Monotonicity in the averaging process

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We investigate an averaging process that describes how interacting agents approach consensus through binary interactions. In each elementary step, two agents are selected at random and they reach compromise by adopting their opinion average. We show that the fraction of agents with a monotonically decreasing opinion decays as $e^{-\alpha t}$, and that the exponent $\alpha = \frac{1}{2} - \frac{1+\ln\ln 2}{4\ln 2}$ is selected as the extremum from a continuous spectrum of possible values. The opinion distribution of monotonic agents is asymmetric, and it becomes self-similar at large times. Furthermore, the tails of the opinion distribution are algebraic, and they are characterized by two distinct and nontrivial exponents. We also explore statistical properties of agents with an opinion strictly above average.

I. INTRODUCTION

The averaging process models a system of agents who reach agreement via compromise. The system consists of N agents. In each step, two agents, say i and j, are chosen at random and their opinions v_i and v_j are replaced by the average

$$(v_i, v_j) \to \left(\frac{v_i + v_j}{2}, \frac{v_i + v_j}{2} \right) . \tag{1}$$

As this elementary step is repeated, the system moves closer and closer toward consensus where all agents have the same opinion.

This averaging process has been studied by statistical physicists [1], applied probabilists [2], computer scientists [3], and social scientists [4] with applications ranging from opinion dynamics [5] to communication algorithms for computer and sensor networks [6, 7] to linguistics [8]. For the pure averaging process (1), the system approaches perfect consensus. However, when interactions are restricted to agents with sufficiently close opinions, the system bifurcates into groups [4, 9–13], with all agents within the same group sharing the same opinion. Another generalization of the averaging process (1) involves partial averaging where the opinion difference is reduced by a fixed multiplicative factor in each interaction. Partial averaging is equivalent to inelastic collisions, and it has been used to model freely evolving and driven inelastic gases [14–23] that satisfy the Maxwell model rules [24].

In this study, we analyze the pure averaging process (1). The system approaches perfect consensus and the difference between the typical opinion and the consensus opinion decays exponentially with time [14]. While the typical opinion follows a unidirectional path en route to consensus, an individual opinion may increase or decrease due to fluctuations. We focus on agents with a monotonically decreasing opinion (similar behavior is exhibited by agents with a monotonically increasing opinion).

We study the fraction $M(t)$ of agents whose opinions have only decreased up to time t, and find that this quantity decays exponentially with time,

$$M(t) \simeq A\, e^{-\alpha t}, \tag{2}$$

in the long-time limit (see Fig. 1). Our main result is that the exponent α is nontrivial

$$\alpha = \frac{1}{2} - \frac{1+\ln\ln 2}{4\ln 2} = 0.271517\ldots . \tag{3}$$

The opinion distribution of monotonic agents becomes self-similar at sufficiently large times. Further, this distribution has two algebraic tails that are characterized by two different exponents. These features of the opinion distribution enable us to determine the exponent α which is selected as the extremal value from a spectrum of possible values.

The rest of this paper is organized as follows. We begin with a brief overview of the averaging process in Sec. II. In Sec. III, we consider monotonic agents with an opinion that only decreases with time. We obtain the exponent (3), and also study the opinion distribution of monotonic agents. In Sec. IV, we investigate the behavior of agents with strictly positive opinion. In Sec. V, we summarize our findings and discuss possible avenues for future work. The generalization of the pure averaging process (1) to partial averaging is outlined in Appendix A, and corrections to the leading asymptotic behavior (2) are discussed in appendix B.

II. THE AVERAGING PROCESS

In the averaging process, there are N interacting agents. In each averaging event, two agents are selected at random, and their opinions evolve according to (1). This pairwise interaction is repeated indefinitely, and time is augmented by $2/N$ after each interaction, so that agents experience one averaging event per unit time. The distribution $F(v, t)$ of agents with opinion v at time t sat-

isfies the *nonlinear* rate equation [14]

$$\frac{\partial F(v,t)}{\partial t} = 2 \int_{-\infty}^{\infty} du\, F(u,t) F(2v-u,t) - F(v,t)\,, \quad (4)$$

in the limit $N \to \infty$. The convolution term reflects the binary nature of (1), and the linear loss rate reflects that agents participate in one interaction per unit time. The averaging process (1) conserves the number of agents and the total opinion, and consequently, the rate equation (4) conserves the two lowest moments of the opinion distribution: the normalized distribution, $\int dv\, F(v,t) = 1$, and the average opinion, $\langle v(t) \rangle = \int dv\, v\, F(v,t) = \text{const}$.

The averaging process (1) is invariant under translation $v \to v + \text{const.}$, as well as dilation, $v \to \text{const.} \times v$. Hence, without loss of generality, we may consider initial distributions with zero average, $\langle v(0) \rangle = 0$, and unit variance, $\langle v^2(0) \rangle = 1$ [25]. Furthermore, we restrict our attention to symmetric distributions as Eq. (4) implies $F(v,t) = F(-v,t)$ when $F(v,0) = F(-v,0)$.

The system approaches consensus, $F(v,t) \to \delta(v)$, as all agents acquire the average initial opinion in the long-time limit. The second moment $\langle v^2(t) \rangle = \int_{-\infty}^{\infty} dv\, v^2 F(v,t)$ quantifies the distance between the typical opinion and the consensus opinion. This quantity decays exponentially with time [14]

$$\langle v^2(t) \rangle = e^{-t/2}\,, \quad (5)$$

as follows from Eq. (4). Moreover, the distribution $F(v,t)$ becomes self-similar in the long-time limit, and the second moment (5) sets the scale for the typical opinion. In particular, the distribution $F(v,t)$ adheres to the scaling form [15]

$$F(v,t) = e^{t/4} \mathcal{F}(V), \quad \text{with} \quad \mathcal{F}(V) = \frac{2}{\pi} \frac{1}{(1+V^2)^2}\,, \quad (6)$$

and the scaling variable $V = v\, e^{t/4}$. This scaling behavior holds in the limits $t \to \infty$ and $v \to 0$. Also, the scaling function is normalized $\int_{-\infty}^{\infty} dV\, \mathcal{F}(V) = 1$.

III. MONOTONICITY

The ultimate opinion of every agent vanishes, $v_i \to 0$ as $t \to \infty$. Further, according to Eqs. (5)–(6), the typical magnitude of the opinion $|v|$ decreases monotonically, $|v| \sim e^{-t/4}$. In this study, we focus on agents with a monotonically decreasing opinion. At time t, we refer to agents with opinion that satisfies the inequality $v_i(t_1) \geq v_i(t_2)$ for all $t_1 \leq t_2 \leq t$ as *monotonic* agents. In the context of opinion dynamics, monotonic agents change their opinion only in one direction, say toward the left of the political spectrum only. According to (1), monotonic agents interact only with agents who have a smaller opinion. We stress that monotonic agents have an opinion that strictly decreases with time, but the sign
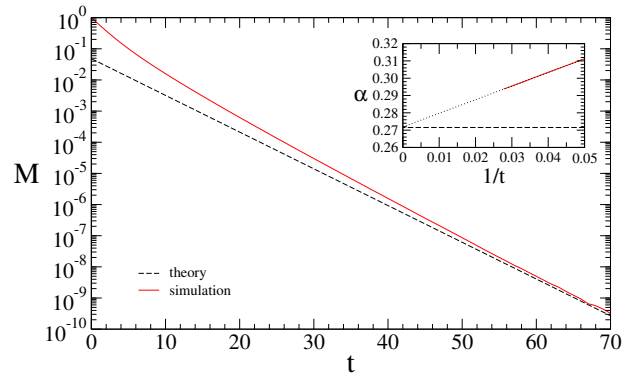

FIG. 1: The fraction of monotonic agents $M(t)$ versus time t. A dashed line with slope given by Eq. (2) is displayed for reference. The inset shows that the local slope $\alpha(t) \equiv -d \ln M / \ln t$ varies linearly with $1/t$. The dotted line shows a linear fit that yields the estimate $\alpha = 0.272 \pm 0.003$.

of the opinion is not constrained. In particular, monotonic agents may start with a positive opinion and end up with a negative one.

We denote by $M(t)$ the fraction of monotonic agents at time t, and by $M(v,t)$ the density of such agents with opinion v. Of course, $M(t) = \int dv\, M(v,t)$. By symmetry, the dual fraction of agents with monotonically increasing opinion equals $M(t)$, and their density is given by $M(-v,t)$. The density $M(v,t)$ is coupled to the total density $F(v,t)$, and it satisfies the *linear* rate equation

$$\frac{\partial M(v,t)}{\partial t} = 2 \int_{v}^{\infty} du\, M(u,t) F(2v-u,t) - M(v,t)\,. \quad (7)$$

The loss term reflects that on average, each agent experiences one interaction per unit time. The gain term in Eq. (7) resembles the gain term in Eq. (4), but the lower limit of integration ensures that the opinion of a monotonic agent may only decrease.

By integrating the master equation (7) over all opinions, we find that the fraction $M(t)$ decreases with time according to the rate equation

$$\frac{dM(t)}{dt} = -\int_{-\infty}^{\infty} du\, M(u,t) \int_{u}^{\infty} dv\, F(v,t)\,. \quad (8)$$

This evolution equation reflects that monotonic agents may only interact with agents having smaller opinions. The fraction $M(t)$ has two bounds, $e^{-t} \leq M(t) \leq 1$. The upper bound follows from the inequality $M(v,t) \leq F(v,t)$. The lower bound reflects that agents experiencing zero interactions are necessarily monotonic. Since agents interact once per unit time, the overall density $N(t)$ of noninteracting agents decays exponentially with time, $N(t) = e^{-t}$. The bounds $\alpha = 0$ and $\alpha = 1$ are realized in limiting cases of the partial averaging process, as discussed in Appendix A.
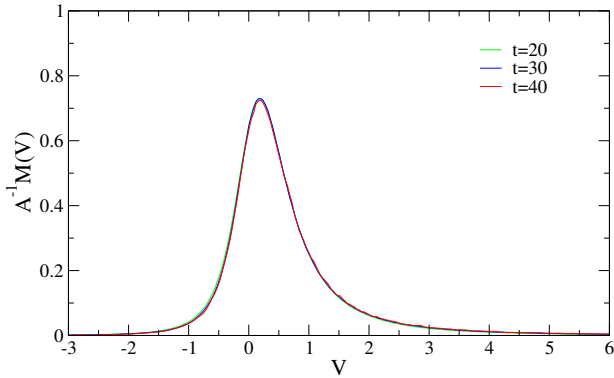
FIG. 2: The normalized scaling function $A^{-1}\mathcal{M}(V)$ versus the scaling variable V. This quantity is the probability distribution function of the scaled opinion V, as follows from Eqs. (2) and (11).

FIG. 3: The exponent α in (13) versus ν. The maximal value $\alpha = 0.271517$ occurs at $\nu = 2.47123$.

When the opinion v is sufficiently large, the dominant contribution to the integral in (7) comes from the vicinity of $u = 2v$. Using the normalization $\int_{-\infty}^{\infty} dv\, F(v,t) = 1$ we arrive at the linear equation

$$\frac{\partial M(v,t)}{\partial t} = 2M(2v,t) - M(v,t),\qquad (9)$$

that holds for sufficiently large v. The derivation of Eq. (9) relies on the fact that the distribution $F(v,t)$ is normalized, but remarkably, the precise form of that distribution is not utilized. Indeed, as follows directly from (1), the outcome of interactions involving an agent with a very large opinion v is not affected by the opinion of the interaction counterpart. Consequently, large opinions are reduced by a factor 2 with each interaction, thereby leading to a simple multiplicative process [26]

$$v \to v/2 \to v/4 \cdots .\qquad (10)$$

Equation (9) merely reflects this multiplicative process.

In the long-time limit, the density $M(v,t)$ of monotonic agents adheres to the scaling form

$$M(v,t) \simeq e^{-\alpha t} e^{t/4}\, \mathcal{M}(V)\qquad (11)$$

with the same scaling variable $V = v e^{t/4}$. By integrating (11), we obtain the exponential decay (2) with $A = \int_{-\infty}^{\infty} dV\, \mathcal{M}(V)$. Figure 2 convincingly shows that the scale $v \sim t^{-1/4}$ also characterizes the opinion distribution of monotonic agents.

By substituting the scaling form (11) into (9), we find that the scaling function $\mathcal{M}(V)$ satisfies

$$\frac{V}{4}\frac{d\mathcal{M}(V)}{dV} + \left(\frac{5}{4} - \alpha\right)\mathcal{M}(V) = 2\mathcal{M}(2V)\qquad (12)$$

when $V \gg 1$. This difference-differential equation admits an algebraic solution, $\mathcal{M}(V) \sim V^{-\nu}$, with
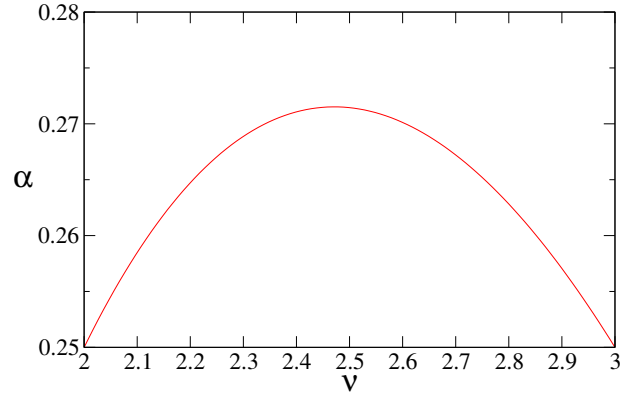
$$\alpha = \frac{5-\nu}{4} - 2^{1-\nu}.\qquad (13)$$

The right-hand side of (13) is bounded from above, see Fig. 3. The maximal value quoted in (3) occurs at

$$\nu = 3 + \frac{\ln \ln 2}{\ln 2} = 2.471233\ldots .\qquad (14)$$

Thus, the algebraic tail for $V \gg 1$ yields the upper bound $\alpha \le \alpha_*$. We postulate that this extremal value is realized, $\alpha = \alpha_*$, for the averaging process. Our numerical simulations give the estimate $\alpha = 0.272 \pm 0.003$ and hence support this theoretical prediction (see the inset to Fig. 1 and also, Appendix B). We note that our assumption that the behavior in the neighborhood of $u = 2v$ dominates the integral in Eq. (7) is consistent with the fact that the tail $M(v) \sim v^{-\nu_*}$ is shallower than the tail $F(v) \sim v^{-4}$. The selection of the extremal value emerging from the dispersion-like relation (13) very much resembles the selection of the extremal propagation velocity in traveling waves that are governed by partial differential equations in deterministic [27–31] and stochastic systems [32–36].

Next, we substitute the scaling form (11) into the master equation (7) and arrive at the linear integro-differential equation

$$\frac{V}{4}\frac{d\mathcal{M}}{dV} + \left(\frac{5}{4} - \alpha\right)\mathcal{M} = 2\int_V^{\infty} dU\, \mathcal{M}(U)\, \mathcal{F}(2V - U)\qquad (15)$$

with $\mathcal{F}(V)$ given in Eq. (6). We stress that this equation governs the scaling equation $\mathcal{M}(V)$ for all values of V, in contrast with Eq. (12) that applies only when $V \gg 1$. The parameter α is an eigenvalue of this equation, and in principle, a solution for the eigenvalue α requires a solution for the entire eigenfunction $\mathcal{M}(V)$. However, in our particular problem, extreme-value analysis suffices.

To understand the behavior when $V \ll -1$, we introduce the change of variables $U = 2V - W$, and thereby recast the right-hand side of Eq. (15) into $2\int_{-\infty}^{V} dW\, \mathcal{M}(2V - W)\, \mathcal{F}(W)$. In the limit $V \to -\infty$, this integral is negligible and Eq. (15) simplifies to $V\mathcal{M}' + (5 - 4\alpha)\mathcal{M} = 0$. Therefore, there is a second algebraic tail, $M(V) \sim (-V)^{-(5-4\alpha)}$ when $V \ll -1$.

Thus, the scaling function $\mathcal{M}(V)$ has two distinct algebraic tails

$$\mathcal{M}(V) \sim \begin{cases} (-V)^{-(5-4\alpha)} & V \ll -1, \\ V^{-\nu} & V \gg 1. \end{cases} \quad (16)$$

Interestingly, both tails of the scaling function $\mathcal{M}(V)$ are shallower than the tails of the scaling function $\mathcal{F}(V)$ as both $\nu < 4$ and $5 - 4\alpha < 4$. Of course, the inequality $M(v,t) \leq F(v,t)$ holds for all v. By comparing the two densities $e^{-\alpha t}\mathcal{M}(V_\pm) \sim \mathcal{F}(V_\pm)$, we find that (6) and (11) hold simultaneously in the scaling region $V_- \ll V \ll V_+$. Both scales V_+ and V_- grow exponentially with time, and hence, the two scaling forms (6) and (11) hold over an exponentially growing range of scaled opinions.

The inequality $\nu < 5 - 4\alpha$ suggests that that scaling function $\mathcal{M}(V)$ is asymmetric (figure 2). Moreover, as the tail in the positive velocity region is shallower than the tail in the negative velocity region, we expect that the fraction of monotonic agents with a positive opinion

$$M_+ = \int_0^\infty dV \, \mathcal{M}(V), \quad (17)$$

exceeds the fraction $M_- = 1 - M_+$ of monotonic agents with a negative opinion. Using numerical simulations we find that $M_+ = 0.74 \pm 0.01$ is roughly three times larger than M_-.

Our Monte Carlo simulations utilize a straightforward implementation of the averaging process. Initially, there are N agents whose opinions are drawn from a uniform distribution: $F(v,0) = 1$ for $|v| < 1/2$ and $F(v,0) = 0$ otherwise. We stress that our main results are independent of the shape of the initial distribution: the same scaling functions $\mathcal{F}(V)$ and $\mathcal{M}(V)$ are realized for compact distributions as well as non-compact distributions [25]. In each simulation step, two agents are selected at random, and their opinions are updated according to the averaging rule (1). Time is augmented by $2/N$ subsequently. We maintain a counter for the number of monotonic agents, and whenever an agent interacts for the first time with an agent having a larger opinion, the counter decreases by one. The simulation results shown throughout this paper represent an average over 10^3 independent Monte Carlo runs in a system of size $N = 10^8$.

We made the following choices to optimize the simulations: (i) we adjust the average initial opinion to zero, (ii) we keep track of agents with monotonically increasing opinions as well as agents with monotonically increasing opinions, and (iii) we rescale the opinion $v \to ve^{1/4}$ once per unit time. The latter rescaling enables direct measurement of $V = ve^{t/4}$, and additionally, it prevents simulation of exponentially small opinions.

IV. POSITIVITY

We also studied a related subset of agents with an opinion that remains strictly above average. Since the average
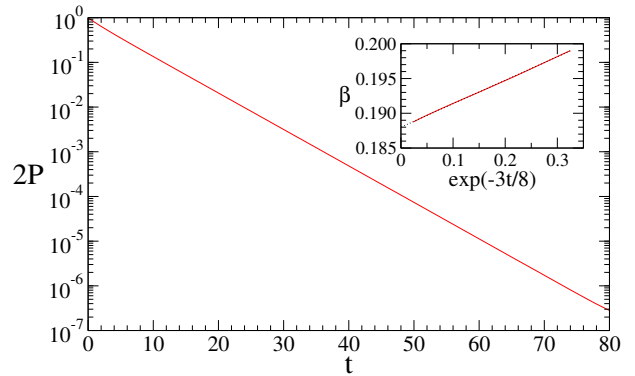

FIG. 4: The normalized fraction of positive agents $2P(t)$ versus time t (since only one half of all agents start with a positive opinion, $2P(0) = 1$). The inset shows $\beta(t) \equiv -d\ln P/\ln t$ versus $\exp(-3t/8)$. A linear fit, shown as the dashed line, yields the estimate $\beta = 0.188 \pm 0.001$.

opinion is zero, these are the agents with a strictly positive opinion. Let $P(t)$ be the fraction of such positive agents at time t. Our numerical simulations show that $P(t)$ decays exponentially with time (Fig. 4)

$$P(t) \simeq B \, e^{-\beta t} \quad \text{with} \quad \beta = 0.188 \pm 0.01. \quad (18)$$

We analyze the density $P(v,t)$ of agents with opinion $v > 0$, from which the overall density follows, $P(t) = \int_0^\infty dv \, P(v,t)$. Of course, there is a dual set of agents who maintain a strictly negative opinion, with an overall fraction $P(t)$ and density $P(-v,t)$. In the context of opinion dynamics, positive and negative agents can be viewed as agents that remain consistently on the left side or the right side of the political spectrum. The density $P(v,t)$ evolves according to the *linear* equation

$$\frac{\partial P(v,t)}{\partial t} = 2 \int_0^\infty du \, P(u,t)F(2v - u, t) - P(v,t), \quad (19)$$

for $v > 0$. This equation reflects that the density $P(v,t)$ is coupled to the overall density $F(v,t)$, and it differs from (7) only in the lower limit of integration. Using (19), we deduce the rate equation

$$\frac{dP(t)}{dt} = -\int_0^\infty du \, P(u,t) \int_u^\infty dv F(v,t) \quad (20)$$

for the fraction $P(t)$. Again, this evolution equation differs from (8) only in the lower limit of integration.

In the long-time limit, the distribution $P(v,t)$ acquires the scaling form

$$P(v,t) \simeq e^{-\beta t}e^{t/4} \mathcal{P}(ve^{t/4}). \quad (21)$$

This form is consistent with the exponential decay (18) with $B = \int_0^\infty dV \, \mathcal{P}(V)$. Our numerical simulations confirm that this scaling behavior applies at sufficiently large times (Fig. 5). The scaling function $\mathcal{P}(V)$ is qualitatively
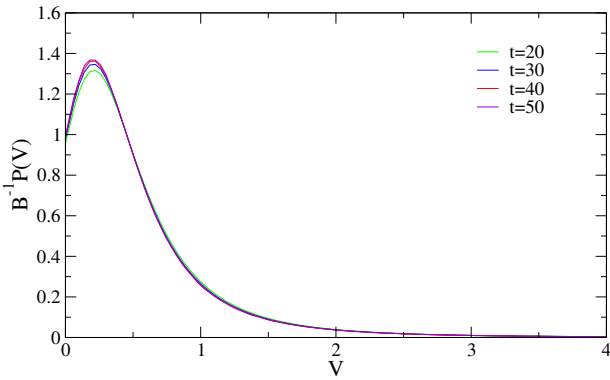
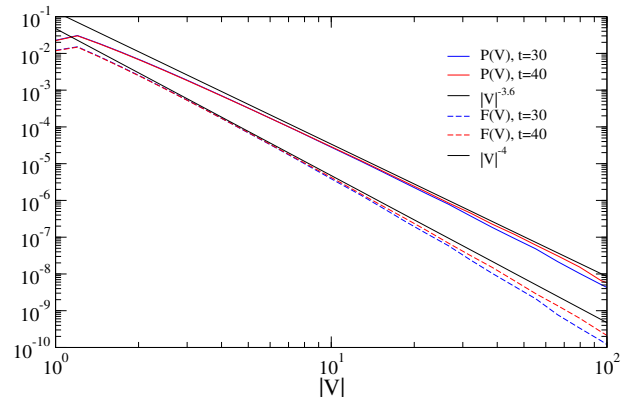FIG. 5: The normalized scaling function $B^{-1}\mathcal{P}(V)$ versus the scaled opinion V.

FIG. 6: The scaling function $\mathcal{F}(V)$ and $\mathcal{P}(V)$ for $V \gg 1$. Also shown as a reference are the theoretical results (6) and (24).

similar to $\mathcal{M}(V)$ (see Fig. 2): both functions are non-monotonic and are maximal at a nonzero value of V. For positive agents, the "depletion" region near $V = 0$ reflects that positive agents with a sufficiently small opinion are less likely to remain positive.

By substituting the scaling forms (6) and (21) into the evolution equation (19), we find that the scaling function $\mathcal{P}(V)$ satisfies the linear integro-differential equation

$$\frac{V}{4}\frac{d\mathcal{P}}{dV} + \left(\frac{5}{4} - \beta\right)\mathcal{P} = 2\int_0^\infty dU\, \mathcal{P}(U)\, \mathcal{F}(2V - U). \quad (22)$$

This equation poses an eigenvalue problem with the scaling function $\mathcal{P}(V)$ being the eigenfunction, and the exponent β being the eigenvalue. The eigenfunction is subject to the constraint $\mathcal{P}(V) > 0$ for all $V > 0$.

To determine the asymptotic behavior of $\mathcal{P}(V)$ when $V \gg 1$, we repeat the approach used in Section III and arrive at an equation that is entirely analogous to (15),

$$\frac{V}{4}\frac{d\mathcal{P}}{dV} + \left(\frac{5}{4} - \beta\right)\mathcal{P} = 2\mathcal{P}(2V). \quad (23)$$

Hence, the tail of the scaling function is algebraic, $\mathcal{P}(V) \sim V^{-\mu}$, and the dispersion relation reads

$$\beta = \frac{5 - \mu}{4} - 2^{1-\mu}. \quad (24)$$

By substituting the Monte Carlo simulation result (18) into the dispersion relation (24), we expect $\mu = 3.58 \pm 0.01$. The numerical simulation results (Fig. 6) give $\mu = 3.6 \pm 0.2$, consistent with (24). The eigenvalue β does not correspond to an extremum in the dispersion equation (24), so it must be determined as the eigenvalue of the full integro-differential equation (22). As was the case for monotonic agents, the large-v tail of the opinion density $P(v, t)$ is algebraic. Furthermore, the algebraic tail of the scaling function $\mathcal{P}(V)$ is shallower than the tail of the scaling function $\mathcal{F}(V)$ as $\mu < 4$.

V. DISCUSSION

In summary, we considered the averaging process and studied sub-classes of agents with an opinion that maintains a certain property throughout the evolution. In particular, we probed the fraction of agents with a monotonically decreasing opinion and the fraction of agents with a positive opinion. These fractions decrease exponentially with time, and the exponents characterizing these decays are eigenvalues of linear integro-differential equations. In the case of monotonic agents, we were able to find the eigenvalue analytically using an extremum selection principle, analogous to velocity selection in traveling waves [27–36]. Both monotonicity and positivity can be viewed as types of persistence, and as is typically the case, nontrivial persistence exponents characterize the time evolution [37–39].

For the averaging process, the opinion distribution becomes self-similar, and in particular, the scaling form (6) is characterized by the second moment (5). However, moments of the opinion distribution exhibit multiscaling and are not characterized by the second moment. The moments decay exponentially with time $\langle v^n \rangle \sim \exp(-\sigma_n t)$ with a nonlinear spectrum of exponents, $\sigma_n/n \neq \sigma_2/2$ when $n > 2$. We anticipate that moments of the opinion distribution of monotonic agents also exhibit multiscaling. Finding the corresponding spectrum of exponents is challenging because the evolution equations that govern the moments of $M(v, t)$ are not closed.

A natural generalization of the averaging process is to multi-component opinions. When the opinion of each agent is a vector, rather than a scalar, one may study monotonicity properties of the magnitude of the opinion vector. Additionally, one can investigate agents with one component of the opinion vector being always larger than all other components.

Monotonicity can also be studied in systems that reach a steady state, and in particular, averaging processes that are forced into a steady state [26]. Monotonicity can be

probed in an even broader class of stochastic processes since the trajectory of any fluctuating quantity may include segments where all changes in the value of the fluctuating quantity occur in the same direction.

Appendix A: Partial averaging

In the partial averaging process, agents make a partial compromise by moving part-way toward each other. The post interaction opinions are linear combinations of the pre-interaction opinions

$$(v_i, v_j) \rightarrow (pv_i + qv_j, qv_i + pv_j) , \qquad (A1)$$

with $0 \leq p \leq 1$ and $p + q = 1$ so that the average opinion is conserved. Each interaction reduces the opinion difference by factor $|p - q|$, as in an inelastic collision [14].

Treatment of the partial averaging process is a straightforward generalization of analysis above. For the random process (A1), the rate equation governing the density $F(v,t)$ of agents with opinion v at time t becomes

$$\frac{\partial F(v,t)}{\partial t} = \frac{1}{q} \int_{-\infty}^{\infty} du \, F(u,t) F\left(\frac{v-pu}{q},t\right) - F(v,t). \quad (A2)$$

We can verify that the distribution remains normalized, $\int dv \, F(v,t) = 1$, and that the average opinion is conserved, $\int dv \, v \, F(v,t) = 0$. We restrict our attention to symmetric distributions, and from (A2), it follows that the second moment decays exponentially with time, $\langle v^2(t) \rangle = \langle v^2(0) \rangle e^{-2pqt}$. In the long-time limit, the opinion distribution follows the scaling form

$$F(v,t) = e^{pqt} \mathcal{F}(V) \qquad (A3)$$

with the scaling variable $V = ve^{pqt}$. Independent of p, the scaling function $\mathcal{F}(V)$ is given by (6).

The density of monotonic agents $M(v,t)$ satisfies

$$\frac{\partial M(v,t)}{\partial t} = \frac{1}{q} \int_v^{\infty} du \, M(u,t) F\left(\frac{v-pu}{q},t\right) - M(v,t). \quad (A4)$$

In the long-time limit, the overall density of monotonic agents decays as in (2), and the density $M(v,t)$ of monotonic agents approaches the scaling form $M(v,t) \simeq e^{-\alpha t} e^{pqt} \mathcal{M}(V)$ with the scaling variable $V = ve^{pqt}$. The tail of the scaling function is algebraic, $\mathcal{M}(V) \sim V^{-\nu}$ when $V \gg 1$, and equation (13) that relates the exponents α and μ becomes
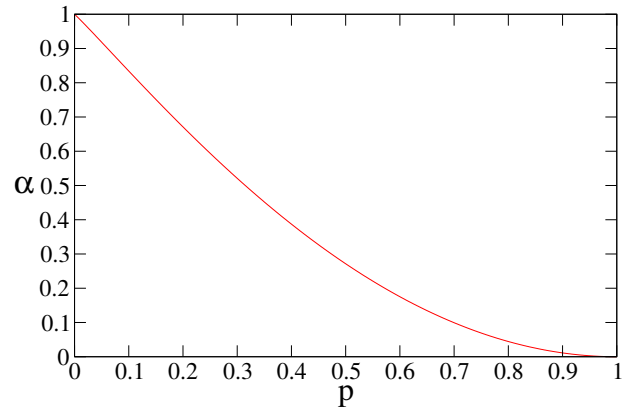
$$\alpha = 1 + pq(1-\nu) - p^{\nu-1} . \qquad (A5)$$

FIG. 7: The exponent α, given by (A6), versus p.

The extremum occurs at (figure 7)

$$\alpha = 1 - pq \frac{1 + \ln\left(\frac{1}{pq} \ln \frac{1}{p}\right)}{\ln \frac{1}{p}} . \qquad (A6)$$

Our numerical simulations confirm that this extremum is indeed selected by the dynamics. The exponent $0 \leq \alpha \leq 1$ decreases monotonically as p increases. The maximal value $\alpha = 1$ is achieved when the interaction is strongest ($p = 0$), and the minimal value $\alpha = 0$ is achieved when the interaction is weakest ($p = 1$).

The exponent ν which characterizes the tail of the opinion distribution is given by

$$\nu = 2 + \frac{\ln\left(\frac{1}{q} \ln \frac{1}{p}\right)}{\ln \frac{1}{p}} . \qquad (A7)$$

The exponent ν increases monotonically with p. Since $\nu < 4$, the tail of the scaling distribution $\mathcal{M}(V)$ remains less steep than the tail of the scaling distribution $\mathcal{P}(V)$.

Appendix B: Correction to the leading asymptotic behavior (2)

In problems admitting traveling wave solutions, the speed w and the wavenumber k of the traveling wave are related via the dispersion relation $w = \Phi(k)$, and the propagation velocity $w = w_* = \Phi(k_*)$ is selected at an extremum of the dispersion curve. This happens in a broad set of problems. Moreover, the asymptotic approach to the speed w_* is remarkably universal

$$w(t) = w_* + \frac{3}{2k_*} t^{-1} + B_2 t^{-3/2} + \cdots . \qquad (B1)$$

The leading t^{-1} correction was derived by Bramson [29] in the context of the Fisher-Kolmogorov equation [27, 28], and then confirmed for many other deterministic [30,

31] and stochastic [32–36] systems. The second $t^{-3/2}$ correction was established in Ref. [30].

In our problem, the dispersion relation is given by Eq. (13). By assuming (B1) is valid, the correction to the leading asymptotic behavior (3) is given by

$$\alpha(t) = \alpha_* + \frac{3}{2\nu_*}\, t^{-1} + B_2 t^{-3/2} + \cdots . \qquad (B2)$$

This form implies an algebraic correction to the leading asymptotic behavior (2) as $M(t) \simeq A\, t^{-a} \exp(-\alpha t)$ with $a = 3/(2\nu_*)$. The inset in Fig. 1 shows that $\alpha(t) \equiv -d \ln M / \ln t$ varies linearly with the inverse time $1/t$. A two-parameter linear fit, accounting only for the leading correction to the asymptotic behavior, yields the value $\alpha = 0.272 \pm 0.003$. However, equation (B2) which accounts for the two leading corrections also involves only two parameters, α and B_2 because α and ν are related by (13). By using (B2) and (13), we obtain the improved estimate $\alpha = 0.2715 \pm 0.0005$ from the very same simulation results. Such an approach is in the spirit of Bob Ziff's work on numerous problems including percolation [40], planar dimer tilings [41], and random sequential adsorption [42], where he perfected the art of using finite-time and finite-size corrections for producing high-precision measurements from Monte Carlo simulations.

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