Theory of Granular Gases

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1. Introuction
2. Freely Cooling Gases
3. Inelastic Maxwell Model

1D: Phys. Rev. Lett. **86**, 1414 (1999)
2D: Phys. Rev. Lett. **89**, 204301 (2002)

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"A gas of marbles"

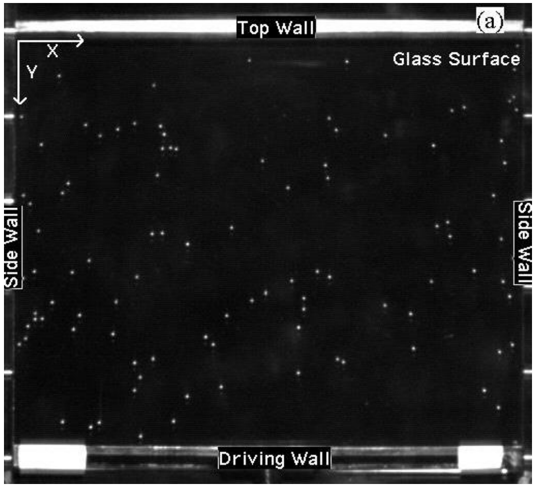
G Collins, Scientific American, Jan 2001

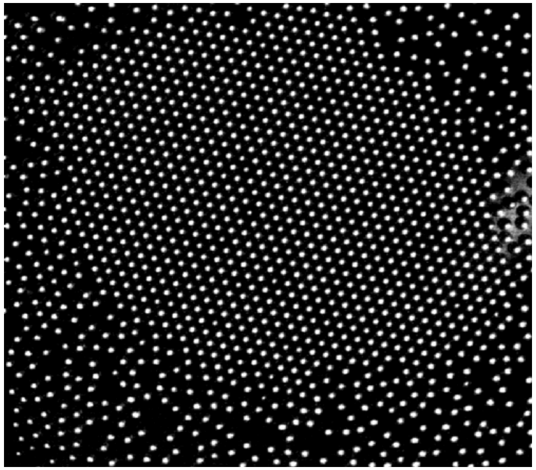
- **Ubiquitous in nature**
  - **Geophysics: sand dunes, volcanic flows**
  - **Astrophysics: large scale formation**
- **Characteristics**
  - **Dissipative collisions**
  - **Hard core interactions**
- **Experimental observations:**

  1. Clustering

  2. Non-Maxwellian velocity distributions

challenge: hydrodynamic description of dissipative gases





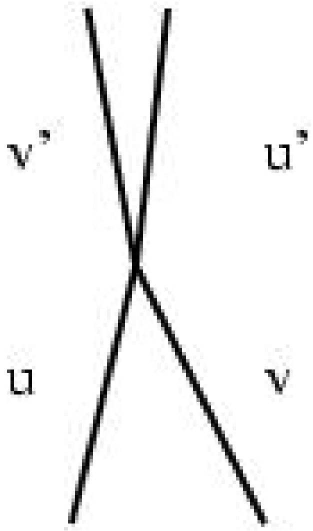
Kudrolli 98, Urbach 98

Inelastic Collisions

- Relative velocity reduced by $r = 1 - 2\varepsilon$

$$\Delta \mathrm{v} \to r\Delta \mathrm{v} \quad \text{or} \quad \mathrm{v} \to \mathrm{v} - \varepsilon\Delta \mathrm{v}$$

- Energy Dissipation $\Delta E \propto -\varepsilon(\Delta \mathrm{v})^2$
- In general d $\Delta \mathbf{v} \to \Delta \mathbf{v} \cdot \mathbf{n}$



Velocities of colliding particles become correlated

Hydrodynamics

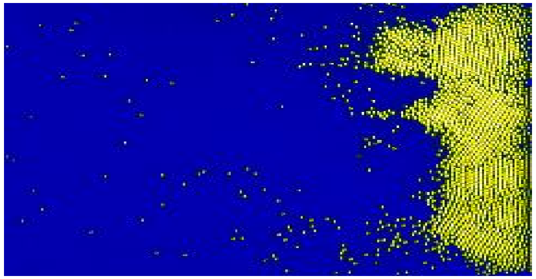
P Haff JFM 134, 401 (1983)

- **Energy dissipation** $\Delta T \propto -\varepsilon(\Delta \mathrm{v})^2 \propto -\varepsilon T$
- **Collision frequency** $\Delta t \propto l / \Delta \mathrm{v} \propto T^{-1/2}$
- **The energy balance equation**
- **Haff cooling law** $\dfrac{dT}{dt} \propto -\varepsilon T^{3/2}$

$$T(t) = (1 + A\varepsilon t)^{-2} \approx \begin{cases} 1 & t << \varepsilon^{-1} \\ \varepsilon^{-2} t^{-2} & t >> \varepsilon^{-1} \end{cases}$$

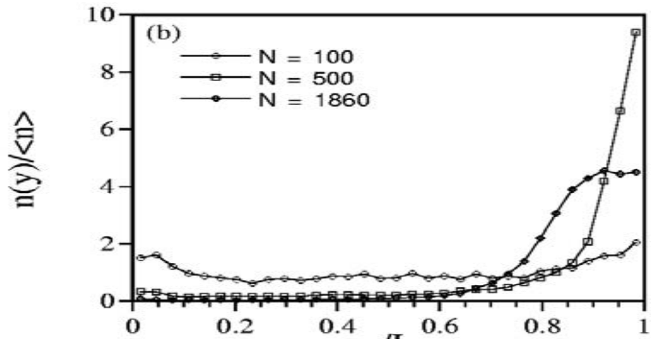
Assume homogeneous gas: a single length, velocity scale

Density Inhomogeneities



Experiment

Horizontally driven steel beads



Hydrodynamic theory

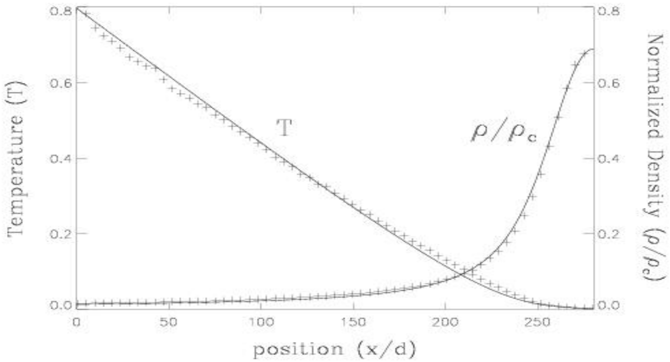
Mass, momentum balance equations

Energy balance equation

$$dq\,/\,dx\;=\;-I$$

Approximate equation of state

$$P=\rho T\,\frac{\rho_c+\rho}{\rho_c-\rho}$$



Agreement – theory, simulation, experiment

Kudrolli & Gollub PRL 78, 1383 (1997)   Grossman, Zhou, ebn PRE 55, 4200 (1997)

Kinetic Theory

Esipov & Poechel JSP 86, 1385 (1997)

- **Boltzmann equation for velocity distribution**

$$\frac{\partial}{\partial t}P(\mathrm{v},t)=\iint d\mathrm{v}_1 d\mathrm{v}_2 |\Delta \mathrm{v}| P(\mathrm{v}_1,t)P(\mathrm{v}_2,t)\left[\delta\left(\mathrm{v}-\mathrm{v}_1+\varepsilon\Delta \mathrm{v}\right)-\delta\left(\mathrm{v}-\mathrm{v}_1\right)\right]$$

- **Consistent with Haff law** $T(t)=\int d\mathrm{v}|\mathrm{v}|^2 P(\mathrm{v},t)$
- **Similarity solution**

$$P(\mathrm{v},\mathrm{t})\to t\Phi(\mathrm{v}t)$$

- **Overpopulated high energy tail**

$$\Phi(z)\propto \exp(-z) \qquad z\to\infty$$

Assume homogeneous gas: ignore velocity correlations

Forced Granular Gases

- **Experiment: vertical vibration**
- **Modeled by: white noise forcing** Menon 99

$$\frac{d\mathrm{v}_j}{\mathrm{dt}} = \eta_j \qquad \langle \eta_j(t)\eta_j(t') \rangle = \delta(t-t')$$

- **Kinetic Theory: diffusion in velocity space, ignore gain term for large velocities** Ernst 97

$$D\frac{d^2}{d\mathrm{v}^2}P(\mathrm{v}) \cong |\mathrm{v}|P(\mathrm{v}) \quad \Rightarrow \quad P(\mathrm{v}) \approx \exp\left(-|\mathrm{v}|^{3/2}\right)$$

Non-Maxwellian High Energy Tails

Maxwellian Distributions

- **Assumption 1: velocity distribution is isotropic**

$$P(\vec{\mathrm{v}}) = P\left(\left|\vec{\mathrm{v}}\right|\right)$$

- **Assumption 2: velocity correlations absent**

$$P(\mathrm{v}_x, \mathrm{v}_y, \mathrm{v}_z) = P(\mathrm{v}_x)P(\mathrm{v}_y)P(\mathrm{v}_z)$$

- **Only possibility: Maxwellian**

$$P(\vec{\mathrm{v}}) = \frac{1}{(2\pi \mathrm{T})^{d/2}} \exp\left(-\frac{\left|\vec{\mathrm{v}}\right|^2}{2T} \right)$$

?

Freely Cooling Granular Gases

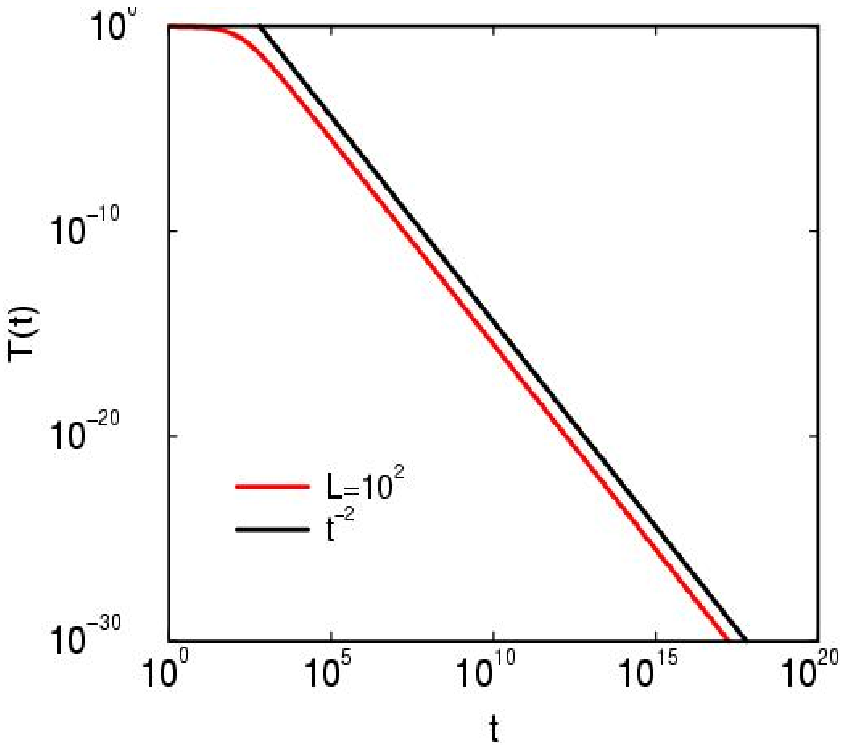
- Initial spatial distribution: uniform
- Initial velocity distribution: $P_0(\mathrm{v})$
- Dimensionless space & time variables
  - Initial mean free path = 1 $x \to x/x_0$
  - Initial typical velocity = 1 $\mathrm{v} \to \mathrm{v}/\mathrm{v}_0$
- Particles undergo inelastic collisions

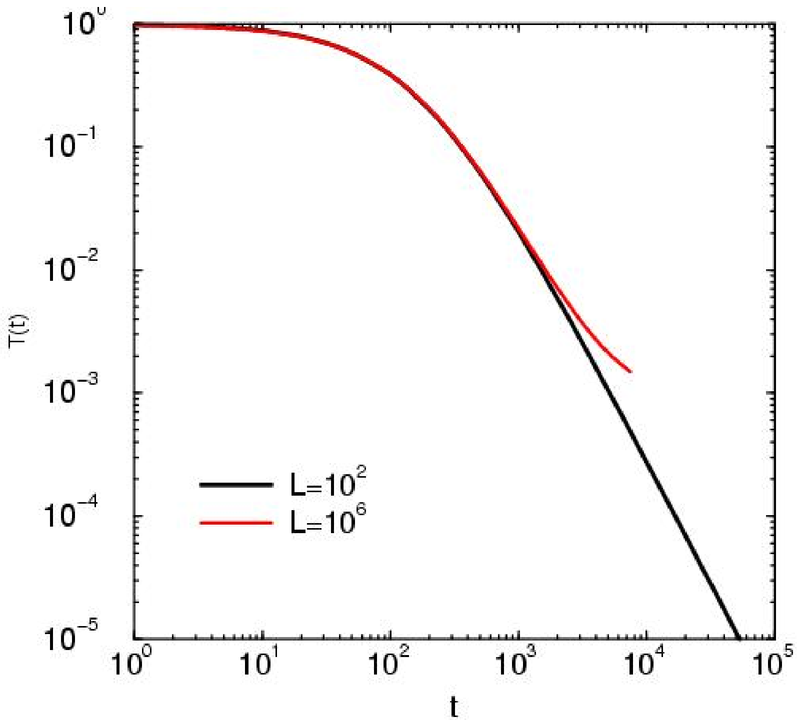
Characteristic length & velocity scales?

$$T(\varepsilon,t) = \left\langle \mathrm{v}^2(t) \right\rangle = ?$$

Hydrodynamic/continuum theory?

Molecular Dynamics Simulations

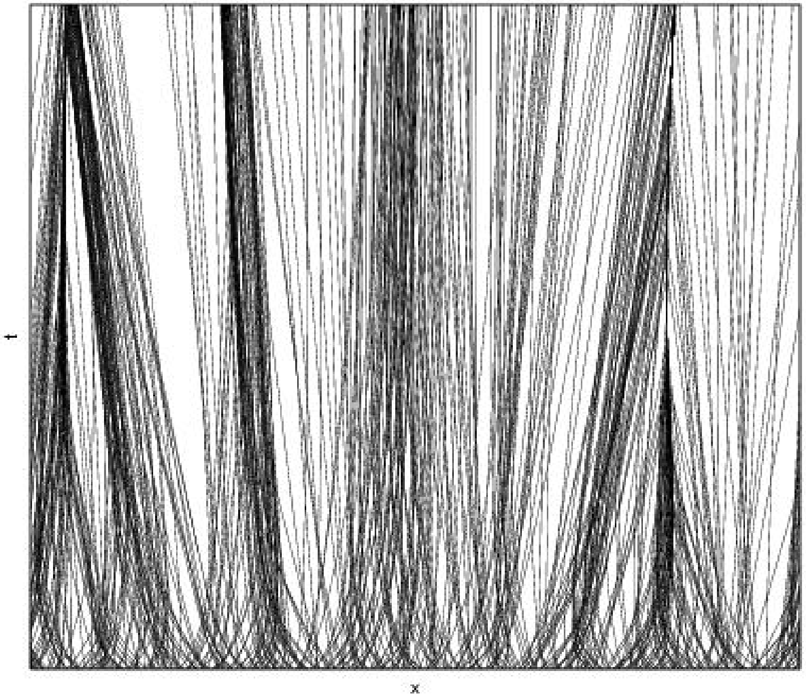




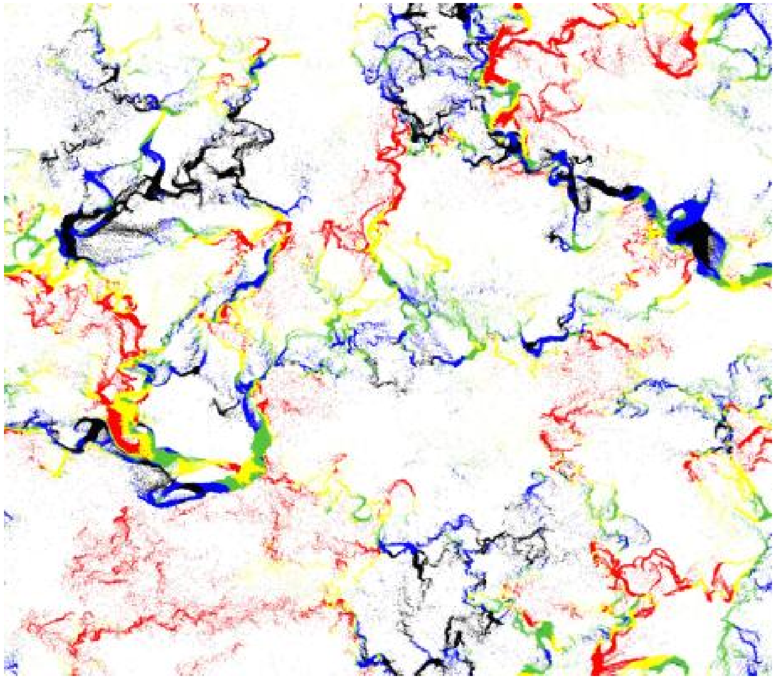
Hydrodynamic/Kinetic Theory hold only for <u>small systems</u> or <u>small times</u>

Inelastic collapse, clustering instability

McNamara & Young PF 4, 496 (1992)



Goldhirsch & Zannetti PRL 70, 1619 (1993)

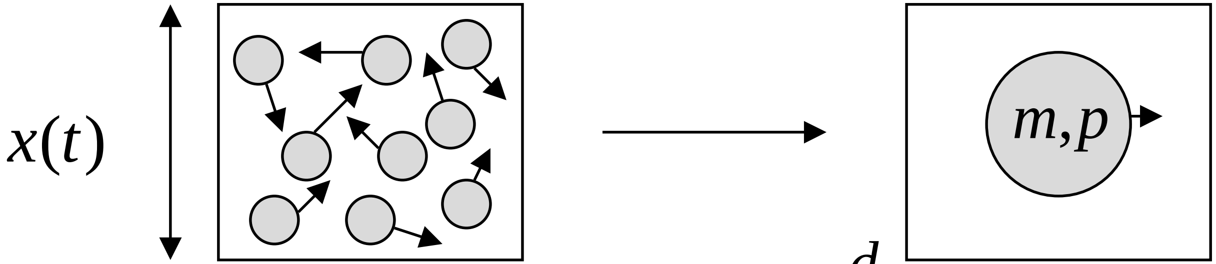


System develops dense clusters

Inelastic collapse: infinite collisions in finite time

Particles positions & velocities become correlated

The Sticky Gas (r = 0)



- **Mass conservation** $m \approx x^d$
- **Momentum conservation** $p \approx \sum_{i=1}^{m} p_i \approx m^{1/2} \approx x^{d/2}$
- **Dimensional analysis** $x \approx \mathrm{v}t$
- **Typical length scale** $x \approx t^{\beta}$ $\quad \beta = \frac{d}{d+2}$
- **Finite system: final state 1 aggregate mass m=N**

$$T(t) \approx \begin{cases} 1 & t \ll 1 \\ t^{-2\beta} & 1 \ll t \ll N \\ N^{-1} & N^{1/2\beta} \ll t \end{cases}$$

Carnavale, Pomeau, Young PRL 64, 2913 (1990)

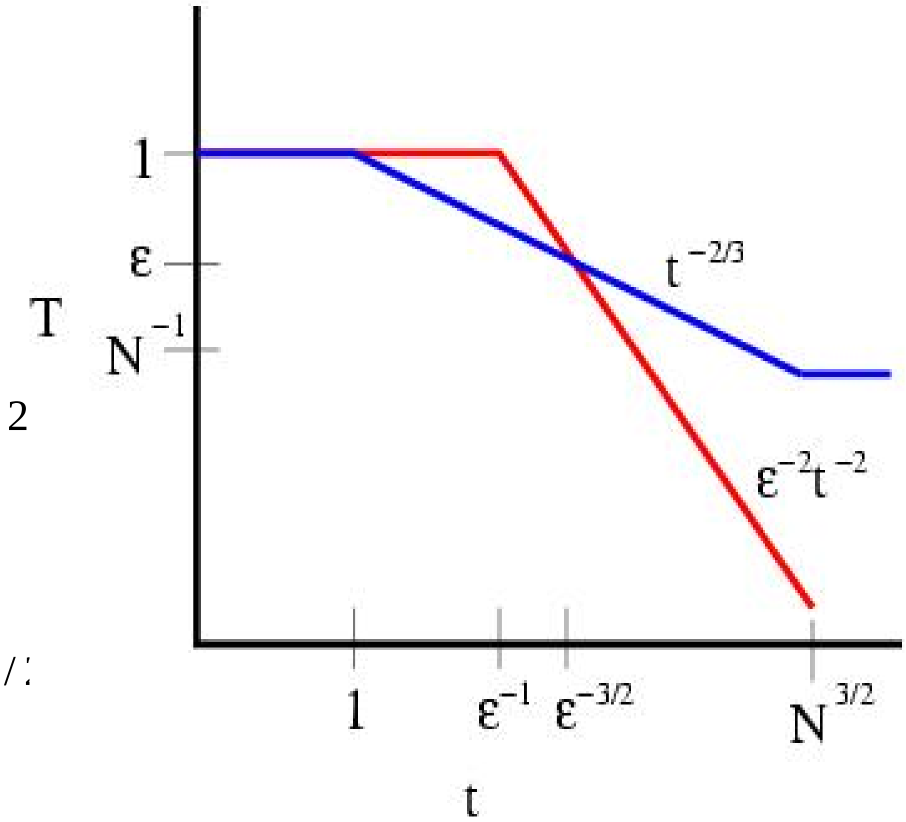
Monotonicity

- $T(\varepsilon,t)$ **Monotonic in** ε,t
- **Sticky gas** $(\varepsilon = 0)$ **is lower bound**
- **Homogeneous when**

$$N << \varepsilon^{-1} \quad \textbf{or} \quad t << \varepsilon^{3/2}$$

- **Clustering when**

$$N >> \varepsilon^{-1} \quad \textbf{and} \quad t >> \varepsilon^{3/2}$$



Asymptotic behavior is independent of inelasticity

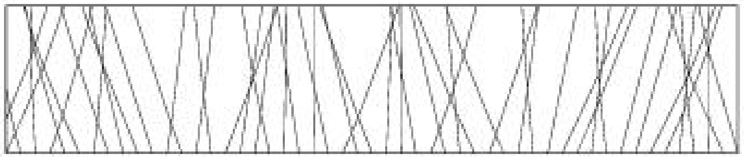
The Crossover Picture

- **Universal, r-independent cooling law**
- **For strong dissipation, clustering is immediate**

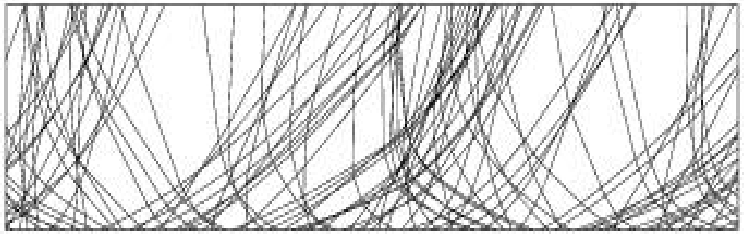
$$T(t) \approx \begin{cases} 1 & t << \varepsilon^{-1} \\ \varepsilon^{-2} t^{-2} & \varepsilon^{-1} << t << \varepsilon^{-3/2} \\ t^{-2/3} & \varepsilon^{-3/2} << t << N^{3/2} \\ N^{-1} & N^{3/2} << t \end{cases}$$

r=0 is fixed point!

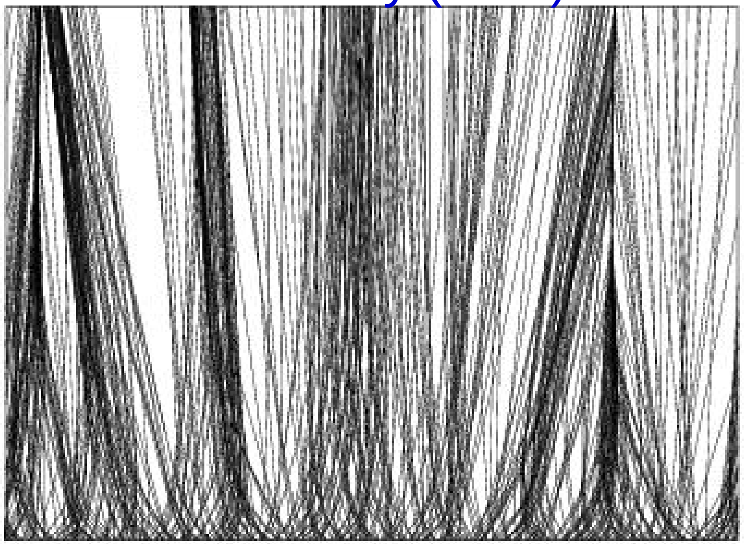
early=elastic (r = 1)



intermediate=inelastic (r = 0.9)

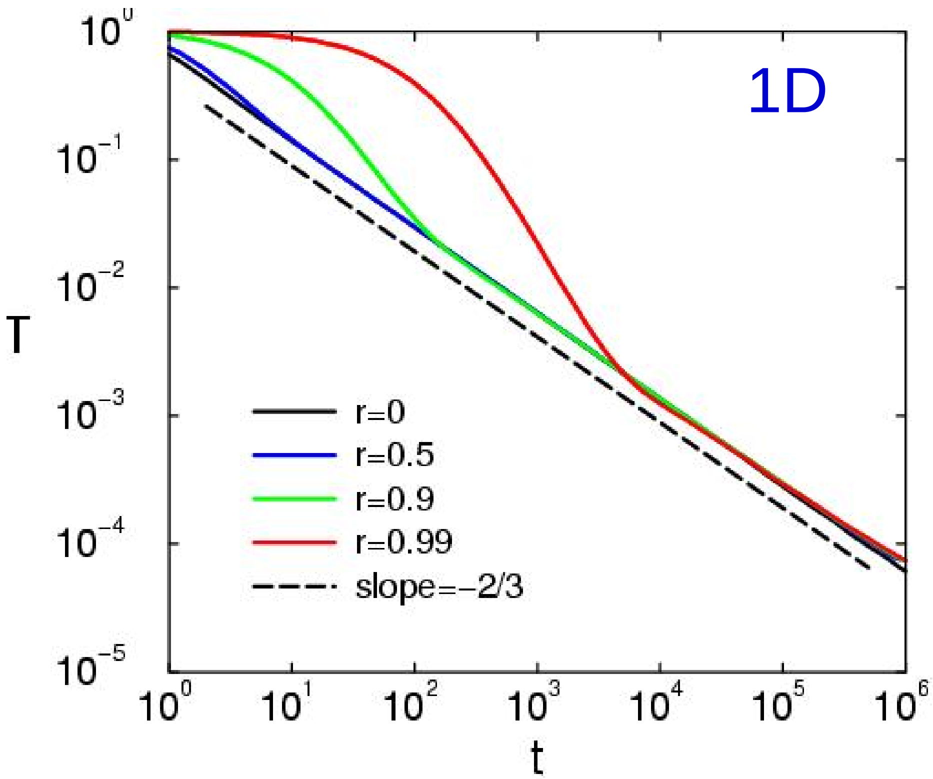


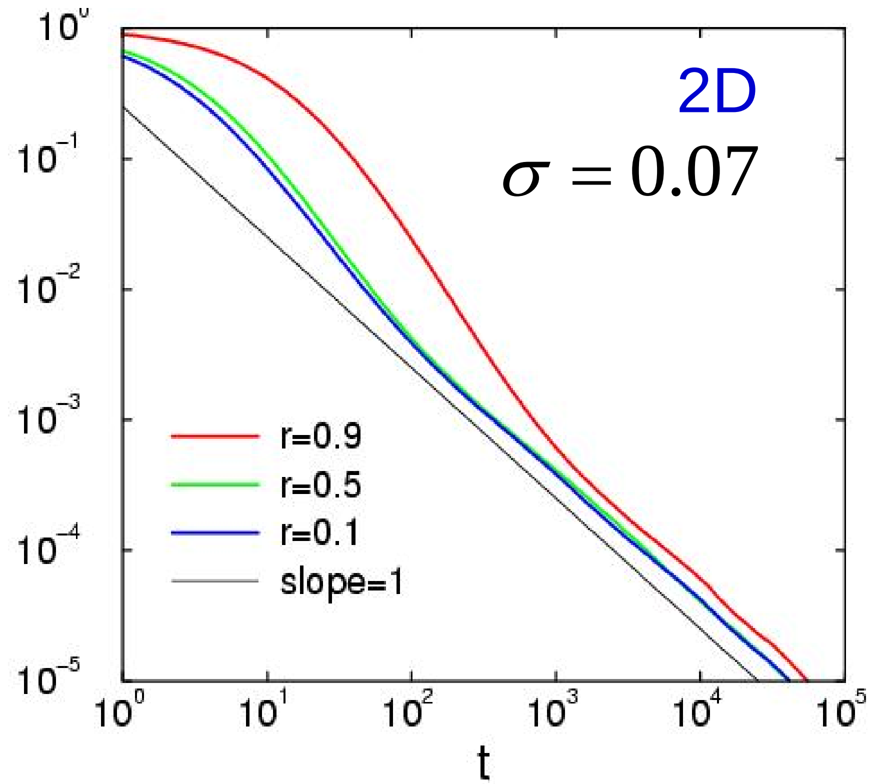
late=sticky (r = 0)



Event Driven Simulations

- **Number of particles: $N_p=10^7$(1D), 10^6(2D)**
- **Number of collisions: $N_c=10^{11}$(1D), 4×10^{10} (2D)**



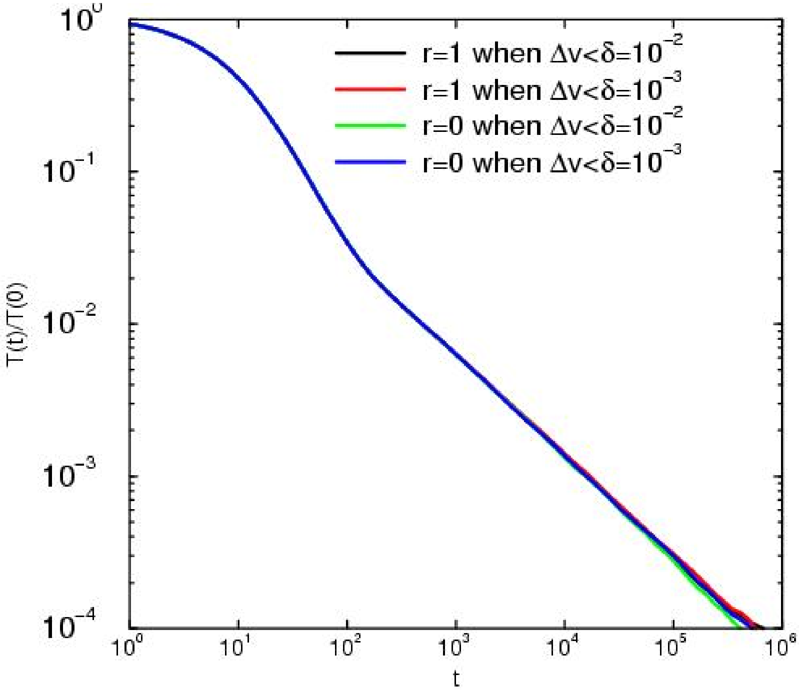


The Simulation Technique

- **Cluster forms via inelastic collapse = finite time singularity**
- **Relax dissipation below small cutoff**

$$r(\Delta \mathrm{v}) = \begin{cases} 1 & \Delta \mathrm{v} < \delta \\ r & \Delta \mathrm{v} > \delta \end{cases}$$

- **Mimics viscoelastic particles**
- **Results are independent of**
  - **Cutoff velocity**
  - **Subthreshold collision law**



Results valid for $\mathrm{v} \ll \delta, \quad t \gg \delta^{-3}$

The Velocity Distribution

- **Self-similar distribution**

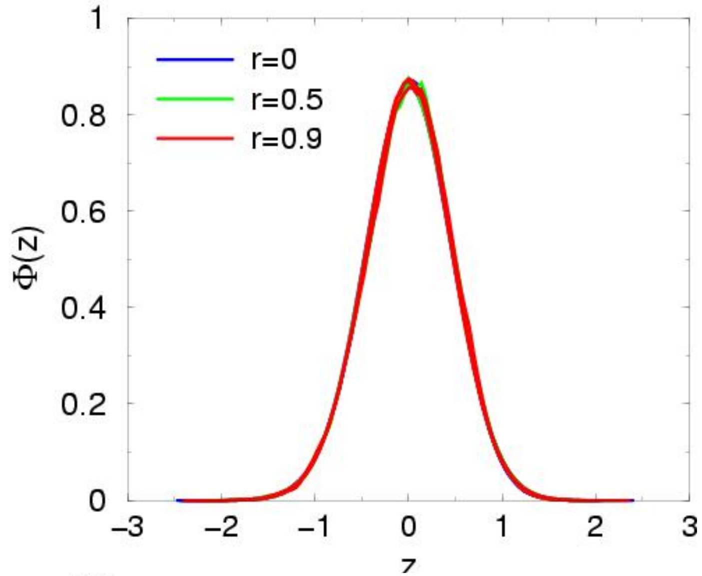
$$P(\mathrm{v},t) \to t^{\beta}\Phi\left(\mathrm{v}t^{\beta}\right)$$

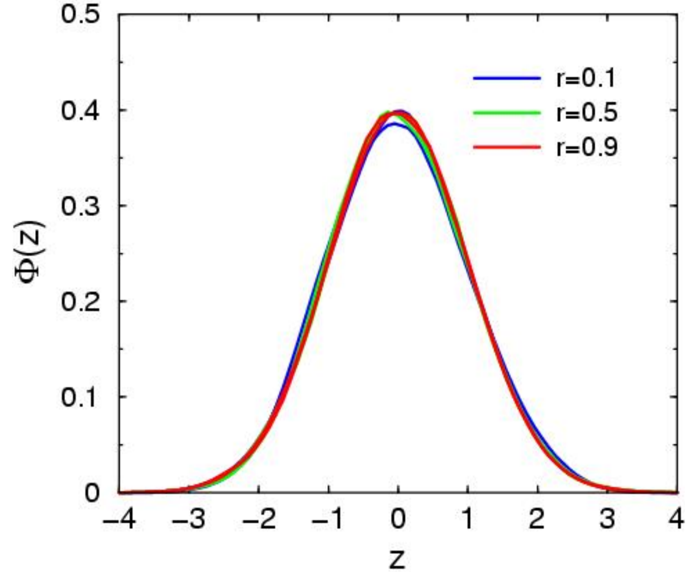
- **Independent of r**
- **Anomalous high-energy tail**

$$\Phi(\mathrm{z}) \approx \exp(\text{-}\mathrm{z}^{1/\beta})$$

$P(r,\mathrm{v},t)$ **is function of**

$z = \mathrm{v}t^{\beta}$ **only!**





The Large Velocity (Lifshitz) Tail

- **Use scaling behavior** $P(\mathrm{v},t) \approx t^{\beta}\Phi(\mathrm{v}t^{\beta})$
- **Assume stretched exponential** $\Phi(z) \approx \exp(-z^{\gamma})$
- **Tail dominated by v=1 particles**
- **t=0: interval of size t empty** $P(\mathrm{v}=1,t) \propto \exp(-t)$
- **Exponent relation** $\beta\gamma = 1$

$$\gamma = \begin{cases} 1 & \text{homogenous} \\ (d+2)/d & \text{clustering} \end{cases}$$

Anomalous velocity distributions

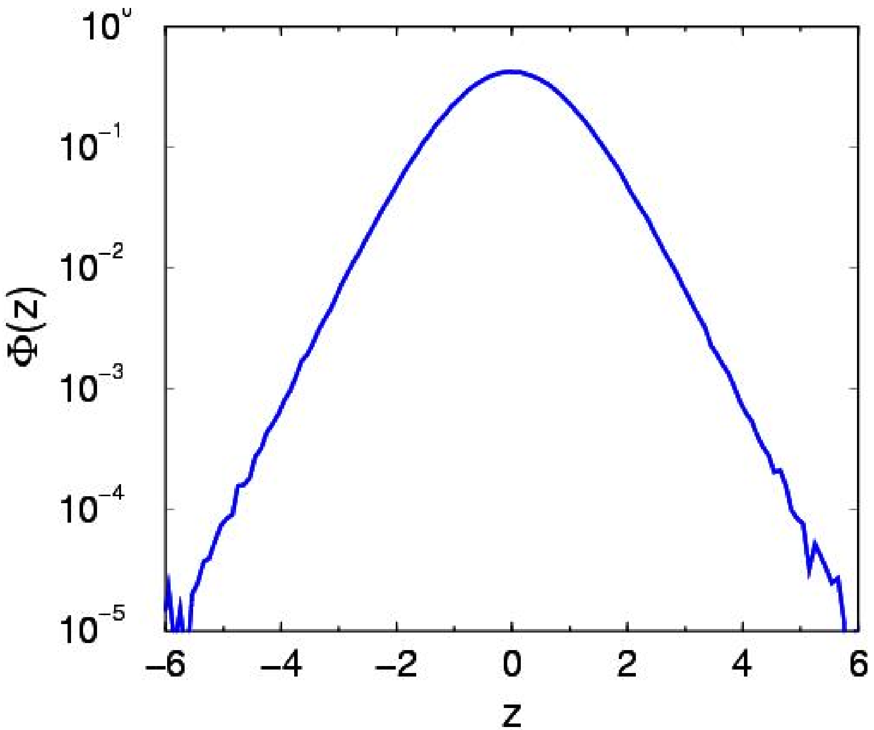
Numerical Verification (2D)

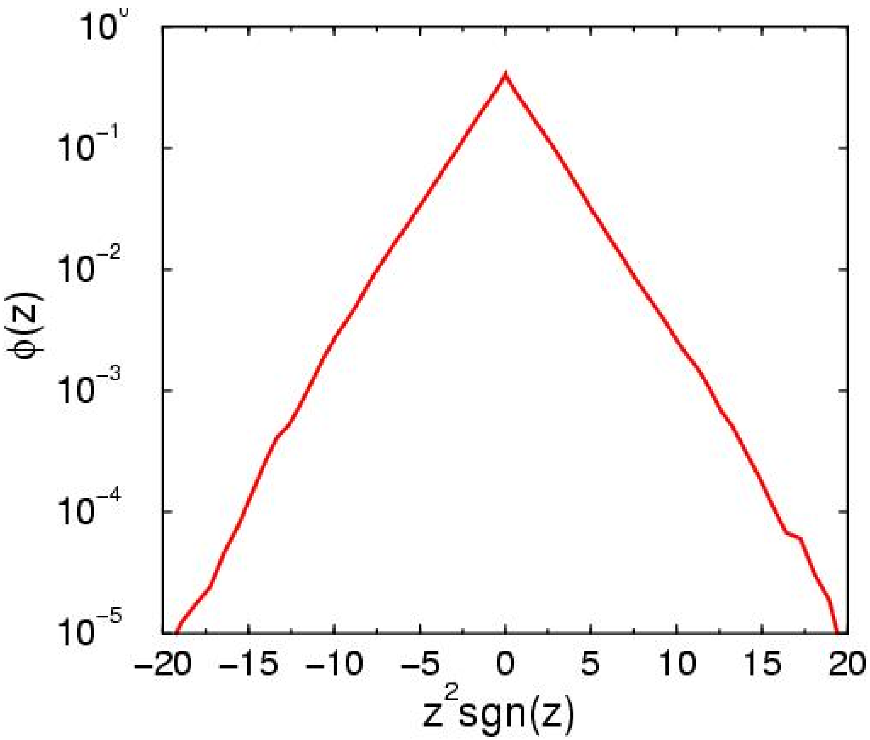
homogeneous

$$\Phi(z) \approx \exp(-|z|)$$

clustering

$$\Phi(z) \approx \exp(-z^2)$$





The inviscid Burgers equation (1D)

Zeldovich & Shandarin, RMP 61, 185 (1989)

- **Nonlinear diffusion equation**

$$v_t + v v_x = \nu\, v_{xx} \quad (\nu \to 0)$$

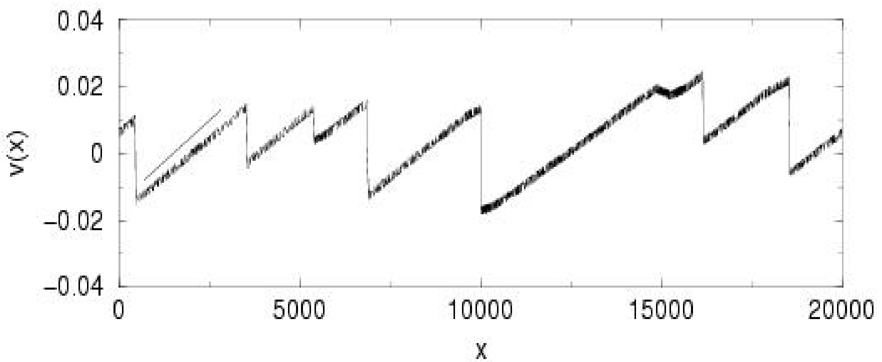
- **Transform to linear diffusion eq.**

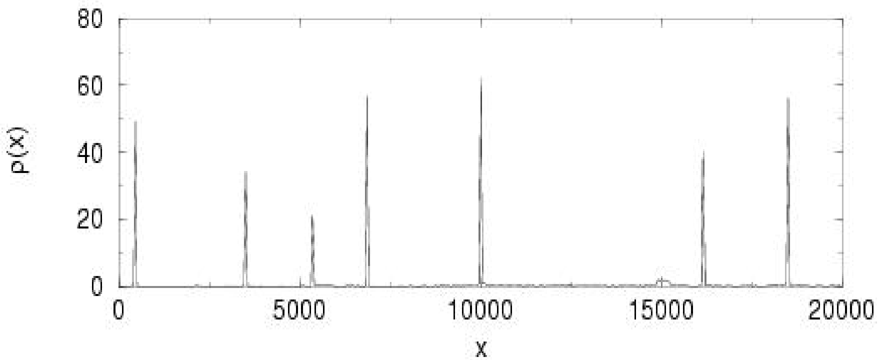
$$u_t = \nu\, u_{xx} \Leftarrow u = -2\nu\, (\ln v)_x$$

- **Sawtooth shock velocity profile**

$$v(x,t) = \frac{x - q(x,t)}{t}$$

- **Shock collisions conserve mass and momentum**
- **Describes sticky gas**





Inviscid Burgers equation = sticky gas = inelastic gas

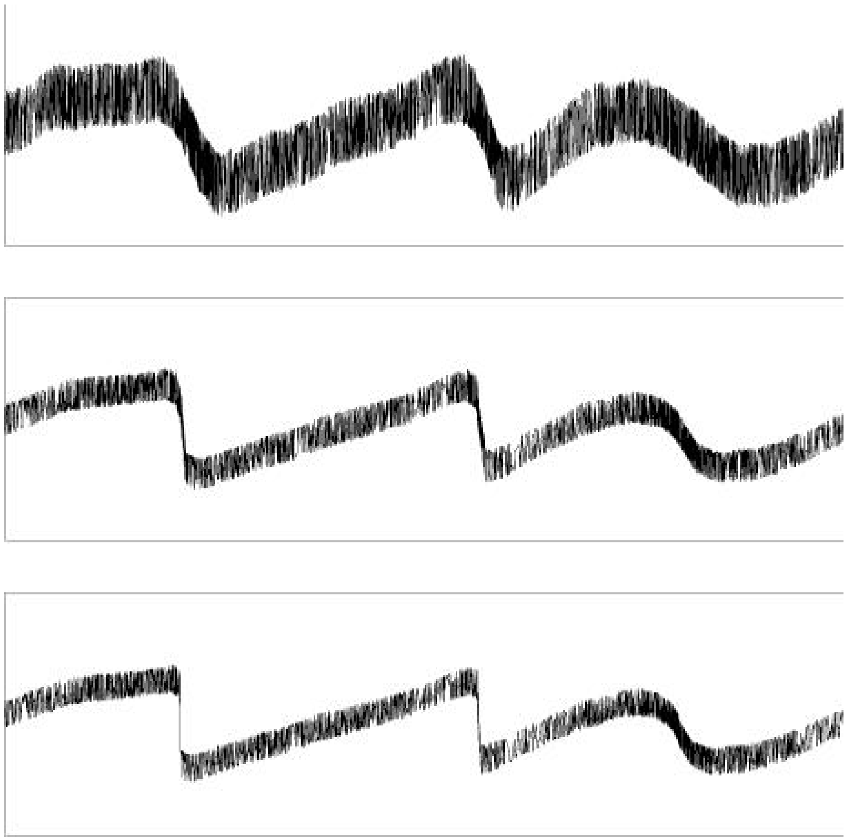
Formation of Singularity

- Collapse=shock formation
- Finite time singularity in

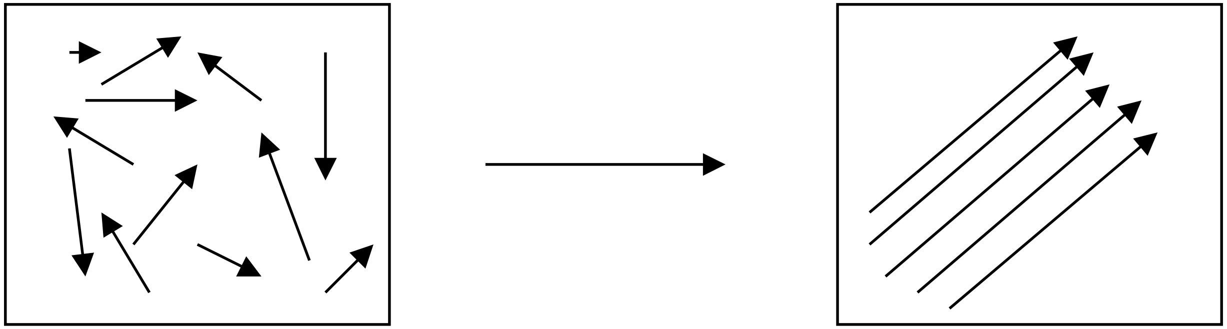
$$\frac{Dv}{Dt} = v_t + vv_x = 0$$

- Infinitesimal viscosity regularizes shocks
- Infinitesimal velocity cutoff regularizes inelastic collapse
- Singular limits

$$\nu \to 0, \quad \varepsilon \to 0$$

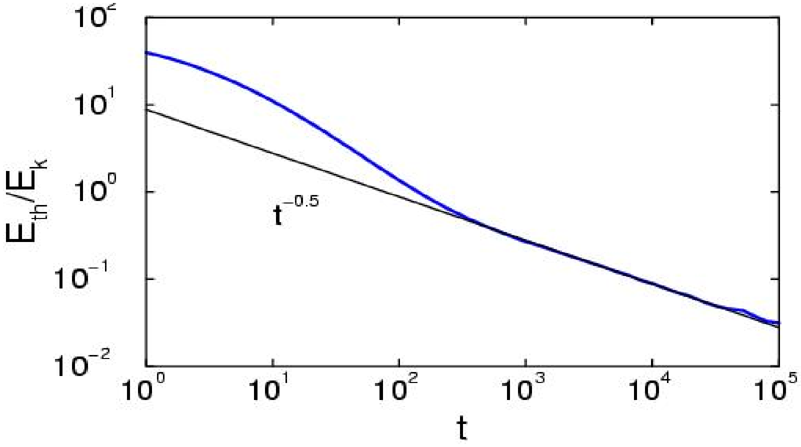


The Burgers equation (2D)



- **The ratio of thermal to kinetic energy vanishes**

$$E_{thermal}/E_{kinetic} \approx t^{-1/2}$$



- **Well defined velocity field**

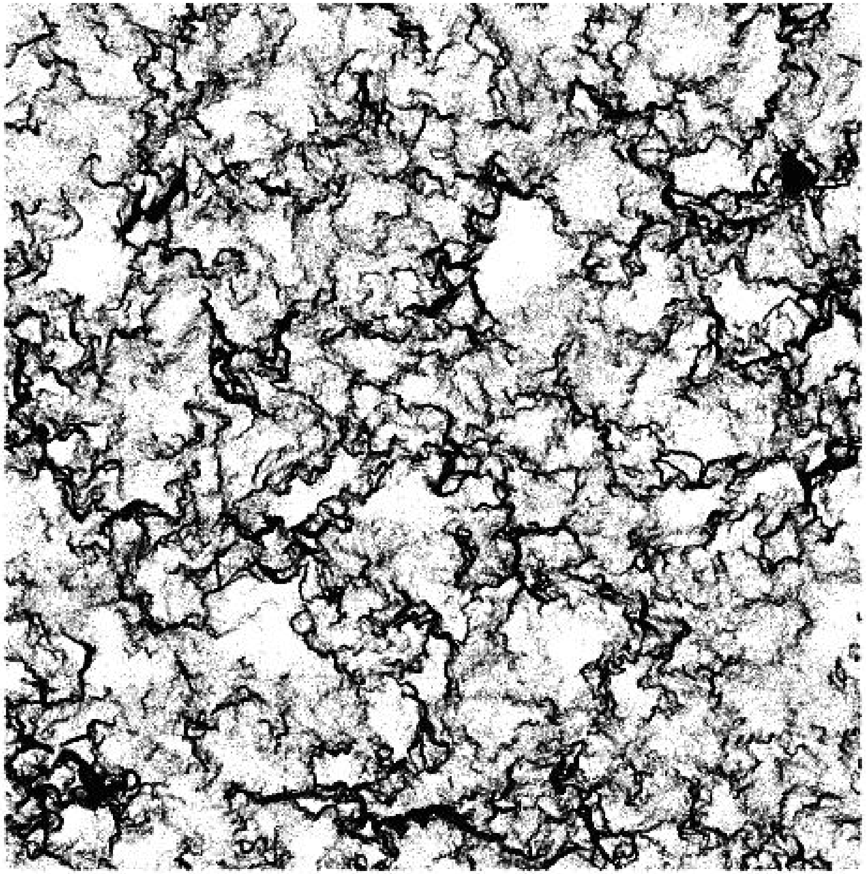
$$\Delta v/v \approx t^{-1/4}$$

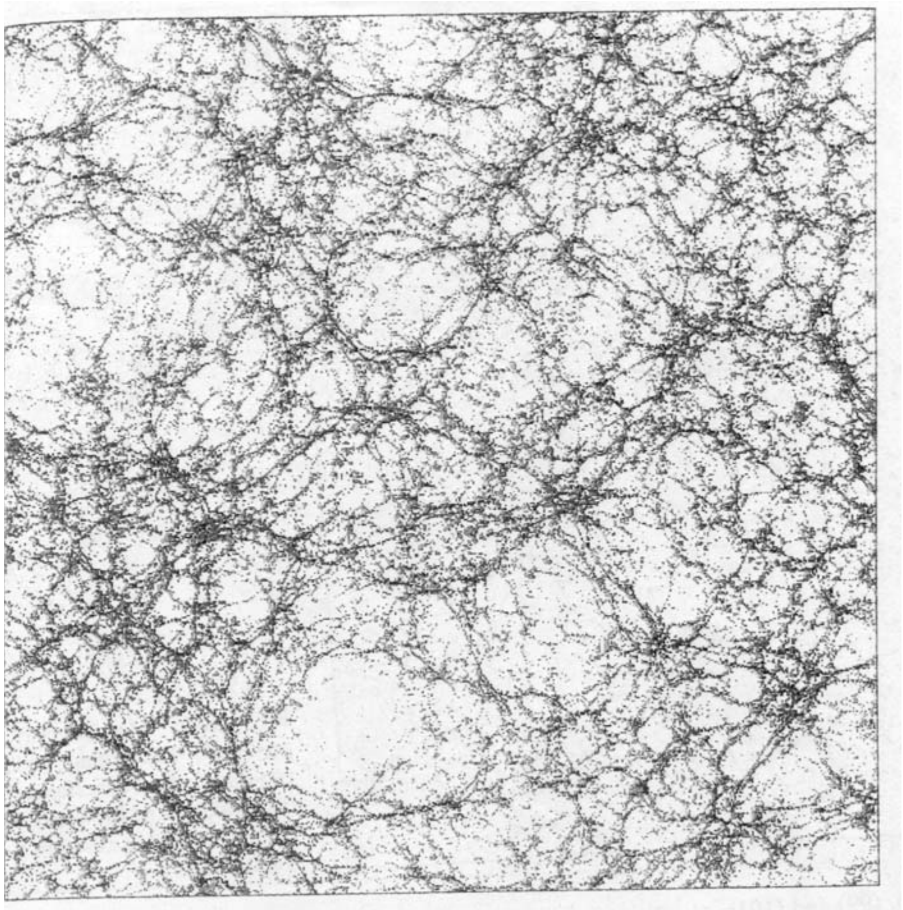
- **Effectively, the pressure normalizes to zero**

$$p \propto T, \quad \nu \propto T^{1/2}$$

Large scale formation of matter in universe?

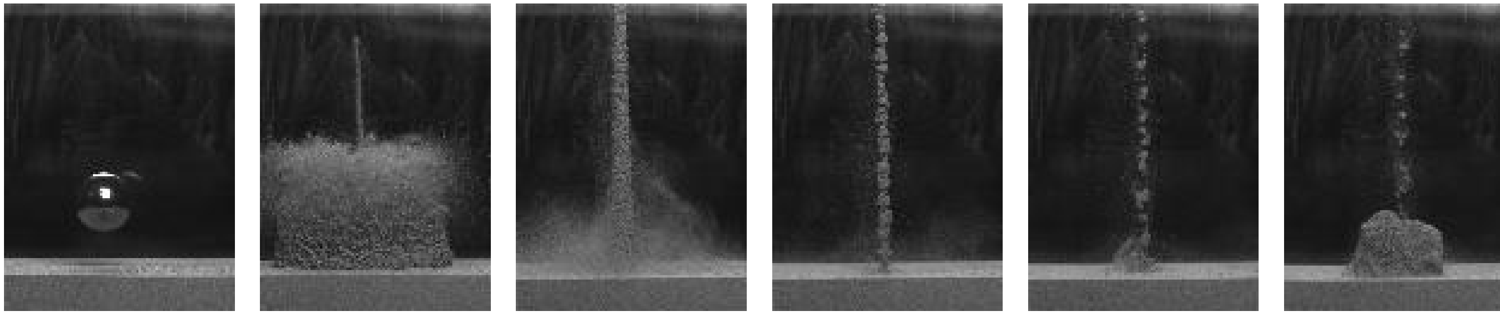
Granular gas vs. Burgers equation

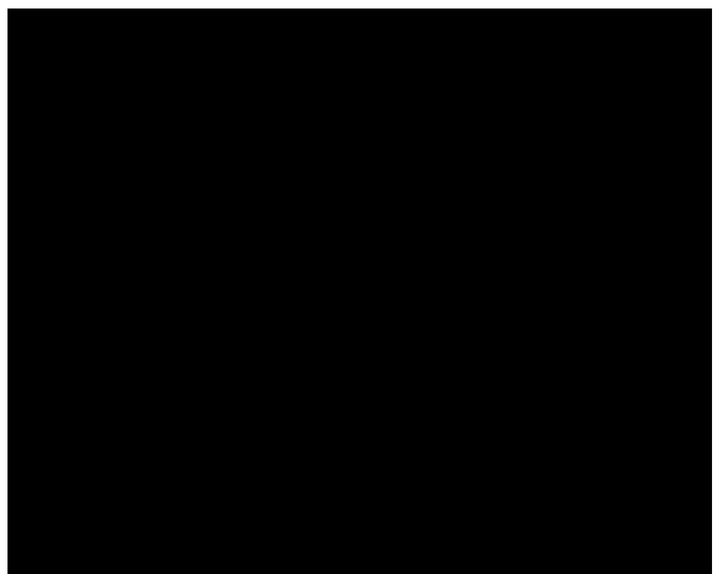




M Vergassola 1995

Granular Jets





D Lohse, A Shen 2001

Conclusions

- Asymptotic behavior
  - Governed by cluster-cluster coalescence
  - Universal: independent of inelasticity
  - Described by Inviscid Burgers equation
- Strong velocity and spatial correlations
- Energy balance equation is only approximate

Still, many open questions remain

Who, then, can calculate the course of a molecule? How do we know that the creation of worlds is not determined by the fall of grains of sand?

Victor Hugo, Les Miserables