The Inelastic Maxwell Model

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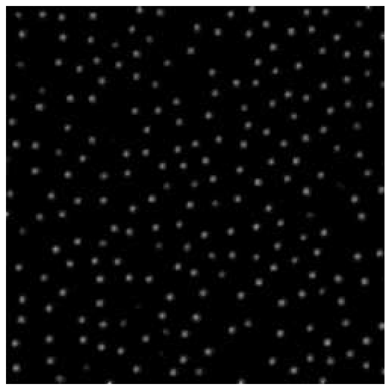
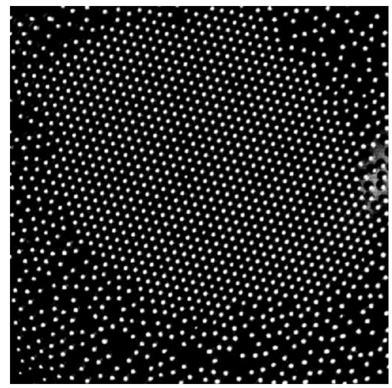
I Motivation: Granular Gases
II Unforced Inelastic Gases
III Forced Inelastic Gases
IV Mixing Processes
V Opinion Dynamics

E. Ben-Naim and P. L. Krapivsky
Lecture Notes in Physics, *cond-mat/0301238*.

Experiments: Granular Gases

- Vibration: vertical, horizontal, electrostatic
  Gollub, Swinney, Menon, Aronson, Kudrolli, Urbach

- non-Maxwellian velocity statistics

$$P(v) \sim \exp\left(-|v|^{\alpha}\right) \qquad 1 < \alpha < 1.5$$

- Clustering, density inhomogeneities

- Collective phenomena: phase transitions, pattern formation, shocks

A nonequilibrium gas

Characteristics

- Hard sphere (contact) interactions
- Dissipative (inelastic) collisions

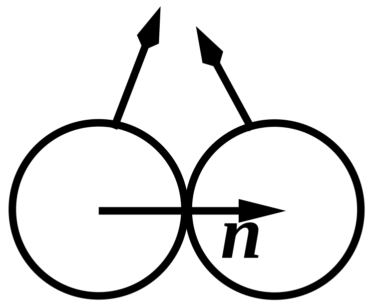
Consequences of energy dissipation

- No energy equipartition
- No ergodicity
- Strong velocity correlations

Challenges

- Kinetic Theory: distributions
- Hydrodynamics: averages
- Sharp validity criteria are missing

Inelastic Collisions



- Momentum is conserved

$$\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{u}_1 + \mathbf{u}_2$$

- Normal component reduced by $r = 1 - 2\epsilon$

$$(\mathbf{v}_1 - \mathbf{v}_2) \cdot \mathbf{n} = -r(\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{n}$$

- Collision rule $\mathbf{g} = \mathbf{u}_1 - \mathbf{u}_2$

$$\mathbf{v}_{1,2} = \mathbf{u}_{1,2} \mp (1 - \epsilon)\,(\mathbf{g} \cdot \mathbf{n})\,\mathbf{n}$$

- Energy dissipation

$$\Delta E = -\epsilon(1 - \epsilon)(\mathbf{g} \cdot \mathbf{n})^2$$

The Elastic Maxwell Model

J.C. Maxwell, Phil. Tran. Roy. Soc **157**, 49 (1867)

- Infinite particle system
- Binary collisions
- Random collision partners
- Random impact directions $\mathbf{n}$
- Elastic collisions $(\mathbf{g} = \mathbf{v_1} - \mathbf{v_2})$

$$\mathbf{v_1} \to \mathbf{v_1} - \mathbf{g} \cdot \mathbf{n}\,\mathbf{n}$$

- Mean-field collision process
- Purely Maxwellian velocity distributions

$$P(\mathbf{v}) = \frac{1}{(2\pi T)^{d/2}} \exp\left(-\frac{v^2}{2T}\right)$$

What about inelastic, dissipative collisions?

Maxwellian Distributions

1. Velocity distribution is isotropic

$$P(\mathbf{v}) = P(|\mathbf{v}|)$$

2. Velocity correlations are absent

$$P(v_x, v_y, v_z) = P(v_x)P(v_y)P(v_z)$$

Only possibility: Maxwellian Distribution

$$P(\mathbf{v}) = \frac{1}{(2\pi T)^{d/2}} \exp\left[-\frac{|\mathbf{v}|^2}{2T}\right]$$

For inelastic collisions, assumption 2 does not hold. Collisions discriminates normal component of relative velocity

The Inelastic Maxwell Model

- **Inelastic collisions** $r = 1 - 2\epsilon$

$$\mathbf{v}_{1,2} = \mathbf{u}_{1,2} \mp (1-\epsilon)\,(\mathbf{g}\cdot\mathbf{n})\,\mathbf{n}$$

- **Boltzmann equation**

$$\begin{aligned}\frac{\partial P(\mathbf{v},t)}{\partial t} \;&=\; \int d\mathbf{n} \int d\mathbf{u}_1 \int d\mathbf{u}_2\, P(\mathbf{u}_1,t)\, P(\mathbf{u}_2,t) \\ &\quad\times \Big\{ \delta\left(\mathbf{v}-\mathbf{v}_1\right) - \delta\left(\mathbf{v}-\mathbf{u}_1\right) \Big\}\end{aligned}$$

- **Fourier transform** Krupp 1967

$$F(\mathbf{k},t) = \int d\mathbf{v} e^{i\mathbf{k}\cdot\mathbf{v}} P(\mathbf{v},t)$$

- **Closed equations** $\mathbf{q} = (1-\epsilon)\mathbf{k}\cdot\mathbf{n}\,\mathbf{n}$

$$\frac{\partial}{\partial t}F(\mathbf{k},t) + F(\mathbf{k},t) = \int d\mathbf{n}\, F\left[\mathbf{k}-\mathbf{q},t\right] F\left[\mathbf{q},t\right],$$

Theory is analytically tractable

One Dimension

- **Scaling of isotropic velocity distribution**

$$P(\mathbf{v},t) \to \frac{1}{T^{d/2}}\Phi\left(\frac{|\mathbf{v}|}{T^{1/2}}\right) \quad \text{or} \quad F(k,t) \to f\left(kT^{1/2}\right)$$

- **Nonlinear and nonlocal** $(T = T_0 \exp^{-2\epsilon(1-\epsilon)t})$

$$-\epsilon(1-\epsilon)f'(x) + f(x) = f(\epsilon x)f\left((1-\epsilon)x\right)$$

- **Exact solution**

$$f(x) = (1+|x|)\,e^{-|x|} \cong 1 - \frac{1}{2}x^2 + \frac{1}{3}|x|^3 + \cdots$$

- **Lorentzian^2 velocity distribution**

$$\Phi(v) = \frac{2}{\pi}\frac{1}{(1+v^2)^2}$$

- **Algebraic tail** Baldassari 2001

$$\Phi(v) \sim v^{-4} \qquad w \gg 1$$

Universal scaling function, exponent

Arbitrary Dimension

- **Freely cooling case**

$$T = \langle v^2 \rangle = T_0 \exp(-\lambda t) \qquad \lambda = 2\epsilon(1-\epsilon)/d$$

- **Governing equation** $x = k^2 T$

$$-\lambda\, x\, \Phi'(x) + \Phi(x) = \int d\mathbf{n}\, \Phi(x\xi)\, \Phi(x\eta)$$

$$\xi = 1 - (1-\epsilon^2)\cos^2\theta, \qquad \eta = (1-\epsilon)^2 \cos^2\theta$$

- **Power-law tails**

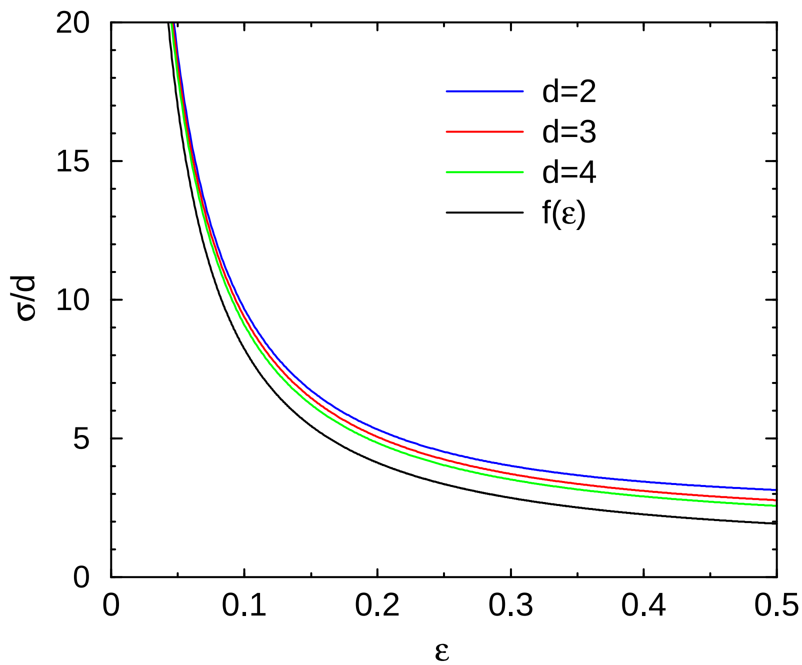
$$\Phi(v) \sim v^{-\sigma}, \qquad v \to \infty.$$

- **Exact solution for the exponent** σ

$$1-\epsilon(1-\epsilon)\,\frac{\sigma-d}{d} = {}_2F_1\left[\tfrac{d-\sigma}{2}, \tfrac{1}{2}; \tfrac{d}{2}; 1-\epsilon^2\right] + (1-\epsilon)^{\sigma-d}\,\frac{\Gamma\left(\frac{\sigma-d+1}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{\sigma}{2}\right)\Gamma\left(\frac{1}{2}\right)}$$

Nonuniversal tails, exponents depend on ϵ, d

The exponent σ



- **Maxwellian distributions:** $d = \infty$**,** $\epsilon = 0$
- **Diverges in high dimensions**

$$\sigma \propto d$$

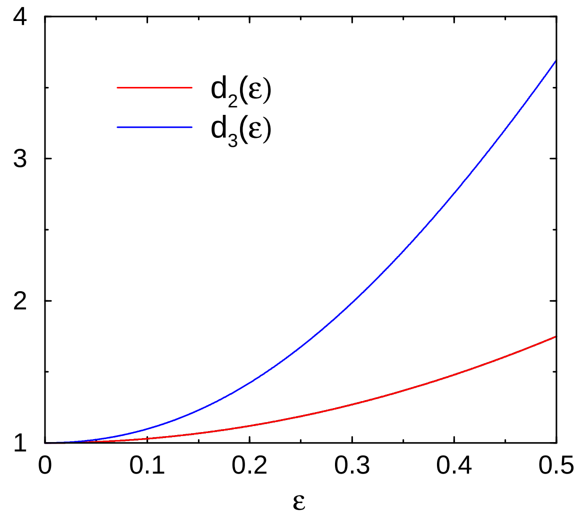
- **Diverges for low dissipation**

$$\sigma \propto \epsilon^{-1}$$

- **In practice, huge**

$$\sigma(d = 3, r = 0.8) \cong 30!$$

Velocity Moments



- Moments of the velocity distribution

$$M_{2n}(t) = \int d\mathbf{v} |\mathbf{v}|^{2n} P(\mathbf{v}, t)$$

- Multiscaling asymptotic behavior

$$M_n \sim \begin{cases} \exp(-n\lambda_2 t/2) & n < \sigma - d, \\ \exp(-\lambda_n t) & n > \sigma - d. \end{cases}$$

- Nonlinear multiscaling spectrum (1D):

$$\alpha_n(\epsilon) = \frac{1 - \epsilon^{2n} - (1-\epsilon)^{2n}}{1 - \epsilon^2 - (1-\epsilon)^2}$$

Sufficiently large moments exhibit multiscaling

Velocity Autocorrelations

- The velocity autocorrelation function

$$A(t_w, t) = \langle \mathbf{v}(t_w) \cdot \mathbf{v}(t) \rangle$$

- Linear evolution equation

$$T^{-1/2} \frac{d}{dt} A(t_w, t) = -(1-\epsilon) A(t_w, t)$$

- Nonuniversal ϵ-dependent decay

$$A(t_w, t) = A_0 [1 + t_w/t_0]^{-2+1/\epsilon} [1 + t/t_0]^{-1/\epsilon}$$

- Memory of initial velocity

$$A(t) \equiv A(0, t) \sim t^{-1/\epsilon}$$

- Logarithmic spreading ("self-diffusion")

$$\langle |\mathbf{x}(t) - \mathbf{x}(0)|^2 \rangle \sim \sqrt{\ln t}$$

Memory/Aging - $A(t_w, t) \neq f(t - t_w)$

The Forced Case

- Add white noise

$$\frac{dv_j}{dt}|_{\rm heat} = \eta_j(t) \qquad \langle \eta_i(t)\eta_j(t')\rangle = 2D\delta_{ij}\delta(t-t')$$

- Steady state $\frac{\partial}{\partial t} \equiv 0$

$$(1+Dk^2)P(k) = P(\epsilon k)P((1-\epsilon)k)$$

- Product solution

$$P(k) = \prod_{i=0}^{\infty}\prod_{j=0}^{i}\left[1+\epsilon^{2j}(1-\epsilon)^{2(i-j)}Dk^2\right]^{-\binom{i}{j}}.$$

- Cumulant solution

$$P(k) = \exp\left[\sum_{n=1} n^{-1}(-Dk^2)^n\psi_n\right]$$

- Fluctuation-dissipation relations

$$\psi_n = \lambda_n^{-1} = [1-(1-\epsilon)^{2n} - \epsilon^{2n}]^{-1}$$

Overpopulated high-energy tails

- Pole closest to origin $k = i/\sqrt{D}$ dominates

$$P(k) \propto \frac{1}{1 + Dk^2} \propto \frac{1}{(k - i/\sqrt{D})}$$

- Exponential tail

$$P(v) \simeq A(\epsilon)\exp(-|v|/\sqrt{D}) \qquad |v| \to \infty$$

- Direct from equation (ignore gain term)

$$D\frac{\partial^2}{\partial^2 v}P(v) \cong -P(v) \qquad |v| \to \infty$$

- Residue at pole yields prefactor

$$A(\epsilon) \propto \exp(\pi^2/12p)$$

$\boxed{\textbf{Non-Maxwellian}}$

Still, Maxwellians may resurface

- Small dissipation limit $\epsilon \to 0$

$$P(k) = \exp(-\epsilon^{-1} D k^2/2) \qquad k \gg \epsilon$$

- Maxwellian for range of velocities

$$P(v) \approx \exp(-\epsilon v^2/D) \qquad v \ll \epsilon^{-1}$$

The small dissipation limit $\epsilon \to 0$

- Maxwell model

$$P(v) \sim \begin{cases} \exp(-\epsilon^{-1} v^2/D) & v \ll \epsilon^{-1} \\ \exp(-|v|/\sqrt{D}) & v \gg \epsilon^{-1} \end{cases}$$

- Boltzmann equation

$$P(v) \sim \begin{cases} \exp(-\epsilon^{a} v^3) & v \ll \epsilon^{-b} \\ \exp(-|v|^{3/2}) & v \gg \epsilon^{-b} \end{cases}$$

- **Limits $v \to \infty$, $\epsilon \to 0$ do not commute!**
- $\epsilon \to 0$ is singular

$$-\epsilon(1-\epsilon)x f'(x) + f(x) = f(\epsilon x) f((1-\epsilon)x)$$

- **Small-ϵ Expansions may not be useful!**

Velocity Correlations

- Definition (correlation between v_x^2 and v_y^2)

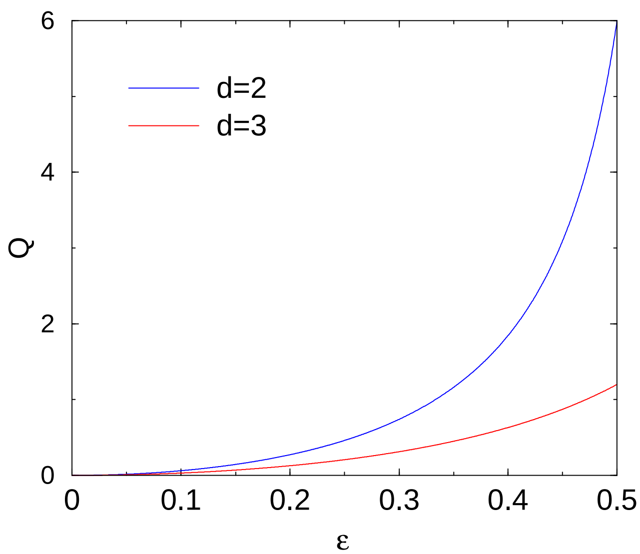
$$Q = \frac{\langle v_x^2 v_y^2\rangle - \langle v_x^2\rangle\langle v_y^2\rangle}{\langle v_x^2\rangle\langle v_y^2\rangle}$$

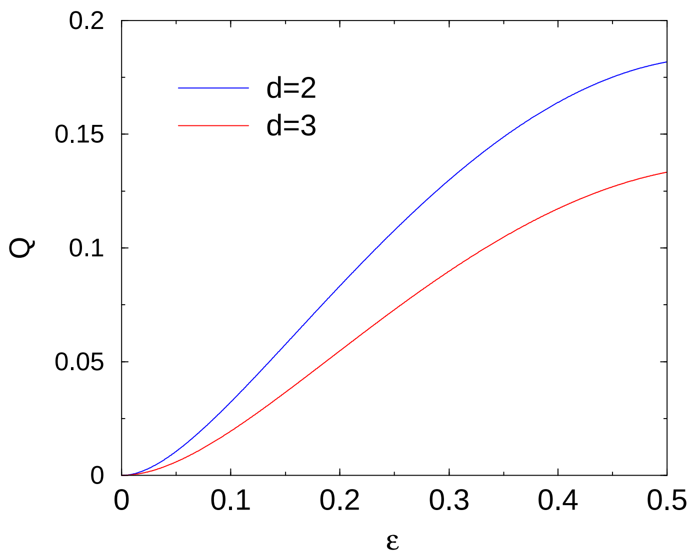
- Unforced case (freely evolving) $P(v) \sim v^{-\sigma}$

$$Q = \frac{6\epsilon^2}{d-(1+3\epsilon^2)}$$

- Forced case (white noise) $P(v) \sim e^{-|v|}$

$$Q = \frac{6\epsilon^2(1-\epsilon)}{(d+2)(1+\epsilon)-3(1-\epsilon)(1+\epsilon^2)}.$$





Correlations diminish with energy input

The "brazil nut" problem

- Fluid background: mass 1
- Impurity: mass m
- Theory: Lorentz-Boltzmann equation
- Series of transition masses

$$1 < m_1 < m_2 < \cdots < m_\infty$$

- Ratio of moments diverges asymptotically

$$\frac{\langle v_I^{2n} \rangle}{\langle v_F^{2n} \rangle} \sim \begin{cases} c_n & m < m_n; \\ \infty & m > m_n. \end{cases}$$

- Light impurity: moderate violation of equipartition, impurity mimics the fluid
- Heavy impurity: extreme violation of equipartition, impurity sees a static fluid

$\boxed{\textbf{series of phase transitions}}$

Conclusions (Maxwell specific)

- Power-law high energy tails
- Non-universal exponents
- Multiscaling of the moments, Temperature insufficient to characterize large moments

Generic features

- Overpopulated tails
- Energy input diminishes correlations, tails
- Multiple asymptotics in $\epsilon \to 0$ limit
- Logarithmic self-diffusion
- Correlations between velocity components
- Spatial correlations
- Algebraic autocorrelations, aging

Mixing Processes

with: Daniel ben-Avraham

- Generalized collision rule

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} p & q \\ q & p \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

- Special case:

1. Inelastic collisions: $p + q = 1$
2. Kac model: $p^2 + q^2 = 1$
3. Inelastic Lorentz gas: $p = 0$
4. Addition/Aggregation: $p = q = 1$

- Momentum not conserved

- Related: averaging, wealth exchange

Scaling Solutions: Algebraic Tails

- Fourier equation

$$\frac{\partial}{\partial t}F(k,t) + F(k,t) = F(pk,t)F(qk,t)$$

- Scaling solution

$$F(k,t) = f\left(ke^{\alpha t}\right) \qquad \Leftrightarrow \qquad P(v,t) \to e^{\alpha t}\Phi(ve^{\alpha t})$$

- Algebraic tails

$$f_{\rm sing}(z) \sim z^{\nu} \qquad \Leftrightarrow \qquad \Phi(w) \sim w^{-\nu-1}$$

- Scaling parameter equation

$$\alpha = \nu^{-1}(1 - p^{\nu} - q^{\nu})$$

- Two possibilities

$$\alpha = \begin{cases} p^{\nu}\ln\frac{1}{p} + q^{\nu}\ln\frac{1}{q} & \nu \leq 2; \\ \frac{1}{2}\lambda_2 = \frac{1}{2}(1 - p^2 - q^2) & \nu \geq 2. \end{cases}$$

$\boxed{\textbf{Every power law } \nu > 0 \textbf{ possible!}}$

Extremum Selection ($\nu < 2$)

- Scaling solution

$$P(v,t) \to e^{\alpha t}\Phi(ve^{\alpha t})$$

- Stationary travelling wave

$$P(\ln v,t) \to F(\ln v + \alpha t)$$

- Exponent relation $\Leftrightarrow$ dispersion relation

$$\alpha = \nu^{-1}(1 - p^\nu - q^\nu)$$

- Extremum selection $\Rightarrow$ exponent equation

$$p^n u \ln\frac{e}{p^\nu} + q^\nu \ln\frac{e}{q^\nu} = 1$$

$$\boxed{\textbf{Similarity solution of second kind}}$$

Steady states: stretched exponentials

- Steady state

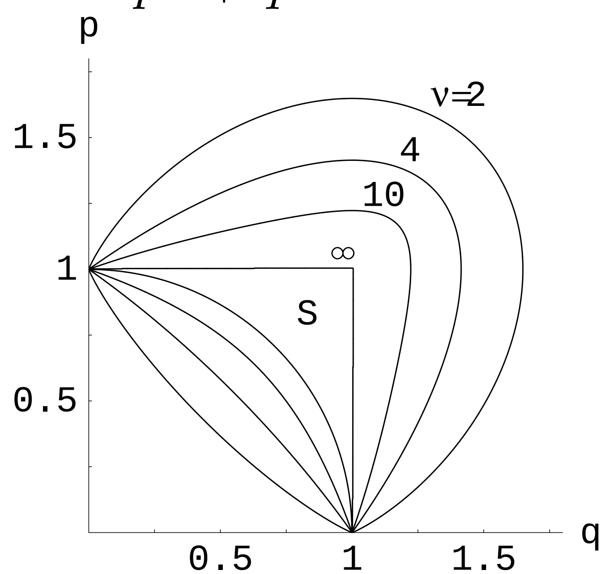
$$F(k) = F(pk)F(qk)$$

- Compatible with

$$F(k) \sim \exp\left[-k^{\mu}\right] \qquad \Leftrightarrow \qquad P(v) \sim \exp[-v^{\gamma}]$$

- Exponent equation: $\mu > 0$, $\gamma > 1$

$$p^{\mu} + q^{\mu} = 1$$



Every sharper than exponential possible

Conservation laws: universal behavior

- Energy conserved: $p^2 + q^2 = 1$

$$\Phi(w) = (2\pi)^{-1/2} \exp[-w^2/2]$$

- Momentum conserved: $p + q = 1$

$$\Phi(w) = \frac{2}{\pi}(1 + w^2)^{-2}$$

Inelastic Lorentz gas ($\nu = 0$)

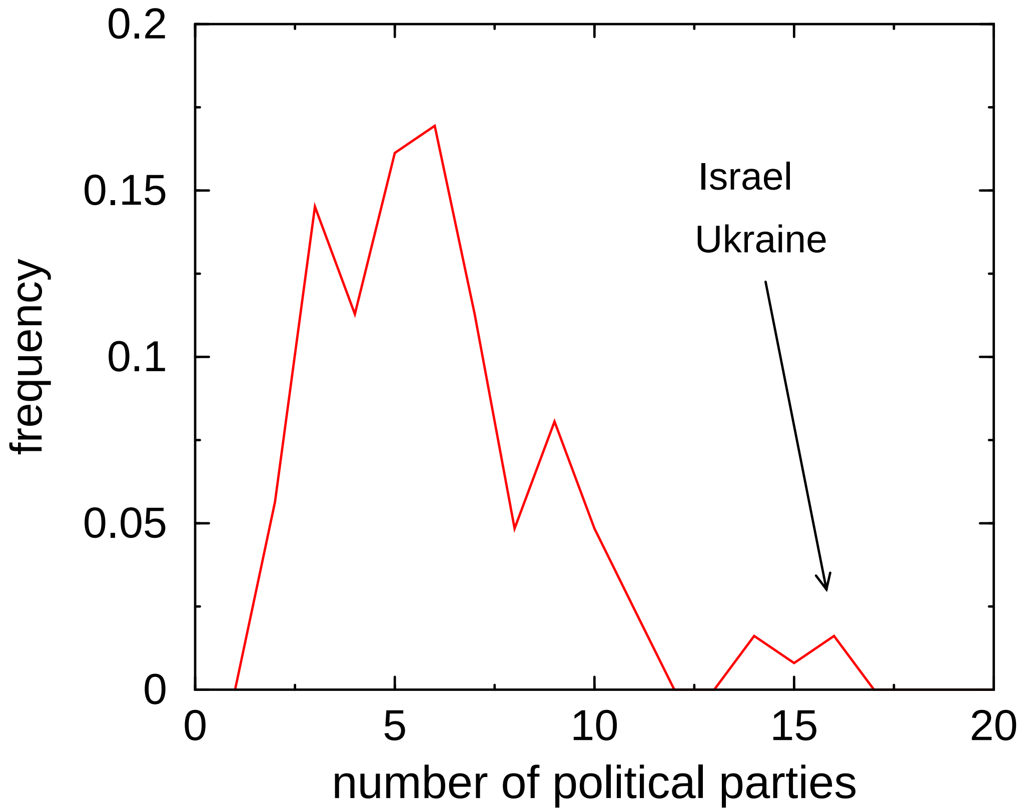
- Unnormalized steady state solution

$$P(v) = v^{-1}$$

- Log-normal distribution:

$$\ln P(v) \propto -(\ln v)^2$$

How many political parties?



- Data: CIA world factbook 2002
- 120 countries with multiparty system
- Average=5.8, Variance=2.9

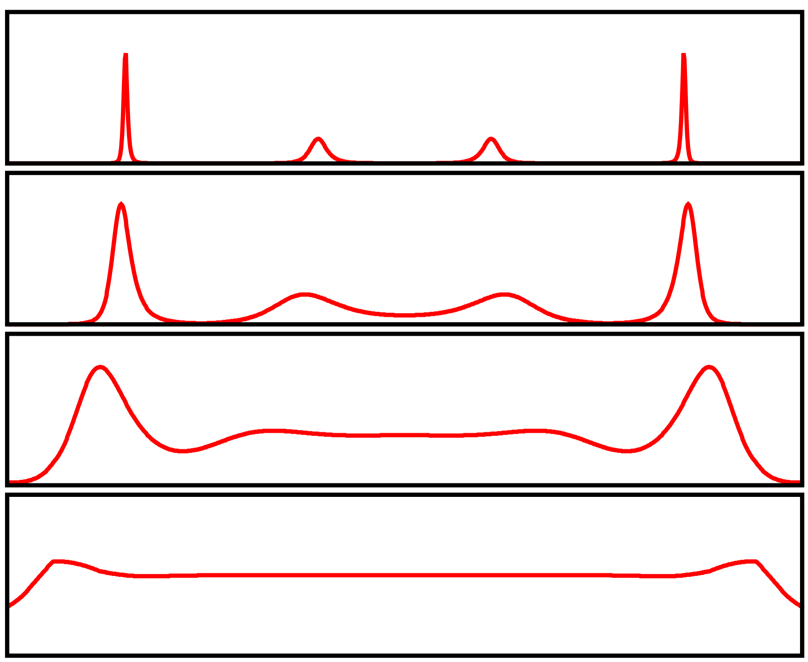
Simple model?

The Compromise Model

- Opinion $-\Delta < x < \Delta$
- Reach compromise in pairs Weisbuch 2001

$$(x_1, x_2) \to \left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}\right)$$

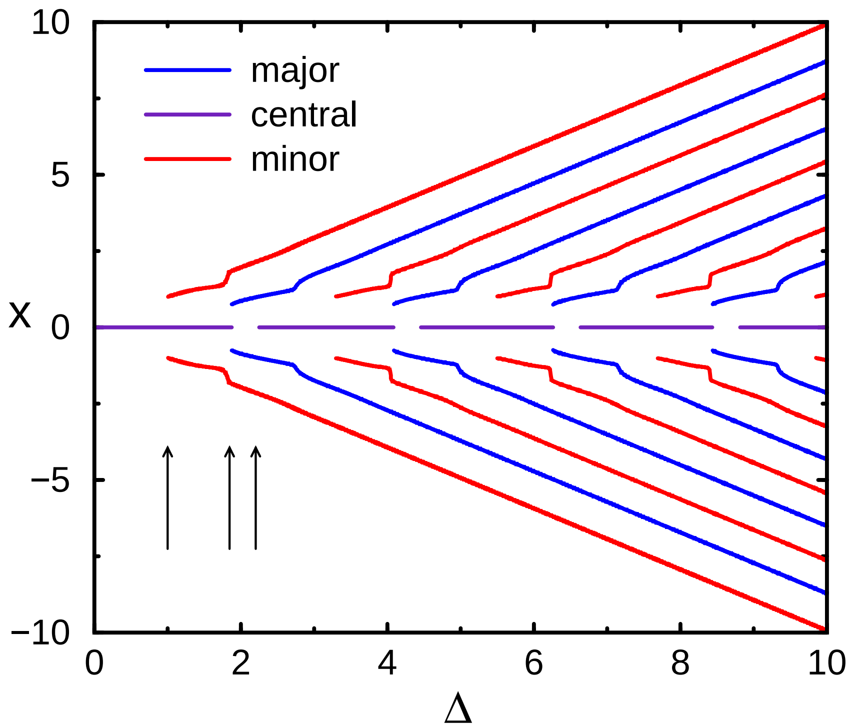
- As long as we are close $|x_1 - x_2| < 1$



$$P_\infty(x) = \sum_i m_i \, \delta(x - x_i)$$

Final State: localized clusters

Bifurcations and Patterns



- Periodic bifurcations

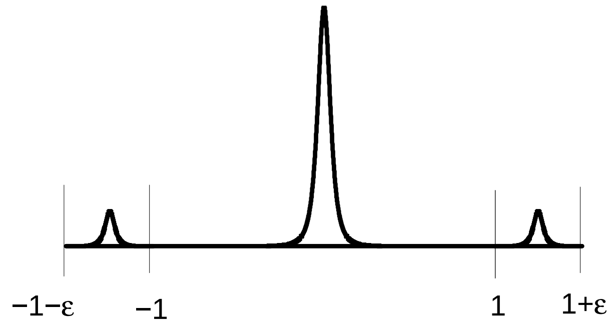
$$x(\Delta) = x(\Delta + L) \qquad L \cong 2.155$$

- Alternating major-minor pattern
- Critical behavior

$$m \sim (\Delta - \Delta_c)^{\alpha} \qquad \alpha = 3 \text{ or } 4.$$

Self-similar structure

Near critical behavior



- Perturbation theory: $\Delta = 1 + \epsilon$
- Central cluster: mass M, $x(\infty) = 0$
- Minor cluster: mass m, $x(\infty) = 1 + \epsilon/2$

$$\frac{dm}{dt} = -mM \quad \rightarrow \quad m(t) \sim \epsilon\, e^{-t}$$

- Process stops when $x \sim e^{-t_f/2} \sim \epsilon$
- Final minor cluster mass

$$m(\infty) \sim m(t_f) \sim \epsilon^3$$

- Argument generalizes to type 3 bifurcations

$$m \sim (\Delta - \Delta_c)^{\alpha} \qquad \alpha = \begin{cases} 3 & \text{type } 1 \\ 4 & \text{type } 3 \end{cases}$$

$$\boxed{\textbf{Masses vanish algebraically near type 1, 3 bif}}$$

Pattern Formation

- Flat state $P(x,t) = 1$ unstable
- Linear stability analysis: $P - 1 \sim e^{i(kx+\omega t)}$

$$\omega(k) = \frac{8}{k}\sin\frac{k}{2} - \frac{2}{k}\sin k - 2$$

- Fastest growing mode $d\omega/dk = 0$

$$L = 2\pi/k \cong 2.25146$$

- Travelling wave $d\omega/dk = \mathrm{Im}[\omega]/\mathrm{Im}[k]$

$$P(x,t) = 1 + e^{-\lambda(x-vt)}e^{i(kx+\omega t)}$$

$$L = 2\pi/k \cong 2.0375$$

- Patterns induced by wave propagating from boundary. However, the emerging period is different $L \cong 2.155$!

Pattern selection intrinsically nonlinear

Outlook

- Polydisperse media: impurities, mixtures
- Lattice gases: correlations
- Hydrodynamics
- Shear flows, Shocks
- Opinion dynamics
- Economics