Fragmentation with a Steady Source

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We investigate fragmentation processes with a steady input of fragments. We find that the size distribution approaches a stationary form which exhibits a power law divergence in the small size limit, $P_\infty(x) \sim x^{-3}$. This algebraic behavior is robust as it is independent of the details of the input as well as the spatial dimension. The full time dependent behavior is obtained analytically for arbitrary inputs, and is found to exhibit a universal scaling behavior.

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Fragmentation underlies numerous natural phenomena [1–5]. The quantity being "split" can be the mass, momentum, or the area, and typically, fragments continue splitting independently of each other. Examples include polymer degradation [6], breakup of liquid droplets [7] and atomic nuclei [8], martensitic transformations [9,10], shattering of solid objects [11,12], and meteor impacts. Fragmentation also arises in several topics of computer science [13–15].

The simplest fragmentation models assume that the rate by which fragments are produced is a function of their size only [16–20]. In this study, we focus on the classic "random scission model" where the cutting is uniform and hence a fragment is cut with a rate proportional to its size. In particular, we are interested in situations where the system is subject to a steady input of fragments. Such "open" systems occur in industrial applications where coarse material is continuously fed into a grinding apparatus to produce a fine powder [21–23].

Fragmentation in open systems has received less attention than fragmentation in closed systems [24,25]. We will show, however, that fragmentation with input is actually *simpler* than the classical counterpart as the system reaches a stationary state which is remarkably robust. Specifically, fragmentation with a steady source is characterized by an algebraic divergence of the size distribution in the small size limit, and this behavior is independent of the particular form of the input. Additionally, the time dependent behavior, obtained analytically for arbitrary inputs, follows a scaling behavior. These two features are shown to be closely related.

We start with a one-dimensional fragmentation process subject to constant input of segments. Here "one-dimensional" means that the fragments are characterized by a single variable which we shall call "length" (the fragments can be viewed as segments). Let the system be initially empty and intervals whose length is within the range $(x, x+dx)$ are added with rate $f(x)\,dx$. Additionally, intervals are cut with a constant spatially homogeneous rate; we set this rate equal to unity without loss of generality. Fragmentation with input has a natural geometric interpretation. Consider the segments as part of an infinite line. The fragmentation process is equivalent to deposition of point "cracks" on the line. The line is initially "immune" to fragmentation, but then segments of length x become "susceptible" to fragmentation with rate $f(x)$. Hence, fragmentation with input is equivalent to inhomogeneous fragmentation of a growing line.

The density $P(x,t)$ of intervals of length x at time t evolves according to the following rate equation

$$\frac{\partial P(x,t)}{\partial t} = -xP(x,t) + 2\int_x^\infty dy\, P(y,t) + f(x). \qquad (1)$$

The negative term on the right-hand side accounts for loss due to fragmentation with the rate equal to the fragment size since the cutting is uniform. The gain term gives the increase in fragments of size x due to cutting of longer fragments. The last term accounts for input of fragments of size x.

The size distribution can be determined by applying the Mellin transformation. The Mellin transform (or moment) of the distribution, $M(s,t) = \int dx\, x^{s-1}P(x,t)$, satisfies

$$\frac{\partial M(s,t)}{\partial t} = \frac{2-s}{s}M(s+1,t) + \hat{f}(s), \qquad (2)$$

where $\hat{f}(s) = \int dx\, x^{s-1} f(x)$ is the Mellin transform of the input density $f(x)$. Although this hierarchy of equations is infinite, its linear nature makes it tractable, as will be seen below.

We first examine what happens when $t \to \infty$. In this limit, the length density should approach the stationary distribution, $P(x,t) \to P_\infty(x)$. Setting the time derivative in Eq. (2) to zero gives the corresponding transform $M_\infty(s) = \left(1 + \frac{2}{s-3}\right)\hat{f}(s-1)$. Note that $\hat{f}(s+n)$ is the Mellin transform of $x^n f(x)$, and $(s-m)^{-1}\hat{f}(s)$ is the transform of $x^{-m}\int_x dy\, y^{m-1} f(y)$. These two facts allow to perform the inverse Mellin transform and yield $P_\infty(x)$ explicitly in terms of the input function

$$P_\infty(x) = x^{-1} f(x) + 2\,x^{-3}\int_x^\infty dy\, y f(y). \qquad (3)$$

In the small size limit, the integral on the right-hand side of Eq. (3) approaches the average length added per unit time, $\lambda = \hat{f}(2) = \int dx\, x f(x)$. Thus, the length density becomes purely algebraic

$$P_\infty(x) \to 2\lambda x^{-3}, \quad \text{when} \quad x \to 0. \tag{4}$$

This behavior is robust as the first term on the right-hand side of Eq. (3) always diverges slower than x^{-3} in the limit $x \to 0$ (otherwise, the total length input rate, $\int dx\, x f(x)$, would be infinite). For a class of input densities, the algebraic behavior may not be necessarily restricted to small sizes. For example, for monodisperse inputs $f(x) = \lambda\delta(x-1)$, the algebraic behavior extends to all sizes $x < 1$, $P_\infty(x) = \lambda x^{-1}\delta(x-1) + 2\lambda x^{-3}$.

The general algebraic behavior should be contrasted with the exponential length distribution found generally in the absence of input. Algebraic distributions have been observed experimentally in fragmentation of solid objects such as rods, spheres, and bricks [11,12]. Although the corresponding exponents measured in these experiments are significantly lower, typically between 1 and 2, it is worth noting that a steady source of fragments can serve as a mechanism for generating algebraic distributions. Curiously, algebraic distributions with an exponent close to 3 were reported recently in social systems which can be viewed as open ones (distributions of citations, of the number of links to sites on the internet, etc.; see e.g. Refs. [26–28]).

The limiting size distribution is ultimately related to the time dependent behavior. This can be demonstrated using the following heuristic argument. From Eq. (2), the total length $L(t) = M(2,t)$ grows linearly with time $\dot{L}(t) = \lambda$, and hence, $L(t) = \lambda t$. Similarly, the total number of fragments, $N(t) = M(1,t)$ satisfies $\dot{N}(t) = \lambda t + \mu$, where $\mu = \hat{f}(1) = \int dx f(x)$ is the number of segments added per unit time, and consequently, $N(t) = \frac{1}{2}\lambda t^2 + \mu t$. These two time dependent results imply that the average fragment length, $\langle x\rangle = L/N$, decreases with time according to $\langle x\rangle \sim t^{-1}$. For the length distribution to follow a scaling form, the corresponding scaling variable must be $x/\langle x\rangle$. The prefactor is fixed by the total number of fragments, $N(t) \sim \lambda t^2$, so the scaling form reads

$$P(x,t) \simeq \lambda t^3 F(xt). \tag{5}$$

This scaling form would be consistent with a time independent limiting distribution only when $F(\xi) \sim \xi^{-3}$, thereby implying the algebraic divergence (4).

The full time dependent solution can be found using the Charlesby method [16]. This method starts with a formal expansion of the Mellin transform,

$$M(s,t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} M_n(s), \tag{6}$$

and proceeds by solving for the functions $M_n(s)$ iteratively. Indeed, substituting the expansion (6) into Eq. (2) and equating similar powers of time yields $M_0(s) = 0$, $M_1(s) = \hat{f}(s)$ and $M_{n+1}(s) = -\frac{s-2}{s} M_n(s+1)$ for $n \geq 2$. Solving this set of equations recursively gives

$$M_{n+1}(s) = (-1)^n \frac{(s-1)(s-2)}{(s+n-1)(s+n-2)} \hat{f}(s+n).$$

To take advantage of the inversion rules used to obtain Eq. (3), we re-write $M_{n+1}(s)$ as

$$M_{n+1}(s) = (-1)^n \left[1 - \frac{n(n+1)}{s+n-1} + \frac{n(n-1)}{s+n-2}\right] \hat{f}(s+n).$$

From Eq. (6), the size distribution can be written as a power series

$$P(x,t) = \sum_{n=0}^{\infty} \frac{t^{n+1}(-x)^n}{(n+1)!} P_n(x), \tag{7}$$

where the inverse transform of $M_{n+1}(s)$ has been conveniently written as $(-x)^n P_n(x)$. The three terms in the above expression for $M_{n+1}(s)$ can be inverted using the rules outlined before Eq. (3). The final expression for $P_n(x)$ reads

$$P_n(x) = f_1(x) + \frac{n(n+1)}{x} f_2(x) + \frac{n(n-1)}{x^2} f_3(x), \tag{8}$$

with

$$\begin{aligned} f_1(x) &= f(x), \\ f_2(x) &= \int_x^\infty dy\, f(y), \\ f_3(x) &= \int_x^\infty dy\, y f(y). \end{aligned} \tag{9}$$

Summing the three terms separately gives the fragment size distribution

$$P(x,t) = \sum_{k=1}^{3} t^k f_k(x) F_k(xt), \tag{10}$$

with the scaling functions

$$\begin{aligned} F_1(z) &= z^{-1}(1 - e^{-z}), \\ F_2(z) &= e^{-z}, \\ F_3(z) &= z^{-3}\left[2 - (2 + 2z + z^2)e^{-z}\right]. \end{aligned} \tag{11}$$

The function $F_3(z)$ has been obtained from the power series $F_3(z) = \sum_{n\geq 0} \frac{(-z)^n}{n!(n+3)}$. One can verify that the previous results for $N(t)$, $L(t)$, and $P_\infty(x)$ agree with this solution. Thus, we have obtained the full time dependent solution for an *arbitrary* time independent input $f(x)$.

The size distribution of Eq. (10) exhibits scaling. Indeed, in the limit $t \to \infty$, $x \to 0$ with the scaling variable $z = xt$ kept finite, the third term in the sum on the right-hand side of Eq. (10) dominates, and the anticipated scaling behavior of Eq. (5) is confirmed with $F(z) = F_3(z)$. Interestingly, the only parameter relevant asymptotically is the overall length input rate λ.

The limiting behaviors of the scaling distribution are

$$F(z) \simeq \begin{cases} \frac{1}{3} - \frac{1}{4}z & z \ll 1, \\ 2z^{-3} & z \gg 1. \end{cases} \tag{12}$$

In particular, the large z behavior implies the correct asymptotic $P_\infty(x) \sim x^{-3}$, in agreement with Eq. (4). Thus, for sufficiently large fragments, $x \gg t^{-1}$, the distribution has already reached the final limiting form, while smaller sizes are still created.

The formal solution (10) has an interesting "staircase" structure, a time power series whose terms are products of time independent functions $f_k(x)$ and time dependent functions $F_k(xt)$. In fact, the solution for the random scission model in the absence of input is also characterized by a similar structure. Indeed, consider the evolution equation

$$\frac{\partial \tilde{P}(x,t)}{\partial t} = -x\tilde{P}(x,t) + 2\int_x^\infty dy\, \tilde{P}(y,t) \tag{13}$$

corresponding to the above fragmentation process in the absence of input. Given the initial conditions $\tilde{P}(x,0) = \tilde{f}(x)$, the solution can be obtained following the same steps that led to Eq. (10). Again, the full time dependent solution is a three term expansion:

$$\tilde{P}(x,t) = \sum_{k=1}^{3} t^{k-1} \tilde{f}_k(x)\, \tilde{F}_k(xt). \tag{14}$$

The time independent functions are given by the same expressions (9) as in the input case (with f_k replaced by $\tilde{f}_k$), while the time dependent functions are different from (11): $\tilde{F}_1(z) = \tilde{F}_3(z) = e^{-z}$, and $\tilde{F}_2(z) = (2-z)e^{-z}$. In the limit $t \to \infty$, $x \to 0$ with $z = xt$ kept finite, the scaling behavior emerges again. Specifically, $\tilde{P}(x,t) \simeq \tilde{\lambda} t^2 \tilde{F}(z)$ with $\tilde{\lambda} = \int dx\, x\tilde{f}(x)$ and the exponential scaling function $\tilde{F}(z) = e^{-z}$.

The above solutions for the input case with empty initial conditions and no input case with arbitrary initial conditions can be used to construct the general solution for Eq. (1). Indeed, the sum of the solutions (10) and (14), $P(x,t) + \tilde{P}(x,t)$, is the solution for a fragmentation process with input $f(x)$ starting from an initial distribution $\tilde{f}(x)$. As expected, the initial conditions are "forgotten" in the long time limit as $P(x,t)$ given by Eq. (10) dominates over $\tilde{P}(x,t)$ given by Eq. (14). In particular, the scaling solution (5) is recovered, and the $P_\infty(x) \simeq 2\lambda x^{-3}$ divergence of the limiting distribution holds in general.

To examine the robustness of the algebraic behavior above, we consider a natural generalization to d spatial dimensions [29–32]. Given that the most interesting long time behavior is independent of the details of the source term, we focus on the simplest monodisperse inputs, namely, unit hypercubes. For instance, in two dimensions we add unit squares with rate λ. A unit square is divided by choosing a point (x_1, x_2) with a uniform probability density, and cutting the original square into four rectangles of sizes $x_1 \times x_2$, $x_1 \times (1-x_2)$, $(1-x_1)\times x_2$, and $(1-x_1)\times(1-x_2)$. Similarly, the process is repeated with rectangular fragments.

Let $P(\mathbf{x}, t)$ with $\mathbf{x} \equiv (x_1, \ldots, x_d)$ be the distribution of fragments of size $x_1 \times \cdots \times x_d$ at time t. This quantity evolves according to the rate equation

$$\left(\frac{\partial}{\partial t} + |\mathbf{x}|\right) P(\mathbf{x},t) = 2^d \int_{\mathbf{x}} d\mathbf{y}\, P(\mathbf{y},t) + \lambda\delta(\mathbf{x}-\mathbf{1}). \tag{15}$$

Here, we used the shorthand notations $\mathbf{1} = (1,\ldots,1)$ and $|\mathbf{x}| = x_1\cdots x_d$. The d-dimensional Mellin transform, $M(\mathbf{s},t) = \int d\mathbf{x}\, x_1^{s_1-1}\cdots x_d^{s_d-1} P(\mathbf{x},t)$ with $\mathbf{s} \equiv (s_1,\ldots,s_d)$, reduces Eq. (15) to

$$\frac{\partial M(\mathbf{s},t)}{\partial t} = \left(\frac{2^d - s_1\cdots s_d}{s_1\cdots s_d}\right) M(\mathbf{s}+\mathbf{1},t) + \lambda. \tag{16}$$

We focus on the limiting size distribution $P_\infty(\mathbf{x})$. Its Mellin transform $M_\infty(\mathbf{s})$ is found from Eq. (16) by setting the time derivative to zero. One gets $M_\infty(\mathbf{s}) = \lambda\left(1 + \frac{2^d}{(s_1-1)\cdots(s_d-1)-2^d}\right)$. Inverting this relation yields [13]

$$P_\infty(\mathbf{x}) = \lambda\left[\delta(\mathbf{x}-\mathbf{1}) + 2^d|\mathbf{x}|^{-1}\Phi_d(\xi)\right], \tag{17}$$

with the shorthand notations

$$\Phi_d(\xi) = \sum_{n=0}^{\infty}\left(\frac{\xi^n}{n!}\right)^d \quad \text{and} \quad \xi = 2\left(\prod_{i=1}^{d} \ln\frac{1}{x_i}\right)^{1/d}. \tag{18}$$

In one dimension, $\Phi_1(\xi) = e^\xi = x^{-2}$, and we recover the one-dimensional result $P_\infty(x) = 2\lambda x^{-3}$. In two dimensions, $\Phi_2(\xi) = I_0(2\xi)$ where I_0 is the modified Bessel function, and in general $\Phi_d(\xi)$ can be expressed in terms of hypergeometric functions.

The small size behavior of $P_\infty(\mathbf{x})$ can be obtained by using the steepest decent method. The leading tail behavior, $\Phi_d(\xi) \simeq (2\pi\xi)^{\frac{1-d}{2}} e^{\xi d}$ for $\xi \gg 1$, corresponds to the case when at least one of the lengths is small, i.e., $x_i \ll 1$. Returning to the original variables, we re-write the above asymptotic as

$$P_\infty(\mathbf{x}) \sim |\mathbf{x}|^{-1}|\ln\mathbf{x}|^{-\frac{d-1}{2d}} \exp\left[2d\,(|\ln\mathbf{x}|)^{1/d}\right],$$

where $|\ln\mathbf{x}| \equiv \prod_{i=1}^d \ln\frac{1}{x_i}$. Thus, the fragment distribution exhibits a "log-stretched-exponential" behavior.

Let us consider the limiting volume distribution $P_\infty(V)$ defined via $P_\infty(V) = \int d\mathbf{x}\, P_\infty(\mathbf{x})\,\delta\left(V - x_1\cdots x_d\right)$. Its Mellin transform, $M_\infty(s) = \int dV\, V^{s-1}P_\infty(V)$, immediately follows from Eq. (16): $M_\infty(s) = \lambda\left[1 + \frac{2^d}{(s-1)^d - 2^d}\right]$. Using the identity $(a^d-1)^{-1} = d^{-1}\sum_{k=0}^{d-1}\zeta^k(a-\zeta^k)^{-1}$, where $\zeta = e^{2\pi i/d}$ is the primitive $d^{\rm th}$ root of unity, we can express $M_\infty(s)$ as a sum over simple poles at $1+2\zeta^k$. Consequently, the inverse Mellin transform is given by a linear combination of d power laws,

$$P_\infty(V) = \lambda\left[\delta(V-1) + \frac{2}{d}\sum_{k=0}^{d-1}\zeta^k V^{-1-2\zeta^k}\right]. \tag{19}$$

One can verify that the volume distribution is real since it equals its complex conjugate. The small-volume tail of the distribution can be obtained by noting that the sum in Eq. (19) is dominated by the first term in the series,

$$P_\infty(V) \simeq \frac{2\lambda}{d}\, V^{-3}, \quad \text{for} \quad V \to 0. \tag{20}$$

Thus, the same V^{-3} algebraic behavior occurs in all spatial dimensions. Clearly, this divergence is general. Indeed, the time dependent evolution equation (15) implies that the overall volume grows linearly, and that the overall number of fragments grows quadratically. Therefore, the heuristic scaling argument leading to Eq. (5) extends to higher dimensions, and consequently, the limiting behavior is given by Eq. (20).

In summary, we have studied random fragmentation processes in the presence of a steady source. We have solved for the full time dependent behavior in terms of the input function. In the long time limit, the size distribution exhibits a universal scaling behavior. The limiting distribution diverges algebraically according to x^{-3} in the small size limit. This behavior is robust. It applies to arbitrary inputs, and it extends to higher dimensions as well. Interestingly, the only asymptotically relevant parameter is the total volume added per unit time. Additionally, we have shown that the scaling behavior can be used to predict the algebraic nature of the final size distribution. Hence, the scaling behavior and the limiting distribution are closely related.

The solution for the time dependent behavior exhibits an interesting staircase structure. The two progressively weaker corrections to the leading behavior are of the order t^{-1} and t^{-2}, respectively. Such staircase structures may be a useful tool for treating similar integro-differential equations which are expected to exhibit a scaling asymptotic behavior. For instance, one can check whether substituting such an ansatz leads to a closed system of equations for the time dependent and time independent functions.

The above results can be extended in a number of ways. One may try to derive a general solution for $P(\mathbf{x}, t)$ in higher dimensions for arbitrary input rate $f(\mathbf{x})$. We anticipate that geometric features of the fragments will be interesting. Indeed, in the no input case the volume distribution exhibits an ordinary scaling behavior while multiscaling asymptotic behavior underlies the full multivariate size distribution. Additionally, a nontrivial set of conservation laws exists as all moments $M(\mathbf{s}^*, t)$ with $\prod_{i=1}^{d} s_i^* = 2^d$ are conserved, as seen from Eq. (16). In the presence of input, the same moments grow linearly in time but multiscaling should still hold asymptotically.

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