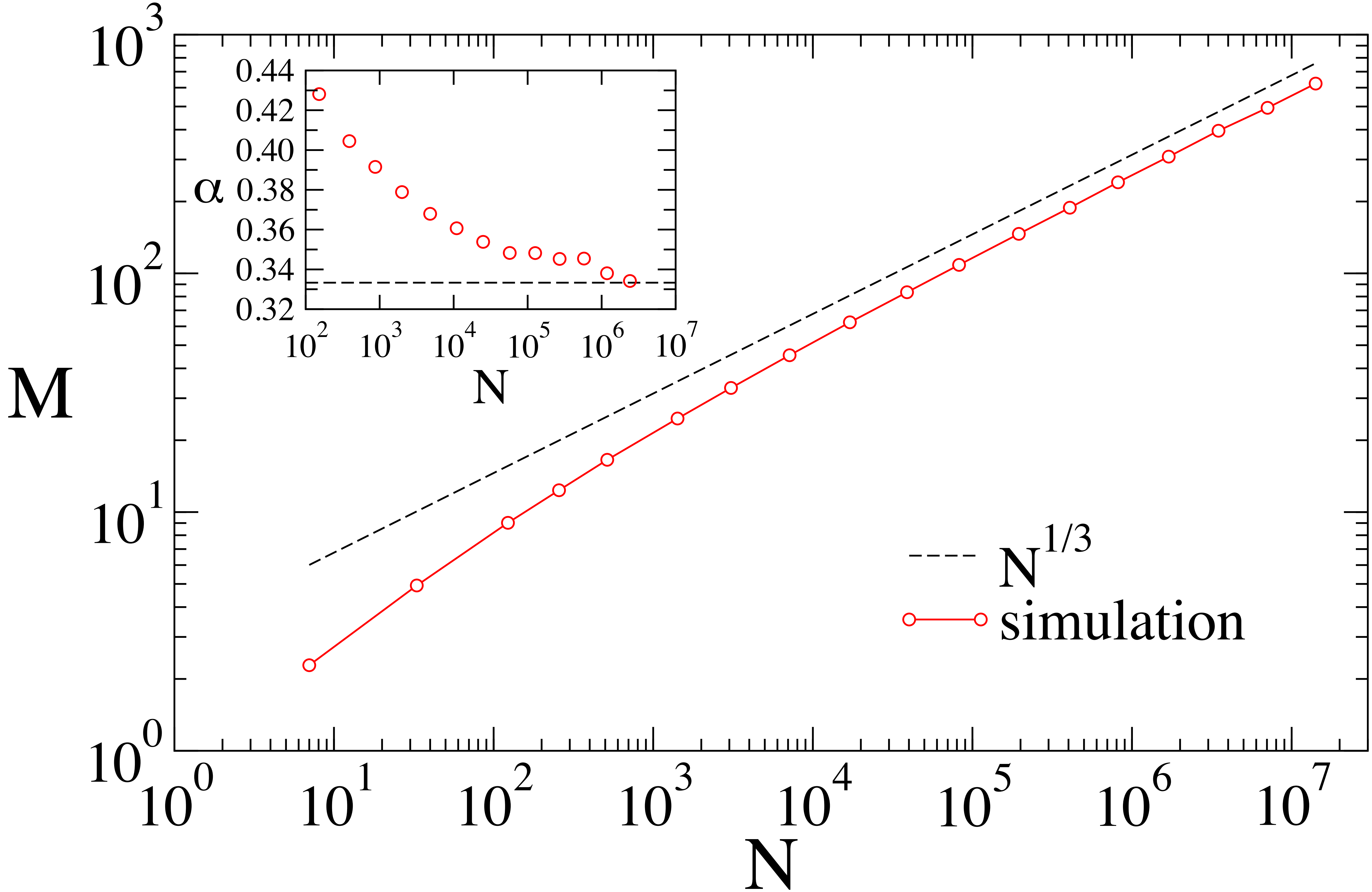
- Diffusion time scale gives number of particles

$$n(T) \sim N^{1/3}$$

EB, Krapivsky 2016

Numerical simulations I

equal populations ($\Delta = 0$)



Fixed number difference ($\Delta \neq 0$)

- Rate equation

$$\frac{dn_-}{dt} = -\frac{n_-(n_- + \Delta)}{N}$$

- Average number of minority particles

$$n_-(t) = N_- \, \frac{\Delta}{N_-(e^{t\Delta/N} - 1) + \Delta}$$

- Average number of surviving minority particles

$$M_- \sim n_-(T) \quad \Longrightarrow \quad M_- \sim N_- \, \frac{\Delta}{N_-(e^{\Delta/N^{1/3}} - 1) + \Delta}$$

- Emergence of critical difference

$$M_- \sim \begin{cases} N^{1/3} & \Delta \ll N^{1/3} \\ \Delta \, \exp\left(-c\,\Delta/N^{1/3}\right) & \Delta \gg N^{1/3} \end{cases}$$

Transition from extinction to survival

Finite-size scaling

- Number of surviving particles

$$M_- \sim N_- \frac{\Delta}{N_-(e^{\Delta/N^{1/3}} - 1) + \Delta}$$

- Scaling laws for surviving number and critical number difference

$$M_- \sim N^{1/3} \quad \text{and} \quad \Delta_c \sim N^{1/3}$$

- Universal scaling form for number of surviving particles

$$M_-/N^{1/3} = G\left(\Delta/N^{1/3}\right)$$

- Scaling function

$$G(x) = \frac{x}{e^{c\,x} - 1}$$

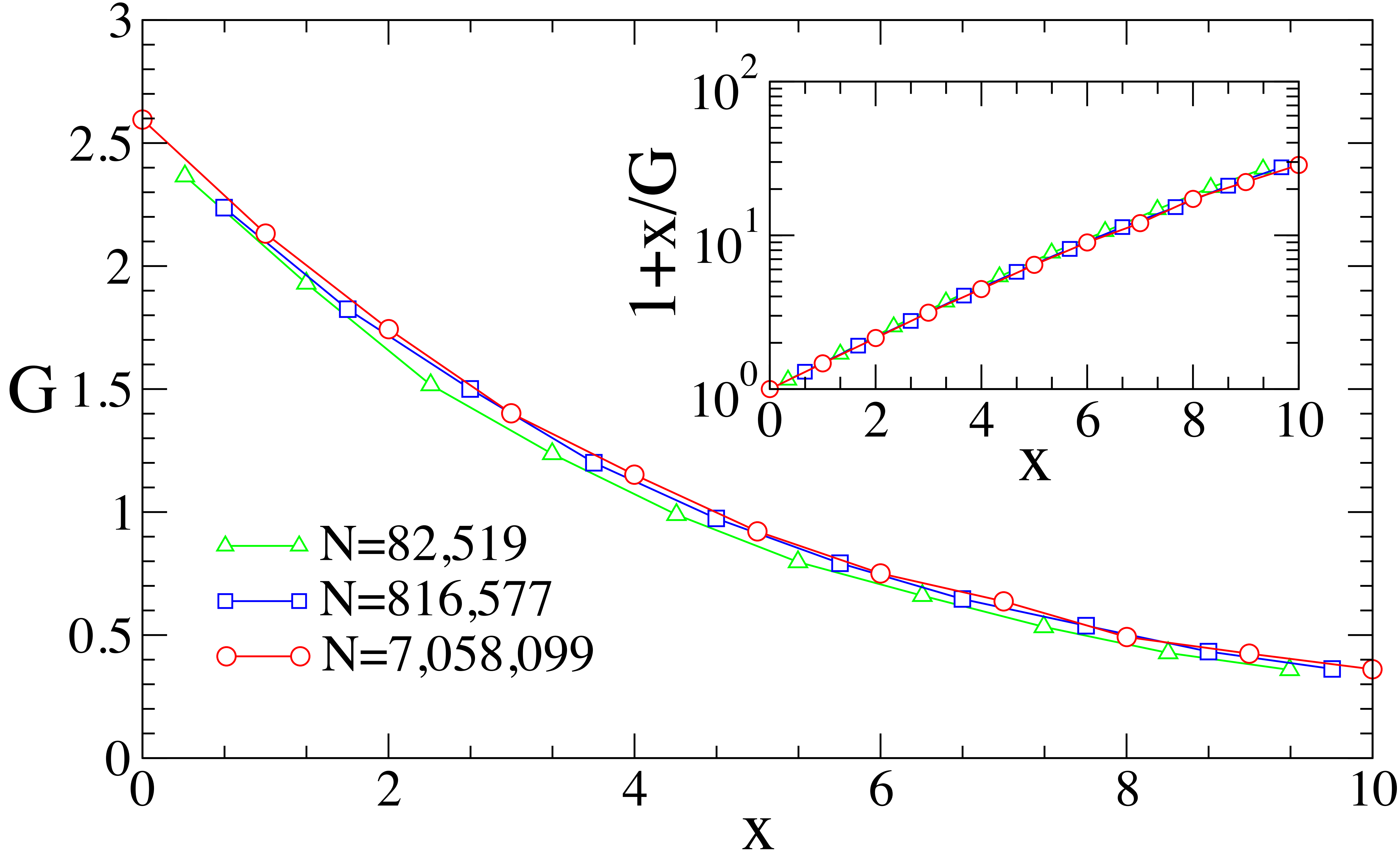
- Two regimes of behavior

$$G(x) \sim \begin{cases} 1 & x \ll 1 \qquad \text{survival} \\ x\,e^{-c\,x} & x \gg 1 \qquad \text{extinction} \end{cases}$$

Critical difference fully characterizes the behavior

Numerical simulations II

finite-size scaling



The critical difference

- There is a critical number difference

$$\Delta_c \sim N^{1/3}$$

- Subcritical difference: minority species survives

$$M_- \sim N^{1/3} \quad \text{when} \quad \Delta \ll \Delta_c$$

- Supercritical difference: minority species becomes extinct

$$M_- \sim \Delta \exp(-c\Delta/N^{1/3}) \quad \text{when} \quad \Delta \gg \Delta_c$$

- In particular, for typical difference (equal concentrations)

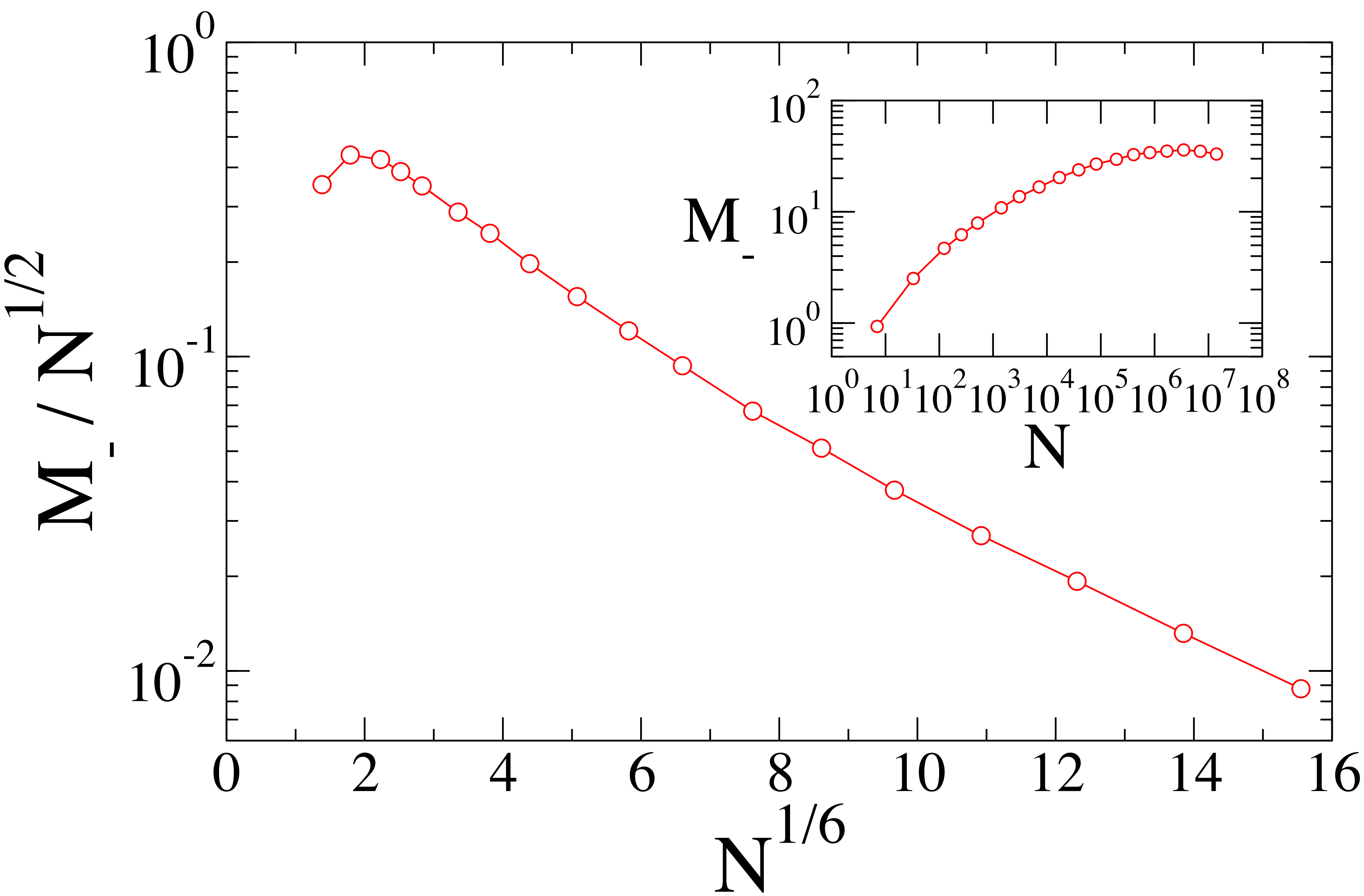
$$M_- \sim N^{1/2} \exp(-cN^{1/6}) \quad \text{when} \quad \Delta = bN^{1/2}$$

Number difference controls the behavior

Numerical simulations III

Typical difference ($\Delta = N^{1/2}$)

$$M_{-}/N^{1/2} \sim \exp(-cN^{1/6})$$



Theory “rescues” simulations!

Equal concentrations

- Massive imbalance, surviving majority population is large

$$\Delta \sim N^{1/2} \quad \Longrightarrow \quad M_+ \sim N^{1/2}$$

- Number difference is normally distributed

$$P(\Delta) = (2\pi N)^{-1/2} \exp[-\Delta^2/(2N)] \quad \to \quad \begin{cases} N^{-1/2} & \Delta < N^{1/2} \\ 0 & \Delta > N^{1/2} \end{cases}$$

- Minority survives with tiny probability

$$\Delta < \Delta_c \quad \text{with probability} \quad N^{-1/2} \times N^{1/3} \sim N^{-1/6}$$

- Minority goes extinct otherwise

$$\Delta > \Delta_c \quad \text{with probability} \quad 1 - N^{-1/6}$$

- Average number of surviving minority particles

$$M_- \sim N^{-1/6} \times N^{1/3} \sim N^{1/6}$$

Lack of self-averaging, huge fluctuations
Two distinct scaling laws for majority and minority

Equal concentrations

- Number difference is normally distributed

$$P(\Delta) = \left(\frac{1}{2\pi N}\right)^{1/2} \exp\left(-\frac{\Delta^2}{2N}\right)$$

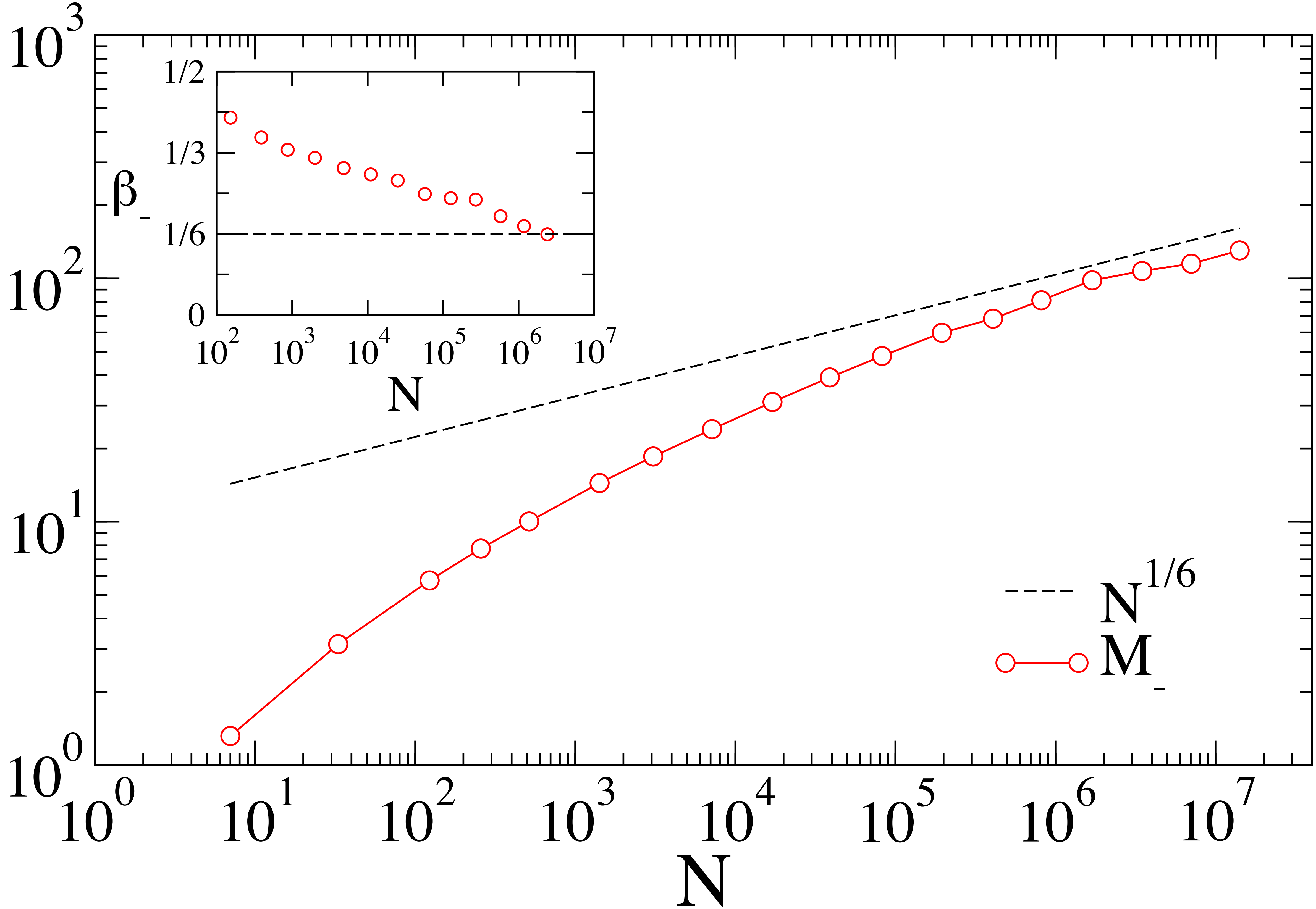
- Separate supercritical and subcritical contributions

$$M_- \sim \int_0^{N^{1/3}} d\Delta \left(\frac{1}{2\pi N}\right)^{1/2} \exp\left(-\frac{\Delta^2}{2N}\right) \times N^{1/3}$$
$$+ \int_{N^{1/3}}^{\infty} d\Delta \left(\frac{1}{2\pi N}\right)^{1/2} \exp\left(-\frac{\Delta^2}{2N}\right) \times \Delta \exp\left(-\frac{\Delta}{N^{1/3}}\right)$$

- System is almost always supercritical (extinction)
- Rare subcritical cases dominate the behavior (survival)

Numerical simulations IV

equal concentrations



General spatial dimensions

- Critical difference, one-tier scaling law

$$\Delta_c \sim N^{\delta} \qquad \delta = \frac{d-2}{d}$$

- Majority population, two-tier scaling law

$$M_+ \sim N^{\beta_+} \qquad \beta_+ = \begin{cases} \frac{1}{2} & d \leq 4 \\ \frac{d-2}{d} & 4 \leq d \end{cases}$$

- Minority population, three-tier scaling law

$$M_- \sim N^{\beta_-} \qquad \beta_- = \begin{cases} 0 & d \leq \frac{8}{3} \\ \frac{3d-8}{2d} & \frac{8}{3} \leq d \leq 4 \\ \frac{d-2}{d} & 4 \leq d \end{cases}$$

Surviving minority population does not grow with N when $d<8/3$

Conclusions

- Diffusion-controlled two-species annihilation, starting with finite number of particles
- Finite number of particles escape annihilation
- Number difference controls the behavior
- Subcritical phase: minority species survives
- Supercritical phase: minority species goes extinct
- Equal concentrations: two distinct scaling laws for minority and majority populations
- Opposite to infinite systems: survival probability is enhanced as the dimension increases
- Exact analytical methods to treat finite number of particles