Space covering by growing rays

P. L. Krapivsky[1] and E. Ben-Naim[2]
[1] *Courant Institute of Mathematical Sciences, New York University, New York, NY 10012-1185, USA*
[2] *The James Franck Institute, The University of Chicago, Chicago, IL 60637, USA*

We study kinetic and jamming properties of a space covering process in one dimension. The stochastic process is defined as follows: Seeds are nucleated randomly in space and produce rays which grow with a constant velocity. The growth stops upon collision with another ray. For arbitrary distributions of the growth velocity, the exact coverage, velocity and size distributions are evaluated for both simultaneous and continuous nucleation. In general, simultaneous nucleation exhibits a stronger dependence on the details of the growth velocity distribution in the asymptotic time regime. The coverage in the continuous case exhibits a universal t^{-1} approach to the jammed state, while an inhomogeneous version of the process leads to nongeneric t^{-p_+} decay, with $0 \le p_+ \le 1$ the fraction of right growing rays.

I. INTRODUCTION

Covering of space by growing objects occurs in numerous natural phenomena; phase separation [1], phase transformation [2], monolayer [3,?] and multilayer [5] adsorption, aggregation [6], and wetting [7] are just few examples. Covering processes can differ by the input mechanism, the covering object growth dynamics, and the interaction between the objects. In reality, collisions may proceed through various physico-chemical processes so the description of a single collision event can be a difficult task. However, in the realm of statistical mechanics one seeks to describe the essential mechanism that underlies the collective behavior, thereby illuminating the complexity resulting from the many-body nature of the problem. Therefore, in this study we will consider models with the simplest possible input rules (all seeds nucleate simultaneously or continuously with constant rate), the simplest ballistic growth law, and a simple collision rule - when an object hits another object, it stops.

This simple model mimics various physical, chemical, and biological processes. For example, a reacted functional group in a 1D polymer chain can poison adjacent unreacted group, this new reacted group then poisons the next one, etc. In many situations the reaction may be *directed* thus giving an example of the ray model we will examine below. Another natural example is a directed biological growth and spreading [8].

Several recent theoretical works model the covering process using ballistically growing objects, see e.g. [2,?,?,?,?,?] and references therein. In other studies (see e.g. [12,?]), the dimension of the objects is smaller than the dimension of the space and hence in such processes space splitting rather than space covering occurs. We study the growth of rays from point seeds, a stochastic process which results in space covering in one dimension. The process proceeds as follows: Rays grow freely with velocity v until they hit other rays; after a collision of a moving tip with another ray, the tip stops and that ray becomes frozen. There are two natural nucleation rules: homogeneous and hetereogeneous. In the heterogeneous model, seeds are nucleated simultaneously. We consider the simplest case of initially uncorrelated spatial distribution of seeds. The velocity is distributed according to an arbitrary velocity distribution $P_0(v)$. For the homogeneous model, the nucleation process also starts at $t = 0$ but proceeds forever: Seeds are nucleated stochastically *both* in space and in time, with a constant rate per unit volume of uncovered space. These two cases have a significant difference – while the homogeneous process is stochastic, the heterogeneous process is deterministic. The spatial dimension d plays an important role in these growth processes. For $d = 1$, growing rays cover a finite fraction of the space, eventually all the space for the homogeneous model. The homogeneous model has some similarity to the Avrami-Kolmogorov nucleation-and-growth process [2], while the heterogeneous one resembles some properties of random sequential adsorption [3]. In contrast, for $d > 1$ the net volume covered by rays is zero, hence some characteristics can be computed trivially, e.g. the number density of rays is equal to t, while the geometric patterns are rich and interesting.

In the following we obtain exact results for arbitrary velocity distribution of the input in one-dimension. Generally, a jamming configuration is approached for long times and we are interested in both the kinetic and the jamming properties. We focus on simple quantities such as the coverage, the velocity distribution of growing rays and the size distribution of the rays in the jammed configuration. The heterogeneous case is considered in section II, and the homogeneous case is considered in section III. We briefly discuss a possible generalization to higher dimensions and describe it using a mean-field technique in section IV.

II. HETEROGENEOUS NUCLEATION

In the heterogeneous case seeds are randomly distributed on a line with concentration c_0 at time $t = 0$.

Each seed gives birth to a ray whose tip moves freely with a constant intrinsic velocity. When the tip of a ray collides with either another seed or another tip, its growth is stopped. The growth velocities are independent of the position and are distributed according to $P_0(v)$, such that $c_0 = \int dv P_0(v)$. We assume that the velocity distribution has a characteristic velocity v_0, *i.e.*, it can be written as $(c_0/v_0)P_0(v/v_0)$ with $\int dz P_0(z) = 1$. It is convenient make a transformation to dimensionless time $c_0 v_0 t \to t$, velocity $v/v_0 \to v$, concentration $c/c_0 \to c$, and velocity distribution $(c_0/v_0)P_0 \to P_0$. Thus, the initial seed concentration is set to unity. In the following we obtain the exact jamming properties of the system as well as its approach towards the jammed state for arbitrary velocity distribution. We then consider the behavior for two special cases, a bimodal velocity distribution and a distribution with a power-law behavior in the small-velocity limit.

Several properties of the jammed configuration can be easily derived from the initial state. Let us introduce the shorthand notations for the velocity distribution and the density of right- and left-moving rays, $P_\pm(v) = P_0(v)\theta(\pm v)$, with $\theta(x)$ the Heaviside step function, and $p_\pm = \int dv P_\pm(v)$, which in turn implies $p_+ + p_- = 1$. The final fraction of uncovered space is

$$\Phi_\infty = p_+ p_-, \tag{1}$$

This follows from a simple observation: For a pair of adjacent seeds, the space between them remains completely uncovered if the left ray moves to the left and the right one moves to the right; for the three other situations, the space between the adjacent seeds will be covered. Of course, the final coverage is given by $1 - \Phi$, with $3/4 \le 1 - \Phi \le 1$. Maximal final coverage is achieved when all rays grow in the same directions, while minimal final coverage occurs for $p_+ = p_- = 1/2$.

One can solve for the kinetics of the covering process by a number of techniques. We will use an approach that proves powerful for the more difficult problem of homogeneous nucleation. This procedure has been applied to few other nucleation-and-growth processes [9,**?**,**?**]. We start by noting that the fraction $\Phi(t)$ of uncovered space can be thought as the probability that some point, say the origin, remains uncovered at time t. The key point of the approach is very simple: One first investigates an auxiliary "one-sided" problem in which seeds are scattered to the left of the origin and no seeds are placed to the right. Having computed the probability $\phi_+(t)$ that the origin remains uncovered up to time t in this one-sided problem, the "two-sided" probability $\Phi(t)$ follows from a clear identity:

$$\Phi(t) = \phi_+(t)\phi_-(t), \tag{2}$$

where $\phi_-(t)$ corresponds to the complementary one-sided problem. To determine $\phi_+(t)$ we note that the origin is covered during the time interval $(t, t + dt)$ with probability $(-d\phi_+/dt)dt$. On the other hand, it happens if the nearest to the origin seed has a positive velocity, say v, and lies in the space interval $(-vt - vdt, -vt)$. Integrating over all positive velocities, we find $(d\phi_+/dt)dt = -\int dv v dt P_+(v)e^{-vt}$. This leads to the rate equation

$$\frac{d\phi_+}{dt} = -\int dv v P_+(v)e^{-vt}, \tag{3}$$

with the initial condition $\phi_+(0) = 1$. Solving Eq. (3) subject to this initial condition gives

$$\phi_+(t) = p_- + \int dv P_+(v)e^{-vt}. \tag{4}$$

Similar expression holds for ϕ_-. Combining these two one-sided problems, we find

$$\Phi(t) = \left(p_- + \int dv P_+(v)e^{-vt}\right)\left(p_+ + \int dv P_-(v)e^{vt}\right). \tag{5}$$

Indeed, the final uncovered fraction of space agrees with Eq. (1).

For sufficiently small times, the uncovered fraction decreases linearly with time according to

$$\Phi(t) \cong 1 - \langle |v| \rangle_0 t, \qquad t \to 0. \tag{6}$$

The prefactor is equal to the average ray speed, and in the following $\langle \cdots \rangle_0 \equiv \int dv \cdots P_0(v)$. Initially, the rays cover the space very effectively, and as the process continues, the overall covering rate decreases depending on the nature of the initial velocity distribution. In the long time limit, the fraction of uncovered space approaches its final value according to

$$\Phi(t) - \Phi_\infty \sim \int_0^\infty dv \tilde{P}(v)e^{-vt} \qquad t \to \infty, \tag{7}$$

with the modified velocity distribution $\tilde{P}(v) = p_+ P_+(v) + p_- P_-(-v)$. The above integral is simply the Laplace transform of $\tilde{P}(v)$, and in the long time limit it is dominated by the velocity distribution near the minimal velocity of the distribution. We conclude that slow rays dominate asymptotically, as demonstrated below for a distribution which behaves as a power-law near the origin.

To determine the size distribution, we first compute $P(v, t)$, the density of rays of velocity v that have not been stopped before time t. This density is simply given by

$$P_\pm(v, t) = P_\pm(v, 0)e^{-|v|t}\phi_\mp(t). \tag{8}$$

Here the exponential factor gives the probability that the interval of length $|v|t$, covered by the growing ray at time t, does not contain other seeds; the latter factor ensures that the point reached by the growing tip at time t, remains uncovered by rays growing from the other half-space. The covering rate, $-d\Phi/dt$, can be calculated from the velocity distribution, $-d\Phi(t)/dt = \int dv|v|P(v,t)$. This result is consistent with the exact solution of Eq. (5).

Eq. (8) allows us to find the density of frozen rays of length l in the final (jammed) configuration, $\rho_\infty(l)$. Frozen rays of length l are those that stopped their growth at time $t = l/|v|$. Therefore, the limiting size distribution is related to the velocity distribution via $\rho_\infty(l) = -\int dt \int dv(\partial P(v,t)/\partial t)\delta(l-|v|t)$. Substituting Eq. (8) and evaluating the integrals gives

$$\rho_\infty(l) = (p_+^2 + p_-^2)e^{-l} + \int dv \int du P_+(v)P_-(u) \times \left[(1-u/v)e^{-l(1-u/v)} + (1-v/u)e^{-l(1-v/u)}\right]. \quad (9)$$

The first term can be easily understood: Two adjacent rays moving in similar directions and initially separated by distance l, give rise to a frozen ray of length l. The exponential factor describes the probability that such an interval is empty and the prefactor accounts for the fraction of parallel moving neighbors. The second term accounts for collisions between rays that grew in the opposite directions. One can verify by direct integration of Eq. (9) that the final fraction of covered space $\int dll\rho_\infty(l)$ equals the previously established value $1 - p_+p_-$, thus providing a useful check of consistency.

In the jammed state, clusters of frozen rays are separated by gaps. Following the above calculations, it might be possible to find the density of n-rays clusters of length l, and even more detailed quantities. One such quantity is $\tilde{\rho}_\infty(l)$, the distribution of gaps of size l at the final state. This distribution is easily found, $\tilde{\rho}_\infty(l) = p_-p_+e^{-l}$ and indeed, it satisfies the normalization condition $\Phi_\infty = \int dll\tilde{\rho}_\infty(l) = p_+p_-$.

We discuss now several specific initial velocity distributions. We first consider the case of bimodal velocity distributions

$$P_0(v) = p_+\delta(v-1) + p_-\delta(v+1), \quad (10)$$

with $p_+ + p_- = 1$. The fraction of uncovered space is found from Eq. (5),

$$\Phi(t) = (p_- + p_+e^{-t})(p_+ + p_-e^{-t}) \quad (11)$$

The approach towards the jammed state is a fast exponential one, $\Phi(t) - \Phi_\infty \simeq (p_+^2 + p_-^2)e^{-t}$ as $t \to \infty$. Hence, the density of active rays is exponentially decreasing with time as well. The jamming length distribution can be evaluated using Eq. (9),

$$\rho_\infty(l) = (p_+^2 + p_-^2)e^{-l} + 4p_+p_-e^{-2l}. \quad (12)$$

The exponential asymptotic behavior holds for polydisperse velocity distributions as long as the distribution vanishes in the vicinity of the origin. To examine the effects of slow rays on the asymptotic behavior, it is useful to consider power-law distributions

$$P_0(v) \sim |v|^\mu, \qquad \text{when} \quad v \to 0, \quad (13)$$

with $\mu > -1$. The long time asymptotics is governed by the large argument Laplace transform of the velocity distribution and consequently an algebraic asymptotic behavior is found for the uncovered fraction:

$$\Phi(t) - \Phi_\infty \sim t^{-1-\mu}, \qquad \text{as} \quad t \to \infty. \quad (14)$$

Furthermore, Eq. (8) indicates that $P(v,t) \sim |v|^\mu \exp(-|v|t)$. In other words, the velocity distribution can be written in a scaling form

$$P(v,t) \sim t^{-\mu} f(z), \qquad \text{with} \quad z = |v|/\langle|v|\rangle. \quad (15)$$

In the above equation the typical velocity decays in time according to $\langle|v|\rangle \sim t^{-1}$ and the scaling function is $f(z) = z^\mu e^{-z}$. The total density of growing rays at time t, $n_r(t)$, decays according to $n_r \sim t^{-\mu-1}$. To see how the exponential behavior turns into a power-law one, we note that $P_0(v) \sim (|v| - v_{\min})^\mu$ for $|v| \to v_{\min}$, with $v_{\min} > 0$, leads to $\Phi(t) - \Phi_\infty \sim t^{-\mu-1}e^{-v_{\min}t}$. This situation, where the smallest velocities dominate, is reminiscent of ballistic aggregation and annihilation processes with continuous velocity distributions [15].

III. HOMOGENEOUS NUCLEATION

In the homogeneous case seeds are nucleated stochastically *both* in space and in time. At time $t = 0$, the system is assumed to be empty, seeds appear constantly with rate of γ_0 on uncovered space, and eventually the system reaches complete coverage. It is again convenient to introduce dimensionless velocity $v/v_0 \to v$, space $x\sqrt{\gamma_0/v_0} \to x$, time $t\sqrt{\gamma_0 v_0} \to t$, and input distribution function $(\gamma_0/v_0)P_0 \to P_0$. In the following, we write the equations describing the coverage and the gap distribution. Although we do not obtain a general explicit solution for the coverage, an asymptotic analysis is carried. For arbitrary distributions, the asymptotic coverage is independent of most details of the input velocity distribution. We obtain explicit results for the kinetic and the jamming properties in the case of bimodal velocity distributions.

We again consider first the one-sided problem: seeds are deposited only to the left of the origin. Repeating the steps used in deriving Eq. (3), $\phi_+(t)$, the probability that the origin remains uncovered at time t satisfies

$$\frac{d\phi_+}{dt} = -\int dv v P_+(v)\, e^{-vt^2/2} \int_0^t d\tau\, e^{v\tau^2/2}\phi_+(\tau). \quad (16)$$

Indeed, the origin can be covered during the time interval $(t, t+dt)$ by a v-velocity seed with positive direction of growth that could have been nucleated at time τ, with $0 < \tau < t$, in the spatial interval $(-v(t-\tau) - vdt, -v(t-\tau))$. Hence, the integration over the velocity and the time variables. The exponential factor ensures that no nucleation have occurred in the spatial interval covered by the ray before the seed appeared and that no nucleation happens in the shrinking part of this interval during the growth of the ray. The last factor ensures that the point $-v\tau$ is uncovered at time τ. We have used the one-sided probability $\phi_+(\tau)$ since the condition that at time τ the point $-v\tau$ was not covered from the right has already been taken into account by the exponential factor. Despite the complex structure of this rate equation, the most interesting aspects of the covering process, *i.e.* the short and long time behavior, can be found for an arbitrary distribution without an explicit solution.

The early behavior of the system can be easily found by setting the second integral to t, and consequently $\phi_\pm(t) \cong 1 - B_\pm t^2/2$ with $B_\pm = \int dv|v|P_\pm(v)$. Hence, the early time behavior is given by

$$\Phi(t) \cong 1 - \frac{\langle |v| \rangle_0}{2} t^2 \qquad t \to 0. \tag{17}$$

This initial coverage is slower than in the homogeneous case since no rays are initially present. Note that in both cases the prefactor equals the average ray speed.

We turn now to the long-time behavior. Asymptotically, the main contribution to the second integral in the right-hand side of Eq. (16) is gained near the upper limit, and this integral is easily estimated:

$$\int^t d\tau e^{v\tau^2/2}\phi_+(\tau) \simeq (vt)^{-1} e^{vt^2/2}\phi_+(t). \tag{18}$$

Substituting this estimate into Eq. (16) and integrating over the velocity, we arrive at the rate equation, $d\phi_+/dt = -p_+\phi_+/t$. As a result, the leading asymptotic behavior for arbitrary input distribution $P_0(v)$ is

$$\phi_\pm(t) \sim t^{-p_\pm}, \qquad t \to \infty. \tag{19}$$

This behavior is remarkable, the one-sided problem exhibits non-generic decay kinetics which is characterized by a simple parameter, the fraction of right (left) moving seeds nucleating per unit time. While the decay for the uncovered fraction for both of the one-sided problems depends on the initial conditions, and specifically on the fraction of left and right growing seeds, the uncovered fraction exhibits a robust decay

$$\Phi(t) \sim t^{-1}, \qquad t \to \infty. \tag{20}$$

The asymptotic uncovered fraction, Φ, is independent of the input distribution, and is reminiscent of the temporal behavior in random sequential adsorption processes [3]. The situation is in contrast with the heterogeneous case where the presence of slow particles reduces the asymptotic covering rate. In the next section, we will show that this robust behavior also emerges from a simple mean-field theory.

The above analysis enables calculation of additional kinetic properties of the system such as $n_s(t)$, the seed density. The seeds creation rate is equal to the available space, and therefore,

$$\frac{dn_s}{dt} = \Phi. \tag{21}$$

Using the asymptotic behavior of $\Phi \sim t^{-1}$, we learn that the seed density grows logarithmically in time,

$$n_s \sim \ln t, \qquad t \to \infty. \tag{22}$$

The velocity distribution function can be calculated following the same line of reasoning that led to Eq. (16). Denoting by $P(v,t)$ the density of growing rays at time t, one finds that for $v > 0$

$$P_\pm(v,t) = P_\pm(v) \int_0^t d\tau e^{-|v|(t^2-\tau^2)/2}\phi_\pm(\tau)\phi_\mp(t) \tag{23}$$

The integration is carried over all possible creation time τ, $0 < \tau < t$. As in Eq. (16), the exponential term ensures that (i) the interval covered by the growing ray remains empty during the time interval $(0,\tau)$, and (ii) the space covered by the ray remains empty during the time interval (τ, t). The factor $\phi_+(\tau)\phi_-(t)$ ensures that the initial position of the seed and the final position of the tip belong to uncovered area.

In the long-time limit, the main contribution to the integral in the right-hand side of Eq. (23) is accumulated near the upper limit. Thus we can use the estimate of Eq. (18) to find

$$P(v,t) \simeq \frac{P_0(v)}{|v|}\frac{\Phi(t)}{t}. \tag{24}$$

The ray velocity distribution is therefore proportional to $|v|^{-1}P_0(v)$ in the late stages of the process, i.e. slow velocities are slightly more favorable. However, the relative enhancement of slow rays is weak in comparison with the heterogeneous case where fast velocities are exponentially suppressed. For all input distributions with finite $\langle |v|^{-1}\rangle_0$ moment, a universal decay of the density of growing rays is found, $n_r \simeq \langle |v|^{-1}\rangle_0 \Phi t^{-1}$. Combining this result with Eq. (20), we see that the ray density decays according to

$$n_r \sim t^{-2}, \qquad \text{when} \quad t \to \infty, \tag{25}$$

for input distributions with finite $\langle |v|^{-1}\rangle_0$.

Similar to the heterogeneous case, it is useful to consider the power law distribution, $P_0(v) \sim |v|^\mu$ for $|v| \to 0$, with $\mu > -1$. For this distribution, the moment $\langle |v|^{-1}\rangle_0$ does not exist when $\mu \le 0$, and we cannot use Eq. (24) in deriving the density of growing rays. Thus we substitute the exact expression of Eq. (23) into the relation $n_r = \int dv P(v,t)$, *first* perform v-integration, and then τ-integration. This yields

$$n_r \sim \begin{cases} t^{-2\mu-2}, & -1<\mu<0, \\ t^{-2}\ln t, & \mu=0, \\ t^{-2}, & \mu>0. \end{cases} \quad (26)$$

The typical ray velocity, defined via $\bar{v} = \int dv|v|P(v,t)/\int dvP(v,t) \equiv \langle|v|\rangle/n_r$, has the following limiting behavior:

$$\bar{v} \sim \begin{cases} t^{2\mu}, & -1<\mu\le 0, \\ 1/\ln t, & \mu=0, \\ \text{const}, & \mu>0. \end{cases} \quad (27)$$

Thus the average velocity decreases when the moment $\langle|v|^{-1}\rangle_0$ is infinite while for all other cases the typical velocity reaches a limiting value. To summarize, despite the general asymptotic behavior found for the coverage and the seed density, the ray density and the velocity distribution exhibit nongeneric behavior. For the ray density the general behavior is $n_s \sim t^{-2}$, and only "pathological" distributions with enough slow rays lead to slower ray density decays.

Another interesting quantity is $\rho(l,t)$, the distribution of frozen rays of length l at time t. This distribution can be readily derived from $P(v,\tau_1,\tau_2)$, the density at time τ_2 of v rays that were nucleated at time τ_1,

$$\rho(l,t) = -\int dv \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \frac{\partial P(v,\tau_1,\tau_2)}{\partial\tau_2}\delta\left(l-|v|(\tau_1-\tau_2)\right). \quad (28)$$

Here, the loss rate of growing rays at time τ_1 is equal to the gain rate of frozen rays, and the delta function ensures proper length of the ray. The conditional density $P(v,\tau_1,\tau_2)$ is given by the integrand of Eq. (23), with the transformation $t\to\tau_1$ and $\tau\to\tau_2$,

$$P_\pm(v,\tau_1,\tau_2) = P_\pm(v)\, e^{-|v|(\tau_1^2-\tau_2^2)/2}\,\phi_\pm(\tau_1)\phi_\mp(\tau_2). \quad (29)$$

The jamming distribution can be found by setting $t=\infty$. Combining the above two equations gives

$$\rho_\infty(l) = \int dvP_+(v)e^{-l^2/2v}\int_0^\infty d\tau_2 e^{-l\tau_2}\phi_-(\tau_2)\times \left[\tau_1\phi_+(\tau_1)+\frac{d\phi_+(\tau_1)}{d\tau_1}\right]_{\tau_1=\tau_2+l/v} + \int dvP_-(v)\cdots \quad (30)$$

The second term is written by exchanging $+$ and $-$. We have not been able to compute the general behavior. However, we obtain below results in the special case of bimodal-velocity distributions.

We present now explicit expressions for the both kinetic and the jamming properties for the bimodal velocity distribution, $P_0(v) = p_+\delta(v-1)+p_-\delta(v+1)$. Differentiating Eq. (16) with respect to t produces the following ordinary differential equation for $\phi_\pm$,

$$\frac{d^2\phi_\pm}{dt^2} + t\frac{d\phi_\pm}{dt} + p_\pm\phi_\pm = 0. \quad (31)$$

This equation is solved subject to the initial conditions $\phi_\pm|_{t=0} = 1$ and $d\phi_\pm/dt\big|_{t=0} = 0$. The solution is expressed through parabolic cylinder functions [6]

$$\phi_\pm(t) = a_\pm \exp\left(-t^2/4\right)\left[D_{-p_\mp}(t)+D_{-p_\mp}(-t)\right], \quad (32)$$

with $a_\pm = 2^{-1+p_\mp/2}\pi^{-1/2}\Gamma(1/2+p_\mp/2)$. The asymptotic behavior agrees with the above analysis,

$$\phi_\pm(t) \sim c_\pm t^{-p_\pm}, \quad (33)$$

with $c_\pm = 2^{-p_\pm/2}\Gamma(1/2+p_\mp/2)/\Gamma(p_\mp)$. The uncovered fraction is evaluated from $\Phi = \phi_+\phi_-$. Thus, the leading asymptotic behavior for the uncovered fraction is $\Phi(t) \sim c_+c_-t^{-1}$. The asymptotic velocity distribution is found from Eq. (24),

$$P(v,t) = c_+c_-\Big(p_+\delta(v-1)+p_-\delta(v+1)\Big)t^{-2}. \quad (34)$$

Moreover, for this special case the time dependent velocity distribution is proportional to the input velocity distribution at all times. Also, the seed density and the ray density are given by $n_s \sim c_+c_-\ln t$, and $n_r \sim c_+c_-t^{-2}$, in agreement with the above theory. Note that the relation $dn_r/dt = -d\Phi/dt$ is satisfied since the ray speed distribution is monodisperse.

For such a simple velocity distribution it is possible to obtain several properties of the length distribution in the jammed configuration. Evaluation of Eq. (30) using the corresponding limiting behaviors $\phi_\pm(0)=1$, and $\phi_\pm \sim c_\pm t^{-p_\pm}$ when $t\to\infty$, gives

$$\rho_\infty(l) \simeq \begin{cases} c_+c_-\, l^{-1}\left(\ln(1/l)-\gamma\right) & l\ll 1, \\ \left(p_+c_+l^{-p_+}+p_-c_-l^{-p_-}\right)\exp\left(-l^2/2\right) & l\gg 1. \end{cases} \quad (35)$$

In the above equation $\gamma \cong 0.5772$ is the Euler constant. In the small-size limit, the jammed distribution $\rho_\infty(l)$ exhibits very weak nonintegrable singularity. One can compute the density of the total number of frozen rays, $F(\epsilon)$, of lengths greater than ϵ:

$$F(\epsilon) = \int_\epsilon^\infty dl\rho_\infty(l) \simeq c_+c_-\ \ln^2(1/\epsilon). \quad (36)$$

A power-law behavior of the form $F(\epsilon)\sim\epsilon^{-D_f}$ would indicate that D_f is the fractal dimension of the pore space of forming pattern. Thus, in the present case $D_f = 0$, although a weak logarithmic singularity still appears.

IV. MEAN-FIELD APPROXIMATION AND HIGHER DIMENSIONS

It is worthwhile to consider possible generalizations of the covering process in higher dimensions. One such generalization [5] assumes that the growing objects are rigid spheres whose radius grows ballistically until a collision occurs. The heterogeneous case is characterized by a superexponential decay of the covered space $\Phi(t) - \Phi_\infty \sim \exp(-vt^d)$. Another natural generalization is to objects which do not cover any volume. For example a growing line in 2D, a growing plane in 3D, *etc.* In this section we study the growth kinetics of such covering processes using approximate (mean-field) equations. For one-dimensional heterogeneous nucleation, the kinetics is highly sensitive to the details of the velocity distribution, suggesting that mean-field type theories fail to describe the kinetics. Hence, we focus on the homogeneous case, where at least in 1D robust behavior was found.

Although the exact solution was obtained in the previous section, it is instructive to study the covering process using an approximate approach. We assume a monodisperse velocity distribution $P_0(v) = \delta(v-1)$, and thus the covering rate is proportional to the density n_r of growing rays, $d\Phi/dt = -n_r$. The ray density itself is estimated from $dn_r/dt = \Phi - n_r(1-\Phi)/l$. Here, rays are gained with a rate equal to the fraction of uncovered space. The estimated loss rate is proportional to the density $1-\Phi$ and inversely proportional to the average length of a gap $l \sim \Phi$. Hence, the uncovered fraction can be estimated from the differential equation

$$\frac{d^2\Phi}{dt^2} + (\Phi^{-1} - 1)\frac{d\Phi}{dt} + \Phi = 0 \tag{37}$$

with the usual initial conditions $\Phi|_{t=0} = 1$, and $d\Phi/dt|_{t=0} = 0$. In fact, both limiting behaviors predicted by this approximation agree with the exact solution. In the early stages of the covering process, $\Phi(t) \cong 1 - t^2/2$, in perfect agreement with (17). When $t \to \infty$, the second derivative term is negligible and Φ can be estimated from $d\Phi/dt = -\Phi^2$. Indeed, the familiar t^{-1} is found for the uncovered fraction. Also, the total density of growing rays can be found from $n_r = -d\Phi/dt$, and the resulting $n_r \sim t^{-2}$ asymptotic behavior is in agreement with Eq. (25). Despite the success of the mean-field approximation, the homogeneous covering process was characterized by nontrivial behavior of the auxiliary one-sided problem, behavior which can not be accounted for by such a simplified theory.

Let us now consider the proposed generalization to higher dimensions. In 2D, The process is defined as follows: Seeds are nucleated with unit rate in free space. A line grows with unit velocity from each seed in a random directions until it collides with another line. Similarly, in arbitrary dimension d, a $d-1$ dimensional hyperplane grows in a random direction and stops upon collisions. (By direction, one calls a normal to the hyperplane). For $d > 1$, zero volume is covered and thus, the seed density is given by $n_s = t$. As the process continues, both open and closed hyper-polygons are created. We assume that the density of such polygons is proportional to the seed density, $n_p \sim n_s \sim t$. Hence, the typical linear size of such objects is $l \sim n_p^{-1/d} \sim t^{-1/d}$. Finally, the growing objects density satisfies the generalization of Eq. (37).

$$\dot{n} = 1 - \frac{nn_s}{l}. \tag{38}$$

The gain term equals unity since no volume is covered by the growth. Solving Eq. (38) we find that asymptotically

$$n \sim t^{-(d+1)/d}, \qquad t \to \infty. \tag{39}$$

In the limit of infinite dimension, a simple t^{-1} behavior is found. In two dimension, the process is equivalent to an isotropic fragmentation process. A numerical simulation in two dimension with infinite growth velocity found evidence that $l \sim t^{-1/2}$ [12], in agreement with the above approximation.

V. DISCUSSION

We have studied a space covering process in one dimension. Exact results for the kinetics and the structure of the system have been presented for both homogeneous and heterogeneous realizations of the process. While for the heterogeneous case the temporal behavior of the coverage is very sensitive to the presence of slowly growing rays, the asymptotic coverage in the homogeneous case is almost independent of the details of the input. In both cases the fraction of left and right moving rays plays a crucial role. In the heterogeneous case, the jamming coverage depends on these fractions, while conditional coverage probabilities in the homogeneous case exhibit a surprising algebraic dependence on these fractions. We also treated a generalization to higher dimension using of a mean-field theory.

It would be interesting to establish a relationship between the present process and the chemical processes with ballistically moving aggregating/annihilating particles. A natural question is whether the nongeneric behavior found for the one-sided covering process occurs for systems of interacting ballistically moving particles.

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