Scaling in Tournaments

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We study a stochastic process that mimics single-game elimination tournaments. In our model, the outcome of each match is stochastic: the weaker player wins with upset probability $q \leq 1/2$, and the stronger player wins with probability $1-q$. The loser is eliminated. Extremal statistics of the initial distribution of player strengths governs the tournament outcome. For a uniform initial distribution of strengths, the rank of the winner, x_*, decays algebraically with the number of players, N, as $x_* \sim N^{-\beta}$. Different decay exponents are found analytically for sequential dynamics, $\beta_{\rm seq} = 1-2q$, and parallel dynamics, $\beta_{\rm par} = 1 + \frac{\ln(1-q)}{\ln 2}$. The distribution of player strengths becomes self-similar in the long time limit with an algebraic tail. Our theory successfully describes statistics of the US college basketball national championship tournament.

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A wide variety of processes in nature and society involve competition. In animal societies, competition is responsible for social differentiation and the emergence of social strata. Competition is also ubiquitous in human society: auctions, election of public officials, city plans, grant awards, and sports involve competition. Minimalist, physics-based competition processes have been recently developed to model relevant competitive phenomena such as wealth distributions [1–3], auctions [4–6], social dynamics [7–10], games [11], and sports leagues [12]. In physics, competition also underlies phase ordering kinetics, in which large domains grow at the expense of small domains that eventually are eliminated [13, 14].

In this study, we investigate N-player tournaments with head-to-head matches. The winner of each match remains in the tournament while the loser is eliminated. At the end of a tournament, a single undefeated player, the tournament winner, remains. Each player is endowed with a fixed intrinsic strength $x \geq 0$ that is drawn from a normalized distribution $f_0(x)$. We define strength so that smaller x corresponds to a stronger player and we henceforth refer to this strength measure as "rank".

The result of competition is stochastic: in each match the weaker player wins with the upset probability $q \leq 1/2$ and the stronger player wins with probability $p = 1-q$. Schematically, when two players with ranks x_1 and x_2 compete, assuming $x_1 < x_2$, the outcome is:

$$(x_1, x_2) \to \begin{cases} x_1 & \text{with probability } 1-q; \\ x_2 & \text{with probability } q. \end{cases} \tag{1}$$

For $q = 0$, the best player is always victorious, while for $q = 1/2$, game outcomes are completely random. We are interested in the evolution of the rank distribution, as well as the rank of the tournament winner.

We find that the rank of the winner, x_*, decays algebraically with the number of players N as

$$x_* \sim N^{-\beta} \tag{2}$$

with the exponent $\beta \equiv \beta(q)$ a function of the upset probability. When the ranks of the tournament players are uniformly distributed, we find different values for sequential and parallel dynamics: $\beta_{\rm seq} = 1-2q$ and $\beta_{\rm par} = 1 + \frac{\ln(1-q)}{\ln 2}$. Moreover, the rank distribution becomes asymptotically self-similar and has a power-law tail. We also extend these results to arbitrary initial distributions. The extreme of this distribution governs statistical properties of the rank of the ultimate winner.

Sequential Dynamics. We formulate the competition process by assuming that each pair of players compete at a constant rate. In this formulation, games are held sequentially, and players are eliminated from the tournament one at a time. The fraction of players remaining in the competition at time t, $c(t)$, decays according to

$$\frac{dc}{dt} = -c^2. \tag{3}$$

Solving this equation subject to the initial condition $c(0) = 1$, the surviving fraction is

$$c(t) = (1+t)^{-1}. \tag{4}$$

The tournament ends with a single player and this occurs at time t_*, that can be estimated from $c(t_*) \sim N^{-1}$. Therefore the time to complete the competition scales linearly with the number of players $t_* \sim N$.

Let $f(x,t)\,dx$ be the fraction of *remaining* players with rank in the range $(x, x+dx)$ at time t. The density $f(x,t)$ obeys the nonlinear integro-differential equation

$$\frac{\partial f(x)}{\partial t} = -2p\, f(x) \int_0^x dy f(y) - 2q\, f(x) \int_x^\infty dy f(y). \tag{5}$$

The first term accounts for games where the favorite wins and the second term for games where the underdog wins. The initial condition is $f(x,0) = f_0(x)$ with $\int dx f_0(x) = 1$. Integrating (5), the total fraction of remaining players, $c(t) = \int dx f(x,t)$, indeed decays according to (3). We note that this master equation is exact in the limit of an infinite number of players and applicable only as long as the fraction of remaining players is finite.

The rank distribution can be obtained by introducing the cumulative distribution $F(x)$, defined as the fraction of players with rank smaller than x,

$$F(x) = \int_0^x dy f(y). \tag{6}$$

The distribution of player ranks is obtained from the cumulative distribution by differentiation, $f(x) = dF(x)/dx$. By integrating the master equation (5), the cumulative distribution obeys the closed nonlinear equation

$$\frac{\partial F}{\partial t} = (2q-1)F^2 - 2qcF. \tag{7}$$

The initial condition is $F(x,0) = F_0(x) = \int_0^x dy f_0(y)$. Substituting $H(x) = 1/F(x)$, we transform (7) to the linear equation

$$\frac{\partial H}{\partial t} = (1-2q) + 2qcH. \tag{8}$$

Integrating this equation with respect to time, we find $H(x) = [H_0(x) - 1](1+t)^{2q} + (1+t)$. Substituting the initial condition $H_0(x) = 1/F_0(x)$, we obtain the cumulative rank distribution

$$F(x,t) = \frac{F_0(x)}{[1-F_0(x)](1+t)^{2q} + F_0(x)(1+t)}. \tag{9}$$

From this, the actual density of player rank is obtained by differentiation

$$f(x,t) = \frac{f_0(x)(1+t)^{2q}}{\left[(1-F_0(x))(1+t)^{2q} + F_0(x)(1+t)\right]^2}. \tag{10}$$

Notice that when the game outcome is random, $q = 1/2$, the normalized distribution of rank does not evolve with time as $f(x,t)/c(t) = f_0(x)$.

Uniform Initial Distribution. Consider first the special case of a uniform initial distribution, $f_0(x) = 1$ for $0 \le x \le 1$, and deterministic games, $q = 0$. Then the initial cumulative distribution is $F_0(x) = x$ for $x \le 1$ and $F_0(x) = 1$ for $x \ge 1$. The time-dependent cumulative distribution (9) is

$$F(x,t) = \frac{x}{1+xt} \tag{11}$$

for $x \le 1$ and $F(x,t) = c(t)$ for $x \ge 1$. Similarly, the rank distribution itself is $f(x,t) = (1+xt)^{-2}$ for $0 \le x \le 1$. As expected, weaker players are more likely to be eliminated as the tournament proceeds and the remaining field becomes stronger. Quantitatively, the average rank of surviving players, $\langle x \rangle = \int dx\, x f(x) / \int dx f(x)$, is

$$\langle x \rangle = t^{-2}\left[(1+t)\ln(1+t) - t\right]. \tag{12}$$

Therefore, the average rank asymptotically decays with time, $\langle x \rangle \simeq t^{-1} \ln t$.

We can write the cumulative distribution in the scaling form $F(x,t) \to t^{-1}\Phi(xt)$, by multiplying and dividing (11) by time. Here, the scaling function is $\Phi(z) = \frac{z}{1+z}$, which approaches unity $\Phi(z) \to 1$ when $z \to \infty$, consistent with total density decay $c \simeq t^{-1}$. In the long time limit, the cumulative distribution retains the same shape as the initial distribution, $\Phi(z) \simeq z$, for $z \ll 1$. The scaling variable $z = xt$ indicates that players with rank larger than the characteristic rank $x \sim t^{-1}$ are eliminated from the tournament.

Let us generalize these results to arbitrary q. In this case, the cumulative distribution is

$$F(x,t) = \frac{x}{(1-x)(1+t)^{2q} + x(1+t)}, \tag{13}$$

for $x \le 1$ and $F(x,t) = c(t)$ otherwise. In the long time limit, we may replace $1+t$ with t, and also replace $1-x$ with 1, since the rank decays with time. Then the cumulative distribution approaches the scaling form

$$F(x,t) \to t^{-1}\Phi\left(x\, t^{1-2q}\right). \tag{14}$$

The scaling function remains as above

$$\Phi(z) = \frac{z}{1+z}. \tag{15}$$

The scaling form (14) implies that the typical rank decays algebraically with time

$$x \sim t^{-(1-2q)}. \tag{16}$$

Interestingly, the exponent governing this decay depends on the upset probability. The larger the upset probability, the smaller the decay exponent. Thus weaker players can persist in a tournament when q approaches 1/2. For completely random games, $q = 1/2$, the exponent vanishes and the strength of the typical surviving player does not change with time.

A similar scaling law characterizes the rank of the tournament winner. From (4), the number of players remaining in the tournament, M, and the initial number of players N, are related by $t \sim N/M$. Using (16), when M players remain, the typical rank is $x \sim (N/M)^{-(1-2q)}$. Substituting $M = 1$, we find that the typical rank of the winner decays algebraically with the total number of players, as in (2), with the exponent

$$\beta_{\rm seq} = 1 - 2q. \tag{17}$$

Therefore, the smaller the tournament or the higher the upset probability the weaker the winner, on average. We note that due to strong fluctuations, the master equation (5) is not applicable when the number of players is of order one, and consequently, our theoretical framework can not be used to obtain the distribution of the tournament winner.

General Initial Distributions. Our findings in the case of uniform distributions suggest that the behavior of the initial distribution in the $x \to 0$ limit governs the long time asymptotics. Let us consider rank distributions with a power-law behavior near the origin,

$$F_0(x) \simeq C\, x^{\mu+1}, \tag{18}$$

as $x \to 0$ with $\mu > -1$ so that the distribution is normalized. The rank density then scales as $f_0(x) \simeq C(\mu+1)x^{\mu}$. Since the rank x decays with time, the term $(1-F_0)(1+t)^{2q}$ in the denominator of (9) can be replaced by t^{2q} and similarly, the term $F_0(x)(1+t)$ can be replaced by $Cx^{\mu+1}t$. Therefore, the scaling form (14) becomes $F(x,t) \to t^{-1}\Phi\left(x\,t^{\frac{1-2q}{\mu+1}}\right)$, with the scaling function $\Phi(z) = Cz^{\mu+1}/(1+Cz^{\mu+1})$. Thus the typical player rank decays with time according to $x \sim t^{-\frac{1-2q}{\mu+1}}$. Similarly, the rank of the winner decays with the number of players as in (2) with $\beta_{\rm seq} = \frac{1-2q}{\mu+1}$.

Like the cumulative distribution, the density of players with given rank also becomes self-similar asymptotically, $f(x,t) \to t^{\beta-1}\phi(x\,t^{\beta})$ with $\beta = \frac{1-2q}{\mu+1}$ and $\phi(z) = \Phi'(z)$. As noted earlier, the shape of the distribution is preserved: $f(z) \sim z^{\mu}$ as $z \to 0$. The large argument behavior is

$$\phi(z) \sim z^{-\mu-2}, \tag{19}$$

as $z \to \infty$. The algebraic decay shows that the likelihood of finding weak players in the tournament is appreciable. Surprisingly, when initially most players are strong they can eliminate each other, leading to an appreciable probability for weak players to survive.

The scaling behavior (2) refers to the typical rank of the winner. The algebraic tail (19) suggests that the average rank may scale differently than the typical rank. For example, for compact uniform distributions ($\mu = 0$), the average is characterized by a logarithmic correction as in (12), $\langle x_* \rangle \sim N^{-(1-2q)} \ln N$.

Parallel Dynamics. Thus far, we addressed sequential games with a single team eliminated at a time. However, actual sports tournaments typically proceed via rounds of parallel play with half of the teams eliminated in each round. We thus consider such round-play tournaments with $N = 2^k$ players. Let $g_N(x)$ be the normalized distribution of the rank of the winner with $\int dx\, g_N(x) = 1$ and let $G_N(x) = \int_0^x dy\, g_N(y)$ be the corresponding cumulative distribution.

Consider first a tournament with $N = 2$ players. Similar to Eq. (5), the rank distribution of the winner is

$$g_2(x) = 2pg_1(x)[1-G_1(x)] + 2qg_1(x)G_1(x). \tag{20}$$

Integrating this equation, we arrive at an explicit expression for the distribution of the rank of the winner $G_2(x) = 2pG_1(x) + (1-2p)[G_1(x)]^2$. Clearly, this nonlinear recursion relation applies to every round of the tournament and therefore,

$$G_{2N}(x) = 2pG_N(x) + (1-2p)[G_N(x)]^2. \tag{21}$$

Iterating this equation starting with $G_1(x)$, we obtain explicit expressions for the distribution of the winner for $N = 2, 4, 8, \ldots$ Explicit expressions can be obtained for the extreme cases of deterministic competitions ($q = 0$) where $1 - G_N(x) = [1 - G_1(x)]^N$ and random competitions ($q = 1/2$) where $G_N(x) = G_1(x)$.

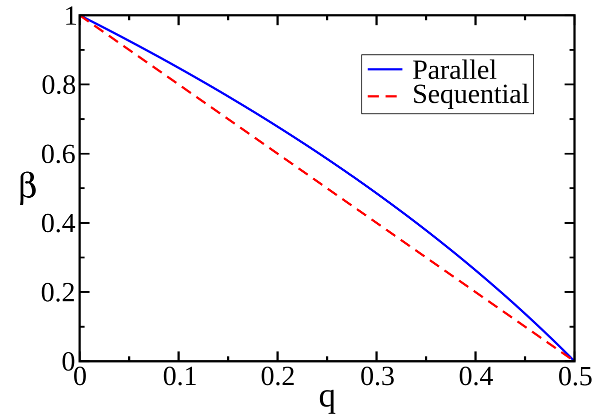

FIG. 1: The decay exponent β versus the upset probability q. Shown are the values for the sequential case (17) and the parallel case (24).

Let us restrict our attention to uniform initial distributions, $G_1(x) = x$ for $x \le 1$. For small-x, we may neglect the nonlinear term in (21) and then, $G_2(x) \simeq (2p)\,x$, $G_4(x) \simeq (2p)^2\,x$, and in general

$$G_{2^k}(x) \simeq (2p)^k\,x. \tag{22}$$

To obtain the asymptotic behavior, we substituting $k = \frac{\ln N}{\ln 2}$ into (22) and then $G_N(x) \simeq N^{\beta}\,x$ with $\beta = 1 + \frac{\ln p}{\ln 2}$. Therefore, the cumulative distribution of the rank of the winner follows the scaling form

$$G_N(x) \to \Psi\left(x\,N^{\beta}\right) \tag{23}$$

when $N \to \infty$. The scaling function is linear, $\Psi(z) \simeq z$, in the limit $z \to 0$, reflecting that the extremal statistics are invariant under the competition dynamics.

The scaling form (23) shows that the rank of the tournament winner decays algebraically with the tournament size as in (2). Surprisingly, the decay exponent

$$\beta_{\rm par} = 1 + \frac{\ln(1-q)}{\ln 2} \tag{24}$$

for parallel dynamics, differs from the decay exponent (17) for sequential dynamics. The two exponents coincide in the extreme cases, $\beta(0) = 1$ and $\beta(1/2) = 0$. The inequality $\beta_{\rm par} \ge \beta_{\rm seq}$ (figure 1) shows that parallel play benefits the strong players. Indeed, in sequential play weak players may survive by being idle. The source of this discrepancy is fluctuations in the number of games. In sequential dynamics, the number of games played by each player is variable while in parallel dynamics the number of games is fixed.

Substituting the scaling form (23) into the recursion (21), the scaling function obeys the nonlinear-nonlocal

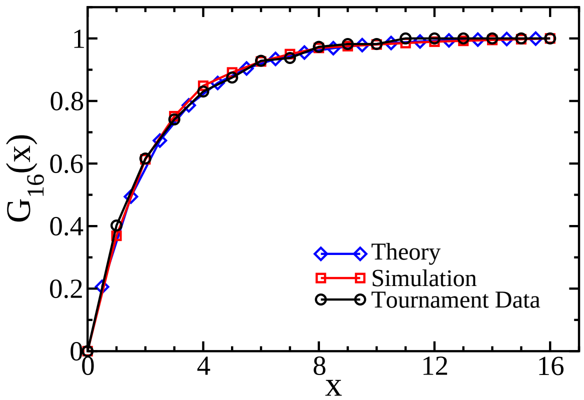

FIG. 2: The cumulative distribution of the rank of the group winner $G_{16}(x)$. The empirical distribution for college basketball (circle) is compared with Monte Carlo simulations (squares), and the parallel dynamics theory (diamonds).

equation

$$\Psi(2pz) = 2p\,\Psi(z) + (1-2p)\Psi^2(z). \tag{25}$$

The boundary condition are $\Psi(0) = 0$ and $\Psi(\infty) = 1$. An exact solution is feasible only when there are no upsets: $\Psi(z) = 1 - e^{-z}$ for $q = 0$. Otherwise, we perform an asymptotic analysis. As shown above, the small-z behavior is generic, $\Psi(z) \simeq z$. At large arguments, we write $U(z) = 1 - \Psi(z)$ and since $U \ll 1$, we can neglect the nonlinear terms and then $U(2pz) = 2qU(z)$. This implies the algebraic decay $U(z) \sim z^{(\ln 2q)/(\ln 2p)}$. As a result, the likelihood of finding weak winners, $g_N(x) \to N^\beta \psi\left(xN^\beta\right)$ with $\psi(z) = \Psi'(z)$, decays algebraically

$$\psi(z) \sim z^{\frac{\ln 2q}{\ln 2p} - 1} \tag{26}$$

as $z \to \infty$. This algebraic behavior is very different from the exponential decay $\psi(z) = e^{-z}$ for deterministic games. In contrast to sequential play, the exponent depends on the upset probability. This large likelihood of finding weak winners reflects that the number of games played by the tournament winner scales logarithmically with the number of teams. For example, as $N = 2^k$, the likelihood that the weakest player wins, $q^k = N^{\ln q/\ln 2}$, is appreciable as it decays only algebraically with N.

Empirical Study. To test our theoretical approach, we studied the US men's NCAA college basketball national championship where 64 teams are divided into 4 groups of 16, with teams in each group ranked 1 (best) to 16 (worst). The winner of each group advances to the "final four". As in the parallel dynamics, half of the teams are eliminated in each round. The schedule, however, is not random: the games are arranged so that if there are no upsets the bottom half is eliminated in each round. We analyzed the results of all 1680 games since this format was established (1979-2006) [15]. We calculated the cumulative rank distribution of the team advancing to the final four, $G_{16}(x)$, with $x = 1, 2, \ldots, 16$ (figure 2). Additionally, we measured the upset frequency $q = 0.275$ by counting the number of games won by the underdog [12].

To compare with the theoretical model, we simulated the NCAA tournament schedule in which the lower-ranked team wins with upset probability q. The parameter q was treated as a tunable variable, and we present results for the value that best matched the empirical data. The simulation results produce a rank distribution that agrees well with the empirical findings (figure 2). The fitted upset probability $q = 0.22$ is close to the observed frequency. Alternatively, we modeled the data by iterating (21) starting with the uniform distribution $G_1(x) = x/16$ using a fitted upset probability of $q = 0.175$ (the theory assumes a random schedule and an approximate uniform distribution). We thus found that the competition model has predictive power that quantitatively captures empirical rank distributions, and enables estimates of upset frequencies from observed rank distributions.

In summary, we studied dynamics of single-elimination tournaments, in which there is a finite probability for a lower-ranked player to upset a higher-ranked player. We obtained an exact solution for the distribution of player ranks for arbitrary initial conditions. Generally, the likelihood of upset winners is relatively large since the tail of the distribution function decays algebraically with rank. The characteristic rank of the winning player decays algebraically with the number of players and the larger the upset probability, the slower this decay (small tournaments are more likely to produce a surprise winner). Different decay exponents are found for sequential and parallel play with the latter generally larger (weak players fare better by avoiding competition). We demonstrated the utility of this model using college basketball results.

Extreme properties of the initial distribution fully governs the asymptotic behavior. In the long time limit, the player distribution becomes self-similar. Both the form of the scaling distribution and the time dependence of the characteristic rank depend only on the small-x behavior of the initial distribution. A similar phenomenology where extremal statistics governs long-time asymptotics was found in studies of clustering in traffic flows [16] and species abundance in biological evolution [17, 18].

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[1] S. Ispolatov, P. L. Krapivsky, and S. Redner, Eur. Phys. Jour. B **2**, 267 (1998).

[2] A. Dragulescu and V. M. Yakovenko, Eur. Phys. Jour. B

17, 723 (2000).
[3] F. Slanina, Phys. Rev. E **69**, 046102 (2004).
[4] R. D'Hulst, G. J. Rodgers, Physica A 294, 447 (2001).
[5] E. Ben-Naim and P. L. Krapivsky, Eur. Phys. Jour. B 25, 239 (2002).
[6] E. Ben-Naim, B. Kahng, J. S. Kim, J. Stat. Mech. P07001 (2006).
[7] E. Bonabeau, G. Theraulaz, and J.-L. Deneubourg, Physica A **217**, 373 (1995).
[8] E. Ben-Naim and S. Redner, J. Stat. Mech. L11002 (2005).
[9] K. Malarz, D. Stauffer, K. Kulakowski, Eur. Phys. Jour. B **50**, 195 (2006).
[10] E. Ben-Naim, F. Vazquez, and S. Redner, Eur. Phys. Jour. B **49**, 531 (2006).
[11] R. Toral and P. Amengual, cond-mat/0505388; P. Amengual and R. Toral, math.PR/0606181.
[12] E. Ben-Naim, F. Vazquez, and S. Redner, J. Quant. Anal. Sports **2**, No. 4, Article 1 (2006).
[13] J. D. Gunton, M. San Miguel, and P. S. Sahni in: *Phase Transitions and Critical Phenomena*, Vol. 8, eds. C. Domb and J. L. Lebowitz (Academic, NY 1983).
[14] A. J. Bray, Adv. Phys. **43**, 357 (1994).
[15] Data source: http://www.shrpsports.com/.
[16] E. Ben-Naim, P.L. Krapivsky and S. Redner, Phys. Rev. E **50**, 822 (1994).
[17] K. Jain and J. Krug, J. Stat. Mech. P040068 (2005).
[18] C. Sire, S. N. Majumdar, D. S. Dean, J. Stat. Mech. L07001 (2006).