Jamming and tiling in aggregation of rectangles

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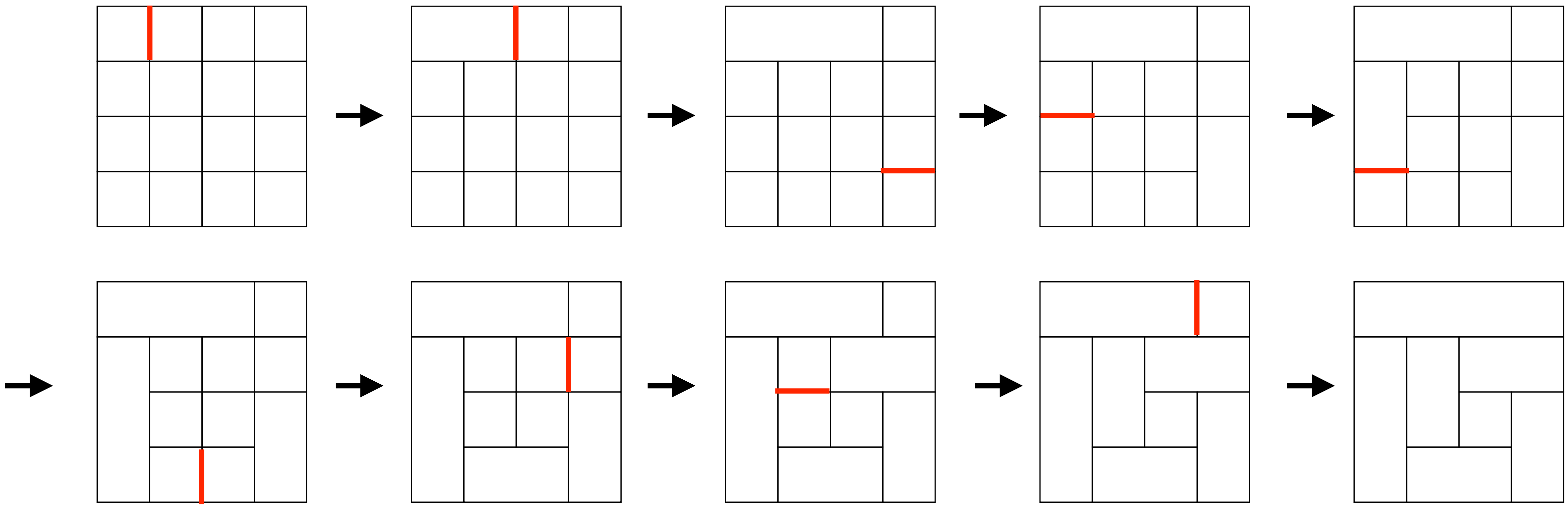
Jamming and tiling in two-dimensions

Pick two neighboring rectangles at random

Merge them if they are compatible

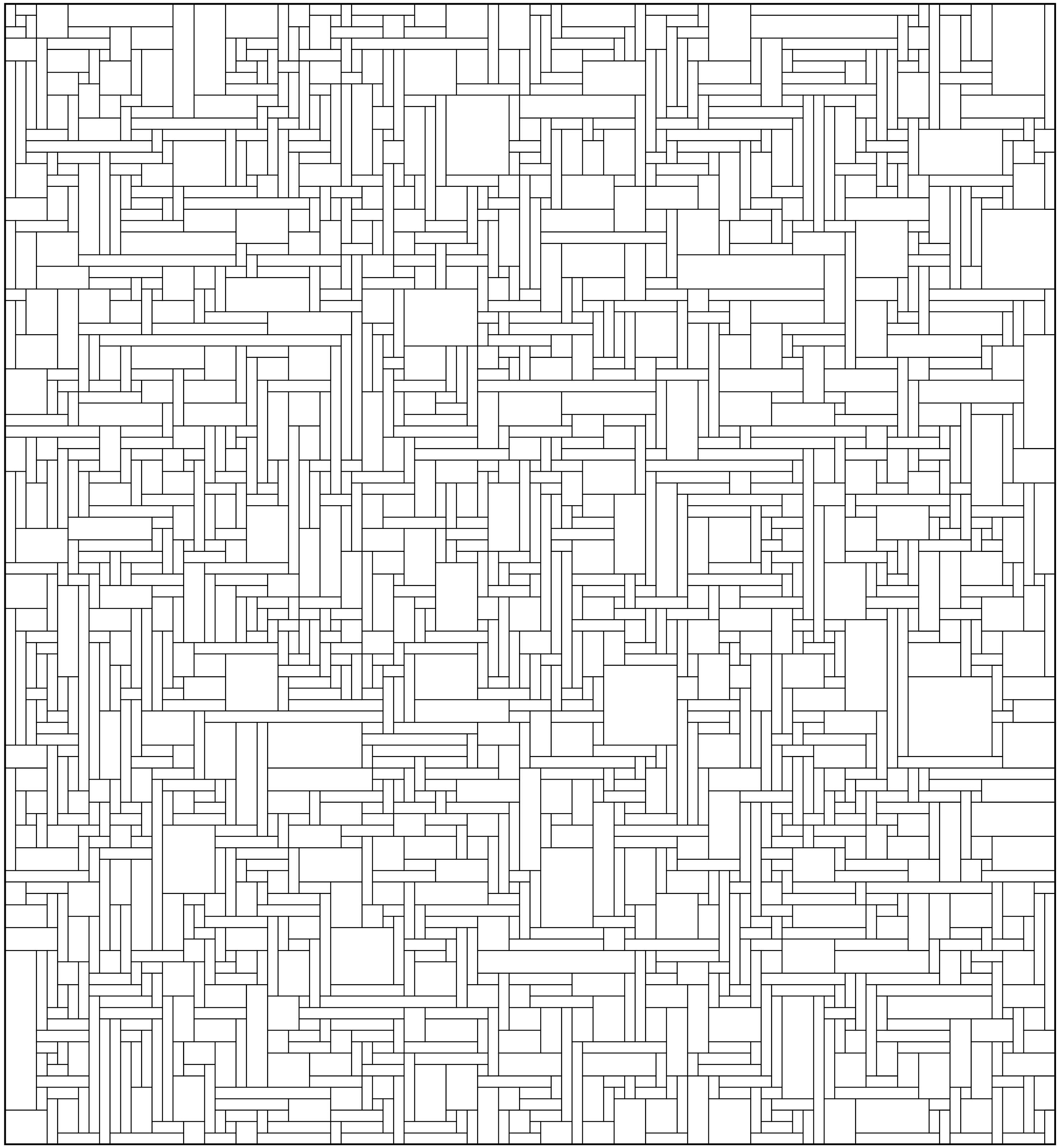


System reaches a jammed state

No two <u>neighboring</u> rectangles are compatible

The jammed state

no two neighbors share a common side



Kastelyan 61
Fisher 61
Lieb 67
Baxter 68
Kenyon 01

Features of the jammed state

- Local alignment
- Finite rectangle density

$$\rho = 0.1803$$

- Finite tile density

$$T = 0.009949$$

- Finite stick density

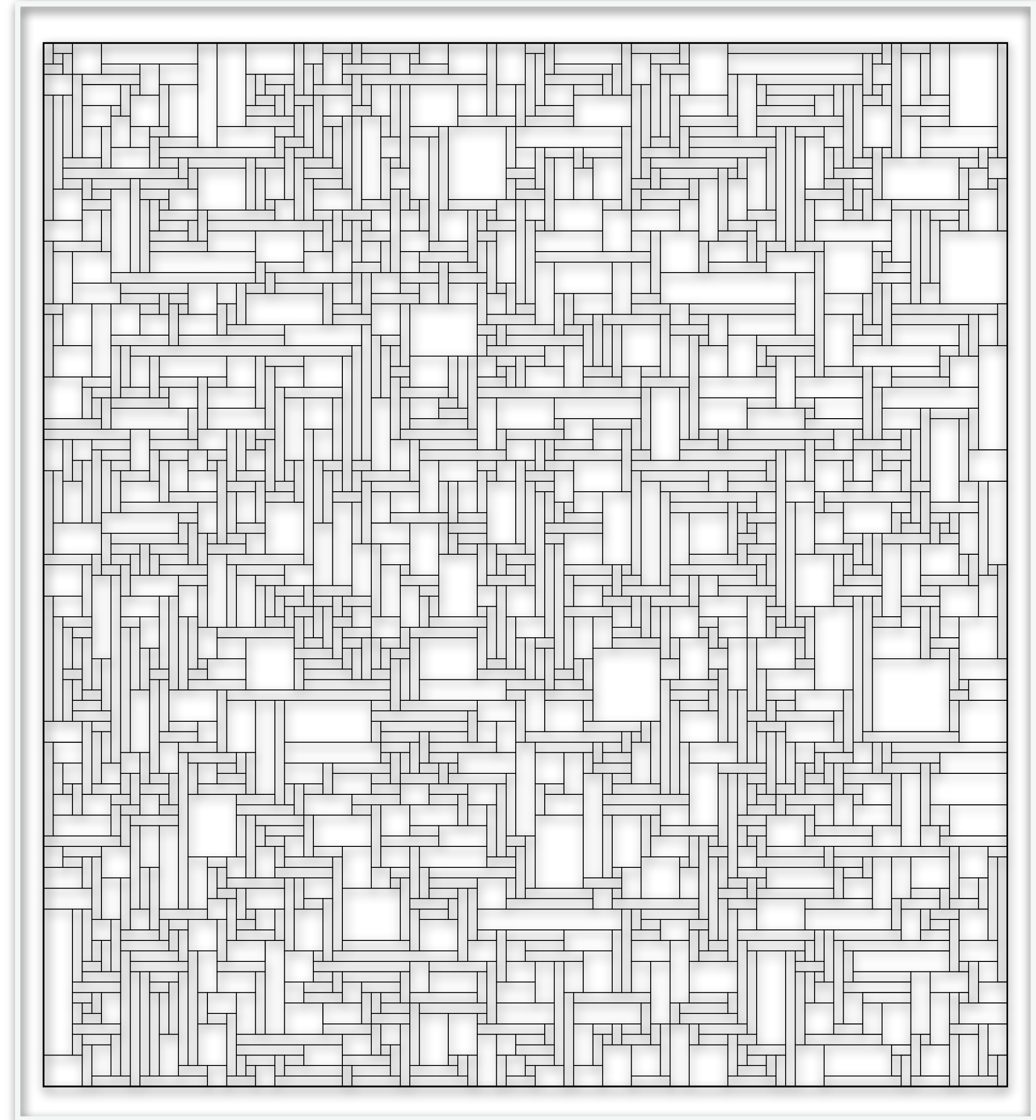
$$S = 0.1322$$

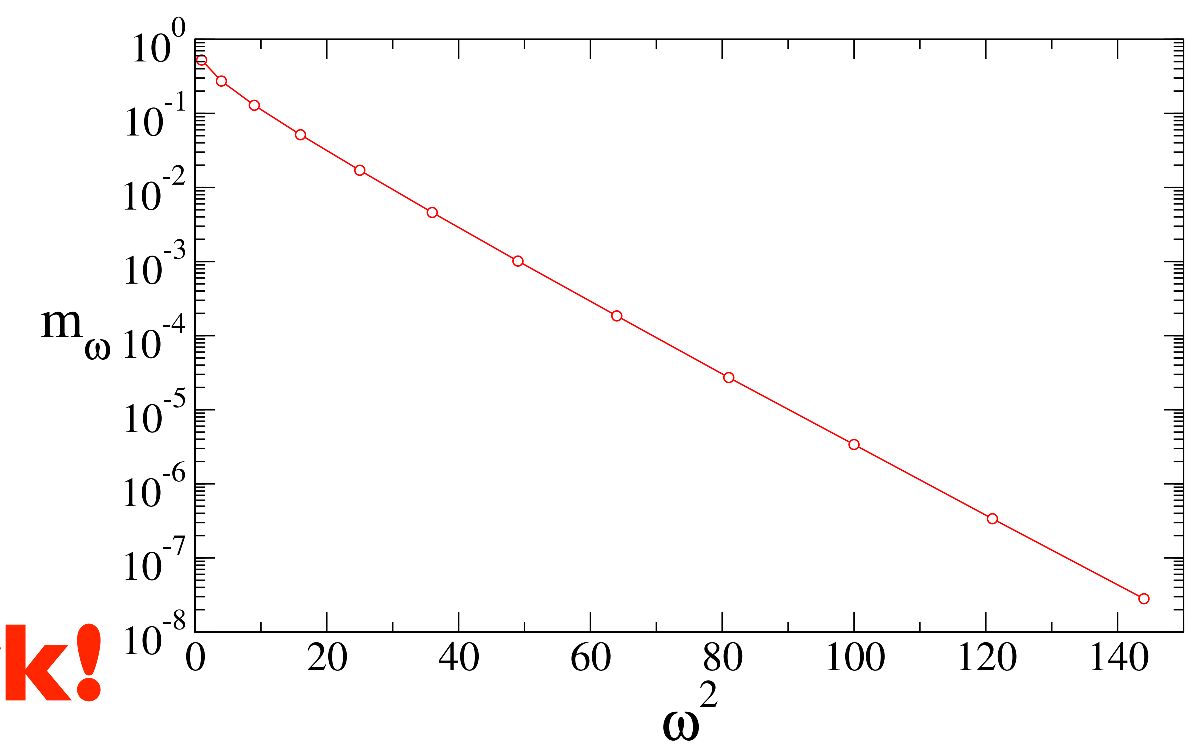
- Finite square density

$$H = 0.02306$$

- Area distribution of rectangles with width *w*

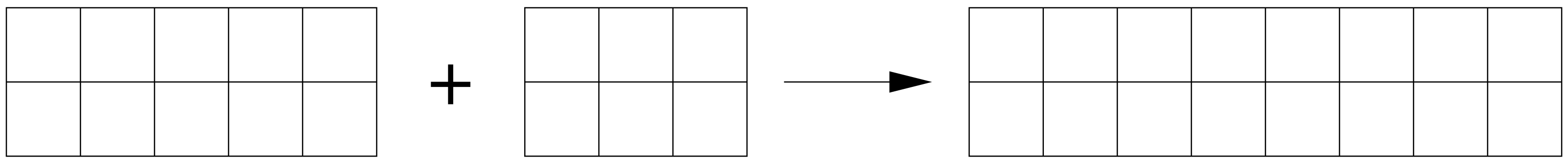
$$m_\omega \sim \exp\left(-\text{const.} \times \ \omega^2\right)$$





No theoretical framework!

Jamming: mean-field version



- Start with N 1x1 tiles (elementary building blocks)
- Pick two rectangles at random
- Pick an orientation at random (vertical or horizontal)
- Merge rectangles if they are perfectly compatible

$$(i_1, j) + (i_2, j) \to (i_1 + i_2, j)$$
$$(i, j_1) + (i, j_2) \to (i, j_1 + j_2)$$

- System is jammed when f rectangles have:

  f distinct horizontal sizes and f distinct vertical sizes

System reaches a jammed state

An example of a jammed state

- Characterize rectangle by horizontal and vertical size

$$(i, j)$$

- Characterize rectangle by maximal and minimal size

$$(\omega, \ell)$$

- Width = minimal size, Length = maximal length

$$\omega = \min(i, j) \qquad \ell = \max(i, j)$$

- Ordered widths of $f=13$ rectangles for $N=10,000$

$$\{1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 7, 9\}$$

Width sequence has gaps!

Number of jammed rectangles

- Average Number of rectangles grows algebraically with N

$$F \sim N^{\alpha}$$

- Nontrivial exponent

$$\alpha = 0.229 \pm 0.002$$

- Typical width of rectangles grows algebraically with N

$$\omega \sim N^{\alpha}$$

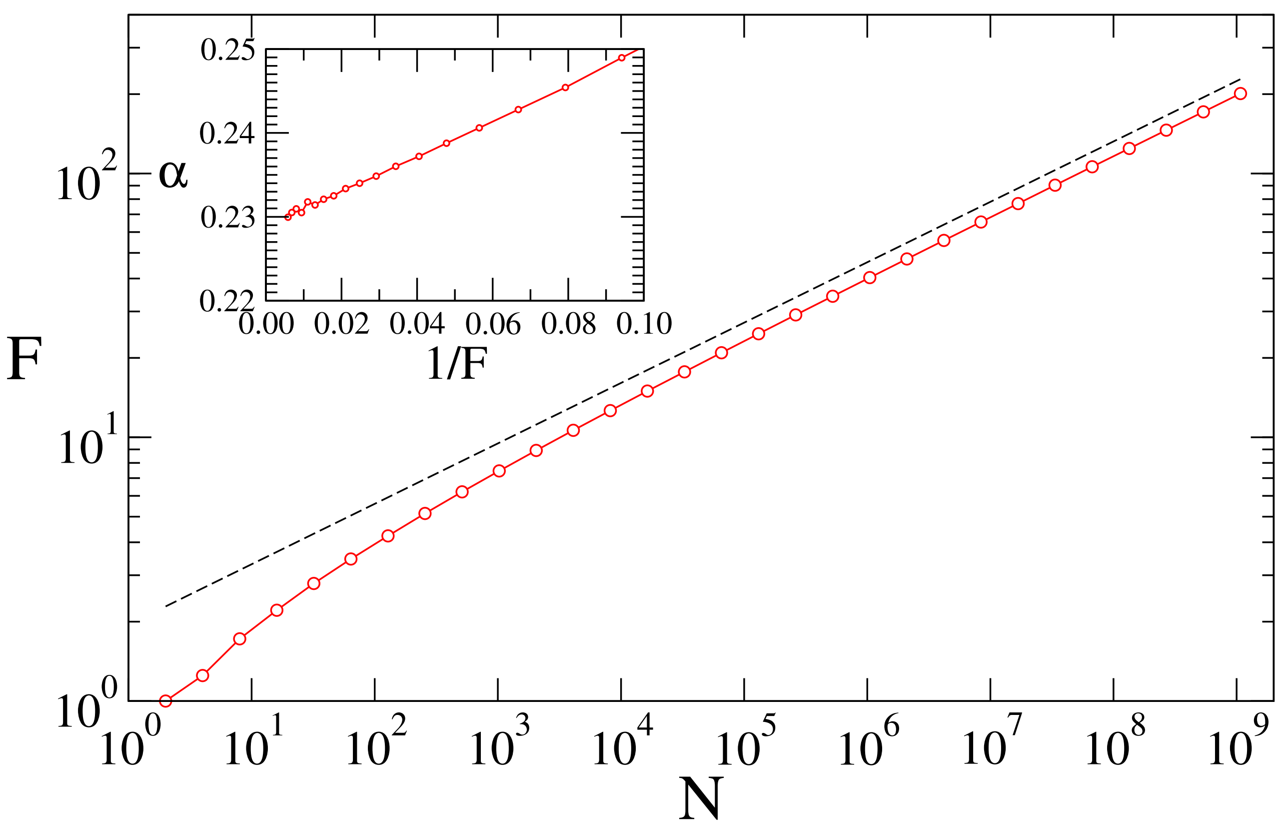
- Area density of rectangles of width w decays as a power law

$$m_{\omega} \sim \omega^{-\gamma} \quad \text{with} \quad \gamma = \alpha^{-1} - 2$$

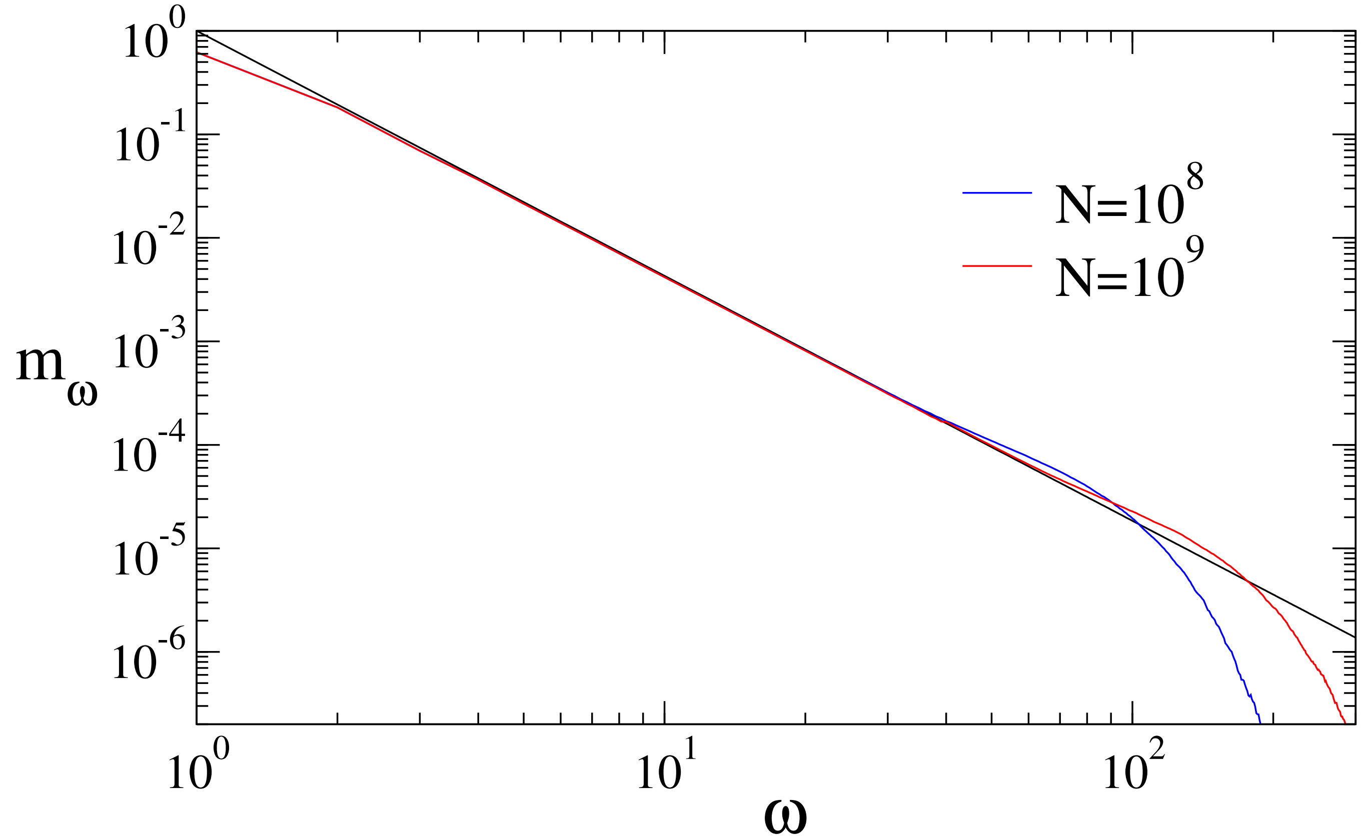
A single exponent characterizes the jammed state

Numerical simulations

$$F \sim N^{\alpha}$$



$$m_\omega \sim \omega^{-\gamma}$$



ω	1	2	3	4	5	6
m_ω	0.622	0.182	0.0694	0.0365	0.0214	0.0139
M_ω	0.622	0.804	0.873	0.910	0.931	0.945

Rectangles with finite width are macroscopic!

Rectangles with width 1,2,3,4,5 contain 95% of area

Still, the area distribution has a broad power-law tail!

Kinetic theory

Smoluchowski 1917

- Straightforward generalization of ordinary aggregation

$$\frac{dR_{i,j}}{dt} = \sum_{i_1+i_2=i} R_{i_1,j}R_{i_2,j} - 2R_{i,j}\sum_{k\geq 1} R_{k,j} + \sum_{j_1+j_2=j} R_{i,j_1}R_{i,j_2} - 2R_{i,j}\sum_{k\geq 1} R_{i,k}$$

- Allows calculation of the density of sticks

$$\frac{dS}{dt} = -S^2 - 2\sum_{i,j} R_{1,j}R_{i,j}$$

- Simple decay for the stick density and jamming time

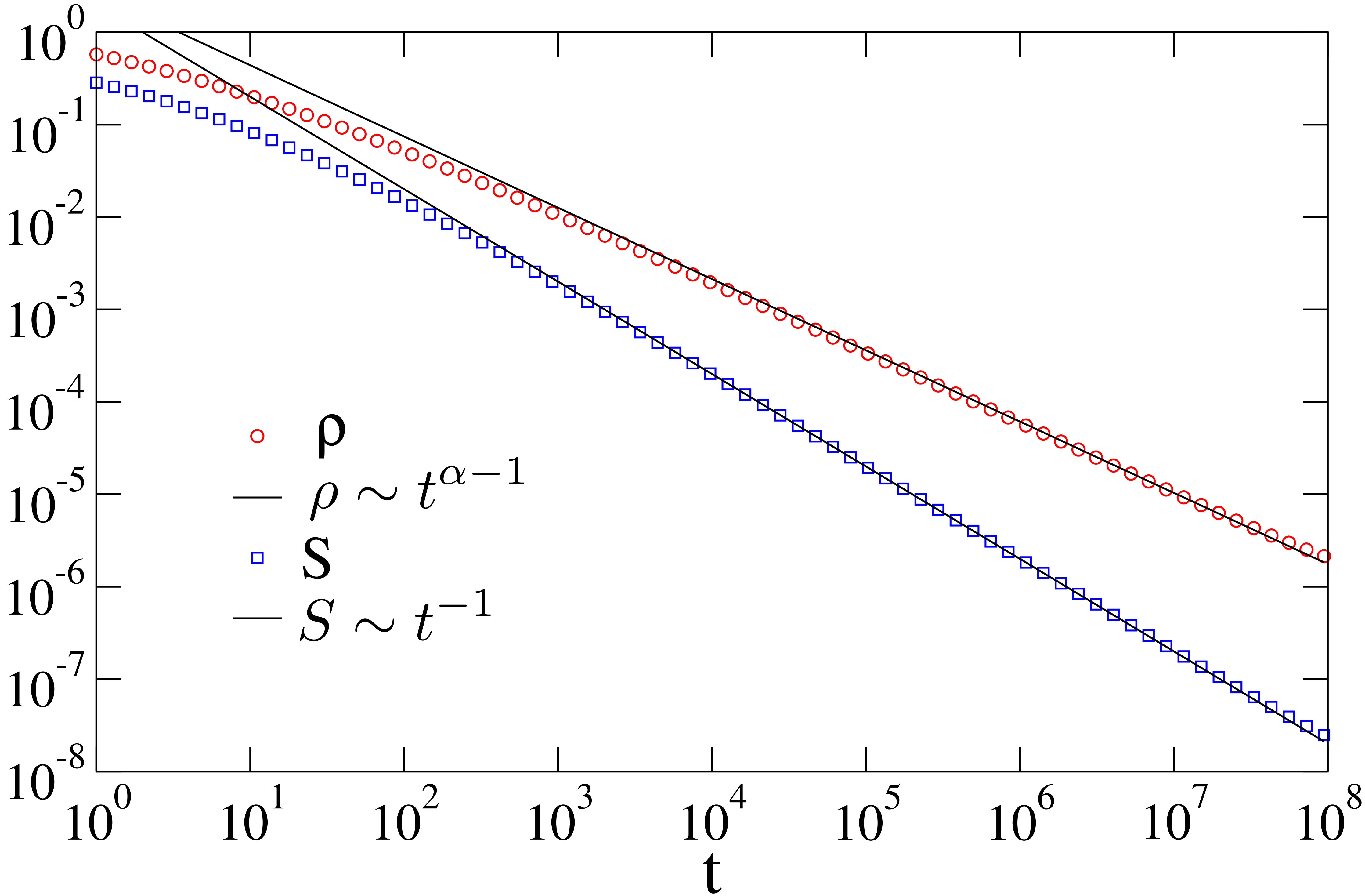
$$S \simeq t^{-1} \quad \Longrightarrow \quad \tau \sim N$$

- Jammed state properties give density decay and width growth

$$\rho \sim t^{\alpha-1} \qquad \text{and} \qquad w \sim t^{\alpha}$$

Jamming exponent characterizes the kinetics, too

Numerical validation

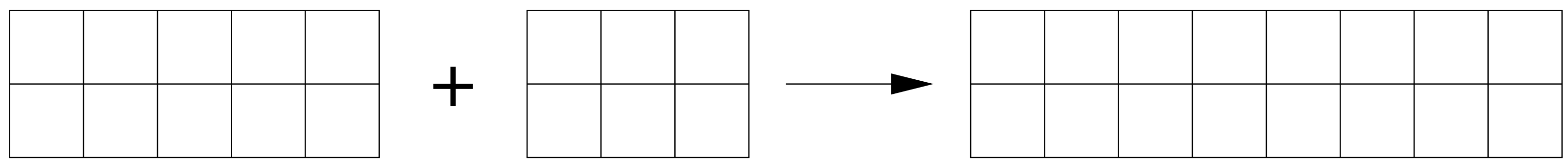


Numerics validate approximation

Suggest two aggregation modes: elongating and widening

Primary aggregation: elongation

- Aggregation between two rectangles of same **width**



- Ordinary aggregation equation (example: sticks)

$$\frac{dR_{1,\ell}}{dt} = \sum_{i+j=\ell} R_{1,i}R_{1,j} - 2SR_{1,\ell} - 2\left(\sum_i R_{i,\ell}\right)R_{1,\ell}$$

- Length distribution as in $d=1$, length grows linearly $l \sim t$

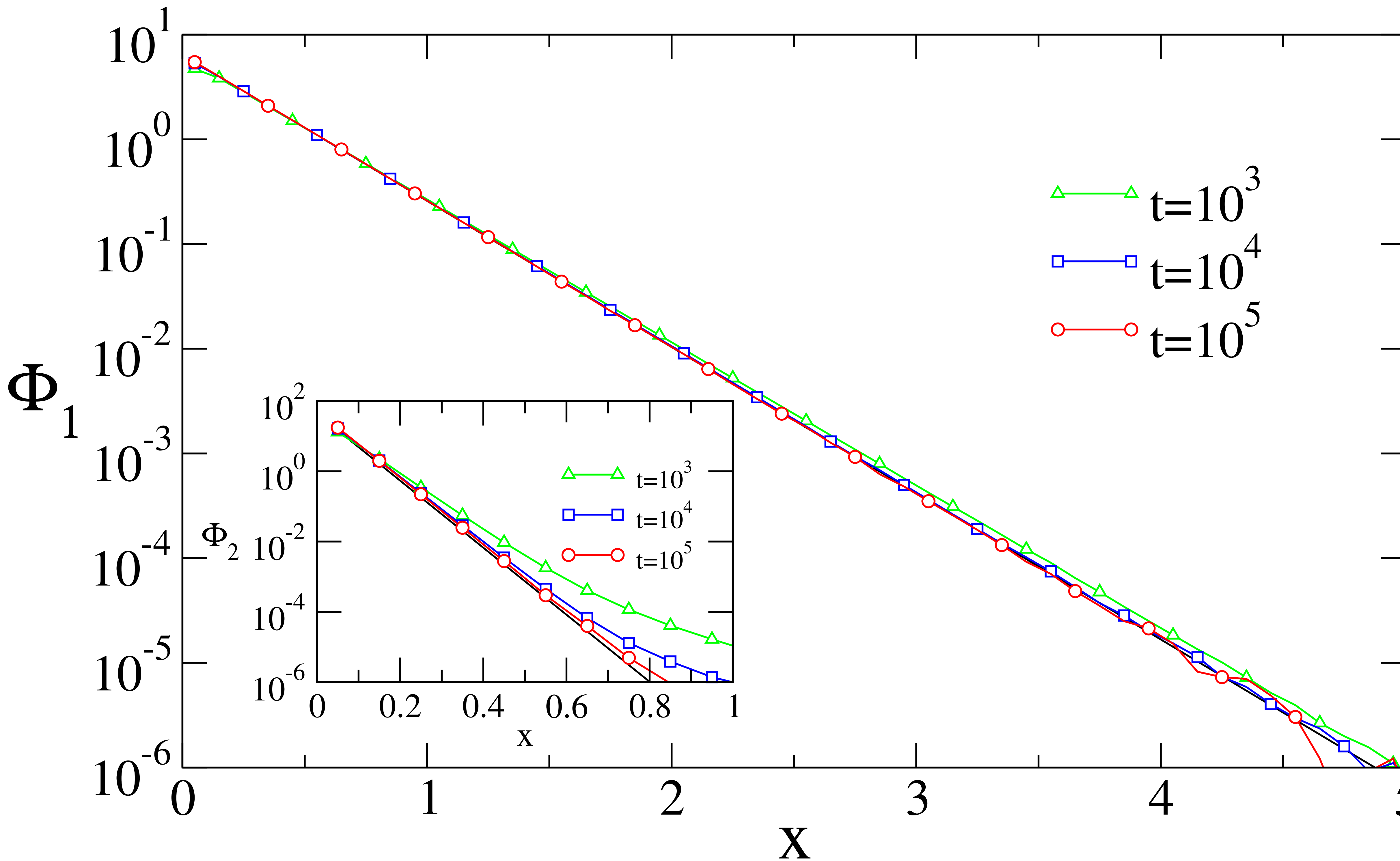
$$R_{1,\ell} \simeq (2/m_1 t^2)\exp(-2\ell/m_1\, t)$$

- Behavior extends to all rectangles with finite width

$$\mathcal{R}_{\omega,\ell}(t) \simeq t^{-2}\,\Phi_\omega\big(\ell\, t^{-1}\big) \quad \text{with} \quad \Phi_\omega(x) = (2\omega/m_\omega)\exp(-2\omega x/m_\omega)$$

Finite width: problem reduces to one-dimensional aggregation
However, total mass for each width is not known

Numerical validation

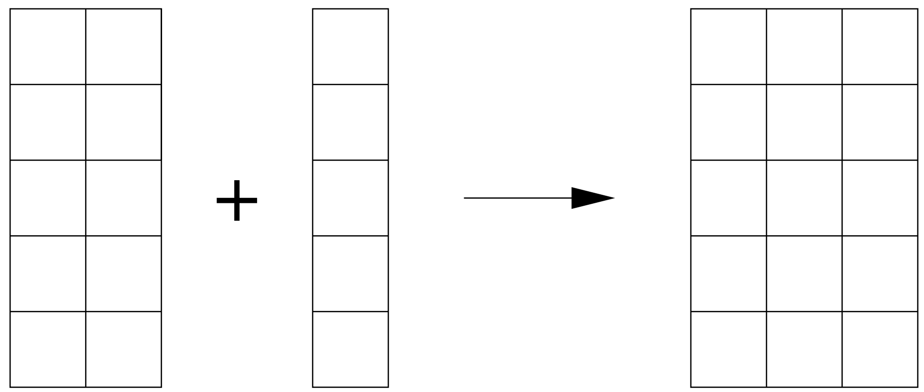


Exponential scaling function

total mass set by the jammed state

Secondary aggregation: widening

- Aggregation between two rectangles of same **length**



- The area fraction is coupled to the size distribution

$$\frac{dm_\omega}{dt} = \frac{1}{2}\sum_{i+j=\omega}\sum_{\ell}\omega\ell\mathcal{R}_{i,\ell}\mathcal{R}_{j,\ell} - \sum_{j}\sum_{\ell}\omega\ell\mathcal{R}_{j,\ell}\mathcal{R}_{\omega,\ell}$$

- Insights about relaxation toward jammed state $\mu_\omega = \frac{2\omega}{m_\omega}$

$$m_\omega(t) - m_\omega(\infty) \simeq C_\omega\, t^{-1} \quad \text{with} \quad C_\omega = -2\omega \sum_{i+j=\omega} \frac{\mu_i \mu_j}{(\mu_i+\mu_j)^2} + 4\omega \sum_{j} \frac{\mu_\omega \mu_j}{(\mu_\omega+\mu_j)^2}$$

Closure & theoretical determination of α remains elusive

Conclusions

- Random aggregation of compatible rectangles
- Process reaches a jammed state where all rectangles are incompatible
- Number of jammed rectangle grows as power-law
- Area distribution decays as a power law
- A single, nontrivial, exponent characterize both the jammed state and the time-dependent behavior
- Primary aggregation: rectangles of same width
- Secondary aggregation: rectangles of same length
- Slow transfer of "mass" from thin to wide rectangles
- Kinetic theory successfully describes primary aggregation process only