Comment on "Kinematic Scaling and Crossover to Scale Invariance in Martensite Growth"

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In a recent Letter [1], Rao *et. al.* proposed a simple model describing martensites formation. In 2D, the model is defined as follows: (i) Segments grow from seeds that are nucleated in a bounded region; (ii) The tips of these segments move with a constant velocity V until they hit either another segment or the boundary; (iii) Any tip grows in one of two possible orthogonal directions. There are two important limiting cases, uniform nucleation and simultaneous nucleation. Uniform nucleation with $V=\infty$ can be viewed as a multifragmentation process [2,3], and both result in similar patterns (compare Figs. 2 of [1] and [3]). In the following Comment, we obtain exact asymptotic properties of the uniform model using the method of Ref. [3]. The process is characterized by an infinite amount of scales and an infinite set of conserved quantities, and thus exhibits multiscaling. These findings disagree with the ordinary scaling behavior of the segment length distribution reported in [1].

The case of uniform nucleation is equivalent to a stochastic process where seeds appear uniformly in space with unit rate, and grow with infinite velocity in the x or the y directions with equal probabilities. For simplicity, we choose the unit square as the transformed region. The distribution function $P(x_1,x_2,t)$, describing rectangles of size $x_1\times x_2$ arising in this kinetic process, satisfies

$$\frac{\partial P(x_1,x_2,t)}{\partial t} = -x_1x_2P(x_1,x_2,t) \qquad (1)$$

$$+x_2\int_{x_1}^{1}dx_1'P(x_1',x_2,t)+x_1\int_{x_2}^{1}dx_2'P(x_1,x_2',t).$$

Performing the double Mellin transform, $M(s_1,s_2,t)=\int dx_1dx_2x_1^{s_1-1}x_2^{s_2-1}P(x_1,x_2,t)$, Eq. (1) reduces to

$$\frac{\partial M(s_1,s_2)}{\partial t}=\left(s_1^{-1}+s_2^{-1}-1\right)M(s_1+1,s_2+1). \qquad (2)$$

Indeed, the total area is $M(2,2,t)=1$ and the number of rectangles is $N=M(1,1,t)=1+t$, in agreement with (2). A remarkable feature of Eq. (2) is that it implies the existence of an infinite number of conservation laws. The moments $M(s_1,s_2,t)$, with s_1 and s_2 on the hyperbola $s_1^{-1}+s_2^{-1}=1$, are time-independent. Thus, in addition to the conservation of the total area, there is an infinite amount of hidden conserved integrals. These integrals are in fact responsible for the absence of simple scaling solutions. Indeed, a solution of the form $P(x_1,x_2,t)=t^wQ(t^zx_1,t^zx_2)$ to Eq. (1) would imply an infinite set of relations, $w=z(s_1+s_2)$ when $s_1^{-1}+s_2^{-1}=1$, which cannot be satisfied by just two scaling exponents, w and z.

The moments of Eq. (2) are exactly solvable in terms of generalized hypergeometric functions. However, we present an asymptotic analysis since for sufficiently large $t\equiv N$, the moments depend algebraically on time, $M(s_1,s_2,t)\sim t^{-\alpha(s_1,s_2)}$. Substituting this form into Eq. (2) yields $\alpha(s_1,s_2)+1=\alpha(s_1+1,s_2+1)$. This difference equation is solved subject to the boundary conditions, $\alpha(s_1,s_2)=0$ on the hyperbola $s_1^{-1}+s_2^{-1}=1$, and one finds

$$\alpha(s_1,s_2)=\frac{s_1+s_2}{2}-1-\sqrt{\left(\frac{s_1-s_2}{2}\right)^2+1}. \qquad (3)$$

The nontrivial nature of the asymptotic behavior is demonstrated by evaluating moments such as $\langle x_1^{n_1}x_2^{n_2}\rangle\equiv\int dx_1dx_2x_1^{n_1}x_2^{n_2}P(x_1,x_2,t)/\int dx_1dx_2P(x_1,x_2,t)$. For example, the ratio $\langle (x_1x_2)^n\rangle/\langle x_1^n\rangle\langle x_2^n\rangle\sim t^{-(\sqrt{n^2+4}-2)}$, *depends* asymptotically on time, while for a scaling distribution $P(x_1,x_2,t)$ such a ratio would approach a nonzero constant. The length distribution $P(l,t)$ can be easily obtained from the size distribution using the evolution equation $\partial P(l,t)/\partial t=\int dx_1dx_2x_1x_2P(x_1,x_2,t)[\delta(l-x_1)+\delta(l-x_2)]/2$. Hence, the corresponding Mellin transform $M(s,t)=\int dl l^{s-1}P(l,t)$ satisfies $\partial M(s,t)/\partial t=M(s+1,2,t)$. This is solved to find $M(s,t)\sim t^{1-\alpha(s+1,1)}$. Substituting Eq. (3) gives

$$\langle l^n\rangle\sim\langle x_1^n\rangle\sim t^{-\left(n+2-\sqrt{n^2+4}\right)/2}, \qquad (4)$$

rather than simple scaling $\langle l^n\rangle\sim t^{-n/2}$ [1]. Thus, the length distribution is characterized by *nontrivial* exponents. For example, one finds $\langle l\rangle\sim t^{-(3-\sqrt{5})/2}\sim t^{-.382}$ and $\langle l^2\rangle\sim t^{2-\sqrt{2}}\sim t^{-.586}$, as confirmed by simulations. Since all the moments still show a power-law behavior, we conclude that the model exhibits a *multiscaling* asymptotic behavior. In contrast with the general behavior, moments of the area $A=x_1x_2$ follow ordinary scaling, $\langle A^n\rangle\sim\langle A\rangle^n\sim t^{-n}$, and the area distribution function approaches a simple scaling form $P(A,t)\simeq t^2e^{-At}$.

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