Self-Similarity in Random Collision Processes

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Random Collision Processes

- Infinite particle system
- Binary collisions
- Random collision partners
- most general **linear** collision rules

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} p & q \\ q & p \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

- Special cases:
  1. Inelastic collisions: $p + q = 1$
  2. Kac model: $p^2 + q^2 = 1$
  3. Inelastic Lorentz gas: $p = 0$
  4. Addition/Aggregation: $p = q = 1$
- Momentum not necessarily conserved

What is nature of velocity distribution $P(v)$?

Kinetic Theory

- **Boltzmann equation**

$$\frac{\partial P(v)}{\partial t} = \int\int du_1 du_2 P(u_1)\, P(u_2) \times \left[\delta(v - pu_1 - qu_2) - \delta(v - u_1)\right]$$

- **Fourier transform**

$$F(k,t) = \int dv\, e^{ikv}\, P(v,t)$$

- **Closed equations**

$$\frac{\partial F(k,t)}{\partial t} + F(k,t) = F(pk,t)F(qk,t)$$

Theory is analytically tractable

Similarity Solutions (first kind)

- Temperature $T = \langle v^2 \rangle$

$$T(t) = T(0)e^{-\lambda t} \qquad \lambda = 1 - p^2 - q^2$$

- Assume temperature characterizes distribution

$$F(k,t) = f\left(kT^{1/2}\right) \quad \Leftrightarrow \quad P(v,t) \to T^{-1/2}\Phi\left(vT^{-1/2}\right)$$

- Giverning equation

$$-\frac{1}{2}\lambda z f'(z) + f(z) = f(pz)f(qz)$$

- Fourier transform is nonanalytic

$$f(z) = \left[1 - f_2 z^2 + \cdots\right] + \left[c\, z^\nu + \cdots\right]$$

- Velocity distribution has an algebraic tail

$$\Phi(w) \sim w^{-\nu-1}$$

- Transcendental exponent, root of

$$1 - p^2 - q^2 = \frac{2}{\nu}\left(1 - p^\nu - q^\nu\right) \qquad \nu > 2$$

Algebraic tail, continuously varying exponent

Similarity solutions (second kind)

- Unknown scale characterizes distribution

$$P(v,t) \to e^{\alpha t}\,\Phi\left(v e^{\alpha t}\right)$$

- Similarity solution $\equiv$ traveling wave

$$P(\ln v,t) \to F(\ln v + \alpha t)$$

- Exponent relation $\Leftrightarrow$ dispersion relation

$$\alpha = \nu^{-1}(1 - p^\nu - q^\nu)$$

- Extremum selection $\Rightarrow$ exponent equation

$$p^\nu \ln\frac{e}{p^\nu} + q^\nu \ln\frac{e}{q^\nu} = 1 \qquad \nu < 2$$

- Relevant in a region of parameters (p,q)
- Relevant for distribution with $M_2(0) = \infty$
- Weak selection $\nu = \min(\nu_{\rm dynamic}, \nu_{\rm initial})$

Similarity solution selected by dynamics

Traveling Waves

- FKPP population dynamics equation

$$u_t = u_{xx} + u(1-u)$$

- Travelling wave

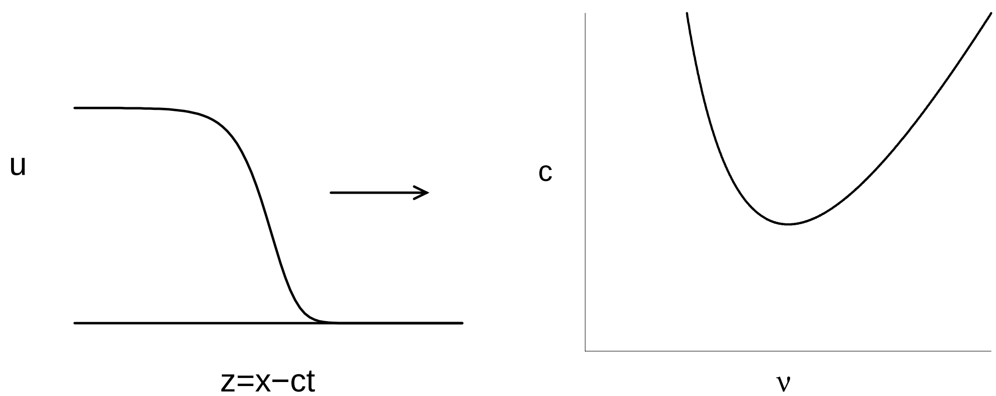
$$u(x,t) \to F(x - c\,t)$$

- Wave equation

$$F_{zz} + cF_z + F(1-F) = 0$$

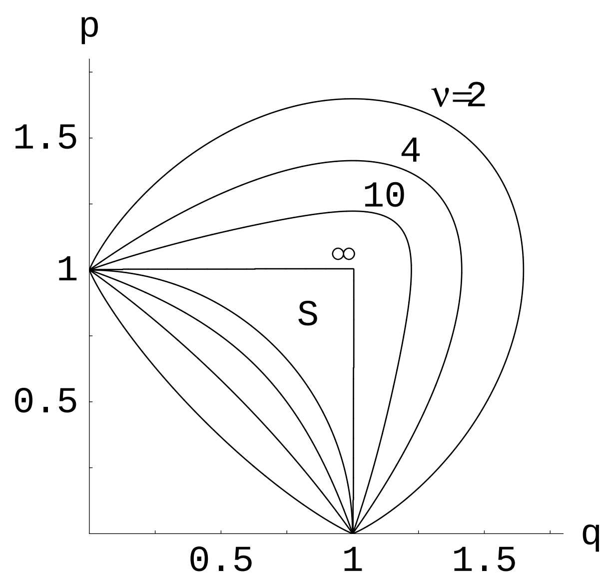
- Linear analysis $\to$ exponential tail $F \sim e^{-\nu z}$

$$c = \frac{\nu^2 + 1}{\nu} \qquad c > 2$$



Minimal velocity is selected: $c = 2$!

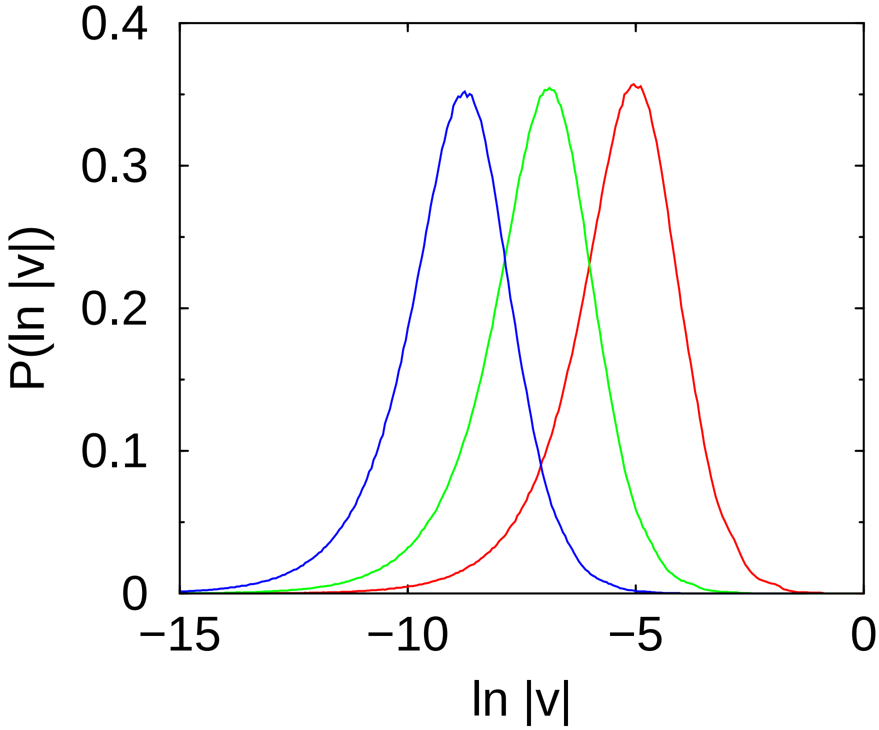
The Phase Diagram

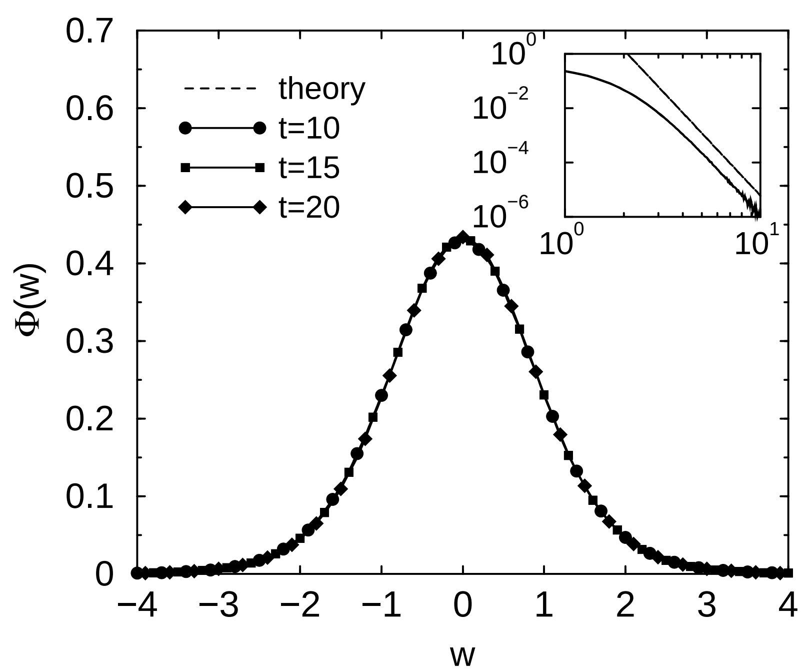


- ν varies continuously with parameters
- Every ν possible in quasi-elastic limit
- Region S: stretched exponentials

$$\boxed{\textbf{Every power-law possible}}$$

Numerical Confirmation





Steady states: stretched exponentials

- Steady state

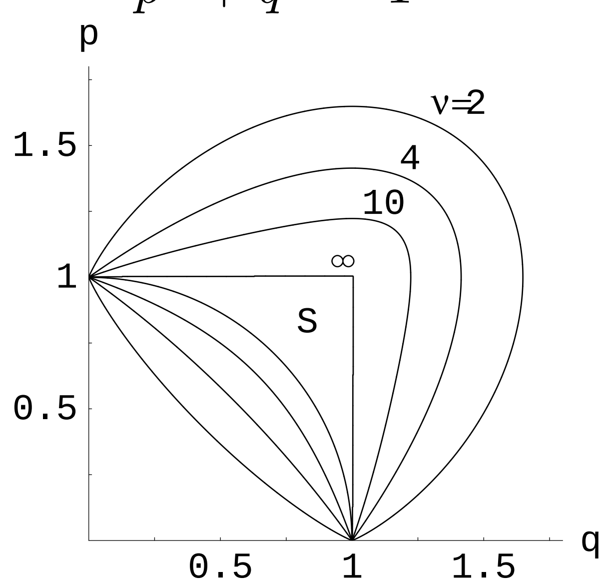
$$F(k) = F(pk)F(qk)$$

- Compatible with

$$F(k) \sim \exp\left[-k^{\mu}\right] \qquad \Leftrightarrow \qquad P(v) \sim \exp[-v^{\gamma}]$$

- Exponent equation: $\gamma = \mu/(\mu - 1) \geq 1$

$$p^{\mu} + q^{\mu} = 1$$



Every sharper than exponential tail possible

Conservation laws: universal distributions!

- Energy (average) conserved: $p^2 + q^2 = 1$

$$\Phi(w) = (2\pi)^{-1/2} \exp[-w^2/2]$$

- Momentum conserved: $p + q = 1$

$$\Phi(w) = \frac{2}{\pi}(1 + w^2)^{-2}$$

Inelastic Lorentz gas ($\nu = 0$)

- Unnormalized steady state solution

$$P(v) = v^{-1}$$

- Log-normal distribution:

$$\ln P(v) \propto -(\ln v)^2$$

Conclusions

- Two types of similarity solutions
- I Temperature characterizes distribution
- II "Hidden" scale characterizes distribution
- Traveling wave, extremum selection useful
- Universal functions when momentum or energy are conserved

Equivalent for inelastic hard spheres?