Multiscaling in Stochastic Fractals

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We introduce a simple kinetic model describing the formation of a stochastic Cantor set in arbitrary spatial dimension d. In one dimension, the model exhibits scaling asymptotic behavior. For $d > 1$, the volume distribution is characterized by a single scale $t^{-1/2}$, while other geometric properties such as the length are characterized by an infinite number of length scales and thus exhibit multiscaling.

The notion of a fractal has been widely used to describe self-similar structures [1]. The simplest way to construct a fractal is to repeat a given operation over and over again. The classical example of such a repetitive consruction is the Cantor's "middle-third erasing" set [1]. Recall the definition of this set: One divides an interval into three equal intervals and then removes the middle interval; on the next step, one repeats the same procedure with the two remaining intervals; *etc.*. The outcome of this process is a counterintuitive uncountable set having a measure ("length") zero. The Cantor set turns out to be a perfect fractal of dimension $D_f = \ln(2)/\ln(3) \cong 0.63093$.

The Cantor set is a *regular* fractal. In contrast, self-similar structures arising in nature are usually random. Moreover, fractals are usually formed by *continuous* kinetic processes while the classical repetitive constructions are discrete in time. In the present letter we introduce a stochastic process which may be considered as a natural kinetic counterpart to the original Cantor construction. The resulting set turns out to be a random fractal of dimension $D_f = (\sqrt{17}-3)/2 \cong 0.56155$. We also investigate d-dimensional random Cantor sets and find that several geometric characteristics such as the average length, surface area, *etc.*, are characterized by different scales. In the following, the existence of multipole kinetic exponents characterizing the process, will be shortly called multiscaling.

In one dimension, our model can be defined as follows. Starting with the unit interval [0:1], cracks are deposited uniformly on the unit interval with unit rate. When two cracks apear on the initial interval, the middle is removed immediately and two new intervals are formed. The process continues independently for the surviving intervals such that whenever a surviving interval contains two cracks, the middle interval is removed. In the classical Cantor process, after n stages we are left with 2^n intervals of length 3^{-n}. In the stochastic process the number of intervals and their lengths at time t are in principle arbitrary. The distribution function $P(x,t)$ describing intervals of length x at time t satisfies the following linear evolution equation,

$$\frac{\partial P(x,t)}{\partial t} = -\frac{x^2}{2} P(x,t) + 2\int_x^1 dy (y-x) P(y,t), \qquad (1)$$

with the initial conditions $P(x,0) = \delta(x-1)$. The loss term on the right-hand side represents the decrease of intervals of length x, x-mers, due to the division process. Each division event consists of choosing two points at random and thus, the overall breakage rate is quadratic in the interval length, while the factor 1/2 arises since the two points are indistinguishable. The gain term represents the increase of x-intervals due to breakups of longer intervals.

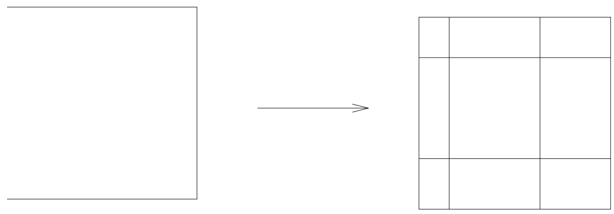

FIG. 1. Fig. 1 Illustration of the process in two dimension.

The formation of the random Cantor set is equivalent to random sequential parking on a line with a uniform distribution of lengths of parking intervals. While the latter problem has been investigated in Ref. [2], we present an alternative solution method [3] that can be easily generalized to higher dimensions. This method focuses on the leading asymptotic behavior of the moments of the length distribution $M(s,t)$, defined by

$$M(s,t) = \int_0^1 P(x,t) x^{s-1} dx. \qquad (2)$$

The rate equation (1) yield the following kinetic equation for the moments,

$$\frac{\partial M(s,t)}{\partial t} = \left[-\frac{1}{2} + \frac{2}{s(s+1)}\right] M(s+2,t). \qquad (3)$$

Asymptotically, the moments exhibit the power-law behavior

$$M(s,t) \simeq A(s) t^{-\alpha(s)}. \qquad (4)$$

By inserting the anticipated power-law behavior into Eq. (3) and solving the resulting difference equations one gets

$$\alpha(s) = \frac{s-\beta}{2}, \quad A(s) = \frac{\Gamma(s)\Gamma\left(\beta+\frac{1}{2}\right)}{\Gamma\left(\frac{s+\beta+1}{2}\right)\Gamma(\beta)} 2^{(\beta-s)/2}. \quad (5)$$

In the above equation we have introduced a shorthand notation $\beta = (\sqrt{17}-1)/2$.

Eq. (4) implies that in the long-time limit $P(x,t)$ approaches the scaling form,

$$P(x,t) \simeq t^{\beta/2}\Phi\left(x\sqrt{t}\right), \quad (6)$$

with the scaling function $\Phi(z)$ being the inverse Mellin transform of $A(s)$. In the limit of small $z, \Phi(z)$ approaches a constant while in the large-z limit, $\Phi(z) \sim z^{-\beta}\exp(-z^2/2)$.

Furthermore, the total number of intervals $N(t)$, $N(t) \equiv M(1,t)$, grows as $t^{(\beta-1)/2}$, while the typical interval size $\langle x\rangle$, $\langle x\rangle = M(2,t)/N(t)$, decays as $t^{-1/2}$. This simple scaling relation follows directly from the rate equations, since the loss rate is quadratic in the interval length. However, it is interesting that the simple rate equation (1) leads to non-trivial asymptotic exponents.

Knowledge of the asymptotic behavior of the average length and the average number enables calculation of the fractal dimension. Since $N \sim \langle x\rangle^{-(\beta-1)}$, the fractal dimension of the stochastic Cantor set is given by

$$D_f = \beta - 1 = (\sqrt{17}-3)/2 \cong 0.56155. \quad (7)$$

The above dimension is smaller than the fractal dimension of the classic Cantor set, $D_f = \ln 2/\ln 3 \cong 0.63093$.

We turn now to the general d-dimensional version of the model. In two dimensions, the model describes the formation of the stochastic Cantor gasket. The governing rule of the model is sketched in Fig. 1 for the two-dimensional situation. Denote by $P(\mathbf{x},t), \mathbf{x} = (x_1,\ldots,x_d)$, the distribution function for (hyper)rectangles of size $x_1 \times \ldots \times x_d$. The rate equation governing $P(\mathbf{x},t)$, is given by a straightforward generalization of Eq. (1),

$$\frac{\partial P(\mathbf{x},t)}{\partial t} = \frac{P(\mathbf{x},t)}{2^d}\prod_{j=1}^{d} x_j^2 + (3^d-1)\int_{x_1}^{1}\ldots\int_{x_d}^{1} P(\mathbf{y},t)\prod_{j=1}^{d}(y_j - x_j)dy_j. \quad (8)$$

Similarly, the moments of the distribution function $P(\mathbf{x},t)$,

$$M(\mathbf{s},t) = \int_0^1\ldots\int_0^1 P(\mathbf{x},t)\prod_{j=1}^{d} x_j^{s_j-1}dx_j, \quad (9)$$

satisfy the kinetic equation

$$\frac{\partial M(\mathbf{s},t)}{\partial t} = \left[-\frac{1}{2^d} + \left(3^d-1\right)\prod_{j=1}^{d}\frac{1}{s_j(s_j+1)}\right] M(\mathbf{s}+\mathbf{2},t), \quad (10)$$

where $\mathbf{s} = (s_1,\ldots,s_d)$ and $\mathbf{2} = (2,\ldots,2)$.

A surprising feature of Eq. (10) is that it implies the existence of an infinite number of conservation laws: on the hypersurface $\prod_{j=1}^{d} s_j(s_j+1) = 2^d\left(3^d-1\right)$, the moments $M(\mathbf{s},t)$ are independent of time. Thus the competition between creation and destruction of the (hyper)rectangles gives birth to an infinite number of integrals of motion. Similar hidden conserved integrals have been found in recent studies of multidimensional fragmentation [3,4]. In contrast to the fragmentation problem where at least one integral - the total volume - is an obvious conserved quantity, in the present model we could not physically explain the appearance of any conserved integral in any dimension.

These integrals play an important role in the dynamics of the system, *e. g.*, they are responsible for the absence of scaling solutions to Eq. (8). Indeed, trying a scaling solution of the form $P(\mathbf{x},t) = t^w Q(t^z\mathbf{x})$, one derives infinitely many scaling relations, $w = z\sum s_j$, which should be valid for *all* points $\mathbf{s}$ on the hypersurface. This infinite set of scaling relations cannot be satisfied by just two scaling exponents, w and z.

In analogy with one-dimensional case, one can expect a power-law behavior of the moments in the general d-dimensional situation: $M(\mathbf{s},t) \sim t^{-\alpha(\mathbf{s})}$ as $t \to \infty$. Substituting this asymptotic form into Eq. (10) we obtain the difference equation for the exponent $\alpha(\mathbf{s})$,

$$\alpha(\mathbf{s}) + 1 = \alpha(\mathbf{s}+\mathbf{2}). \quad (11)$$

In addition, on the hypersurface $\prod_{1\leq j\leq d} s_j(s_j+1) = 2^d\left(3^d-1\right)$, one has $\alpha(\mathbf{s}) = 0$. The solution to Eq. (11) with this boundary condition is given by the formal expression

$$\alpha(\mathbf{s}) \equiv \alpha(\mathbf{s}^* + k\mathbf{2}) = k, \quad (12)$$

where the point $\mathbf{s}^*$ lies on the hypersurface. Geometrically, the exponent $\alpha(\mathbf{s})$ gives a (normalized) distance from the point $\mathbf{s}$ to the hypersurface in the $\mathbf{1} = (1,\ldots,1)$ direction.

For ordinary scaling distributions the exponent $\alpha(\mathbf{s})$ should be linear in the variable $\sum s_j$. This property is equivalent to the existence of a single length scale in the system. However, in our stochastic process the exponent $\alpha(\mathbf{s})$ is a function of *all* of its variables. This manifests the non-trivial scaling properties of the process. On the other hand, since all the moments still show a power-law behavior we conclude that the model exhibits a multiscaling asymptotic behavior.

As a manifestation of the existence of multiple length scales in the system let us consider the ratio of the average volume $\langle V\rangle, \langle V\rangle = M(\mathbf{2},t)/N(t)$, to the d^{th}power of the average length $\langle l\rangle, \langle l\rangle = M(2,1,\ldots,1,t)/N(t)$. For scaling distributions characterized by a single scale, the ratio $\langle V\rangle/\langle l\rangle^d$ *cannot depend* on time. Compute now this ratio for the present system. First using Eq. (12) we derive the asymptotic behavior of the total number of (hyper)rectangles $N(t), N(t) \equiv M(\mathbf{1}, t)$:

$$N(t) \sim t^{(\beta_d-1)/2}, \quad \beta_d = \frac{\sqrt{1+24(1-3^{-d})^{1/d}}-1}{2}. \tag{13}$$

In the above equation β_d is the larger root of the equation $[\beta_d(\beta_d+1)]^d = 2^d(3^d-1)$. Analogously, one finds that $M(\mathbf{2}, t) \sim t^{(\beta_d-2)/2}$ and that $M(2,1,\ldots,1,t) \sim t^{(\gamma_d-1)/2}$ with γ_d being the largest positive root of algebraic equation $(\gamma_d+1)(\gamma_d+2)[\gamma_d(\gamma_d+1)]^{d-1} = 2^d\left(3^d-1\right)$. Finally we obtain

$$\frac{\langle V\rangle}{\langle l\rangle^d} \sim t^{-\mu_d}, \quad \mu_d = \frac{1+d(\gamma_d-\beta_d)}{2}, \tag{14}$$

indicating that the ratio *depends* on time. Computing the exponent μ_d yields μ_d=0, 0.05534, 0.06566, and 0.07054 for d=1,2,3, and 4, respectively.

Note that in the limit of infinite dimension the exponent saturates at $\mu_\infty = \frac{1}{2} - \frac{3}{5}\ln 2 \cong 0.0841117$. Since the exponent μ_d measures the deviation between the asymptotic behavior of the length and the volume we conclude that this discrepancy becomes more pronounced as the spatial dimension increases. It is also easy to find that different directions behave independently in the limit of infinite dimension. Mathematically, it follows from the relation

$$\langle\prod_{j=1}^{\infty} x_j^{n_j}\rangle = \prod_{j=1}^{\infty}\langle x_j^{n_j}\rangle, \tag{15}$$

which is valid if $n_j = 0$ for all indices j except a finite number. Thus the average of a finite product decouples onto a product of single-variable averages. However, further decoupling is impossible: *e. g.*, $\langle x^2\rangle \neq \langle x\rangle^2$. The general asymptotic formula for the n^{th}moment in the limit $d \gg 1$ reads:

$$\langle l^n\rangle \sim t^{-\nu_n}, \quad \nu_n = \frac{3}{5d}\ln\left[\left(1+\frac{n}{2}\right)\left(1+\frac{n}{3}\right)\right]. \tag{16}$$

This equation again indicates the presence of an infinite number of length scales.

In principle, it is possible to write down an explicit solution to Eq. (10) in terms of the generalized hypergeometric function ${}_{2d}F_{2d}$ and then obtain a formal expression for the distribution function $P(\mathbf{x}, t)$ by performing the inverse d-fold Mellin transform. However, such a complete solution is very cumbersome even in one dimension [2]. Thus, we restrict ourselves to the *volume* distribution function $P(V, t)$,

$$P(V,t) = \int_0^1 \ldots \int_0^1 P(\mathbf{x},t)\delta\left(\prod_{j=1}^d x_j - V\right)\prod_{j=1}^d dx_j, \tag{17}$$

for which we can derive more complete results. The moments of the volume distribution function $P(V, t)$ are just the diagonal moments $M(s, \ldots, s; t)$. Asymptotically, they exhibit the power-law behavior

$$M(s,\ldots,s;t) \simeq A_d(s)t^{(\beta_d-s)/2}, \tag{18}$$

with prefactor $A_d(s)$ satisfying the difference equation

$$A_d(s+2) = A_d(s)2^{d-1}(s-\beta_d)\left[1-\frac{2^d(3^d-1)}{s^d(s+1)^d}\right]^{-1}, \tag{19}$$

and the boundary condition $A_d(s = \beta_d) = 1$.

Eq. (18) indicates that in the long-time limit the volume distribution function approaches the scaling form

$$P(V,t) \simeq t^{\beta_d/2}\Phi_d\left(V\sqrt{t}\right), \tag{20}$$

with $\Phi_d(z)$ being the inverse Mellin transform of $A_d(s)$. The most interesting asymptotic behavior of $\Phi_d(z)$ may be found from the corresponding asymptotics of $A_d(s)$. By solving Eq. (19) asymptotically we obtain

$$A_d(s) \sim \begin{cases} s^{-\beta_d/2}2^{ds/2}\Gamma\left(\frac{s}{2}\right), & \text{for } s\to\infty, \\ s^{-d}, & \text{for } s\to 0. \end{cases} \tag{21}$$

By performing the inverse Mellin transform we find the following limiting behaviors of the scaling function $\Phi_d(z)$:

$$\Phi_d(z) \sim \begin{cases} z^{-\beta_d}\exp(-z^2/2^d), & \text{for } z\to\infty, \\ \ln^{d-1}(1/z), & \text{for } z\to 0. \end{cases} \tag{22}$$

Hence for all $d > 1$ the volume distribution function diverges logarithmically in the small-volume limit.

Compare now the asymptotic behavior (13) for the number of (hyper)rectangles with $\langle V\rangle \sim t^{-1/2}$ for their average volume. These asymptotics imply $N \sim \langle V\rangle^{-(\beta_d-1)}$ providing the following value of the fractal dimension of the d-dimensional stochastic Cantor set:

$$D_f = d(\beta_d-1) = d\frac{\sqrt{1+24(1-3^{-d})^{1/d}}-3}{2}. \tag{23}$$

Hence $D_f \cong 1.860804, 2.954859, 3.985106$ for d=2, 3, 4, respectively. For comparison: the regular Cantor set has dimension $D_f = \ln(3^d-1)/\ln 3$, *i. e.*, $D_f \cong 1.89279, 2.96565, 3.98869$ for d=2, 3, 4, respectively. It is easy to check that the fractal dimension of the stochastic Cantor set is always smaller than the corresponding value for regular set.

Formally, the moments provide a complete analytical description of the division process. However, the snapshot of the system at the later stages remains intriguing. Figure 2 represents a single realization of the process at time $t = 1000$. A simple Monte Carlo algorithm was chosen where pairs of points are deposited on the unit square with unit rate. If both points belong to the same rectangle, then that rectangle is divided into 9 rectangles as illustrated in Figure 1. Finally, the central rectangle is removed from the system. In Figure 2 the existing rectangles are shaded. This unexpectedly rich pattern arising in such a simple kinetic process can be viewed as a consequence of the fact that the process is not fully self-similar. Instead, the pattern is formed of sets of different scales which are spatially interwoven. Figure 2 also shows that a number of removed rectangles have large aspect ratio. This qualitative observation is in a good agreement with a power-law behavior of the moments of aspect ratio, $\langle (x_1/x_2)^n \rangle \sim t^{\lambda_n}$ with $\lambda_n = -\alpha(1+n, 1-n) - (\beta_2 - 1)/2$. One can check that $\lambda_n \geq 0$ and hence the aspect ratio grows with time.

In conclusion, we have investigated a kinetic process describing the formation of stochastic counterparts to the Cantor set ($d = 1$), Cantor gasket ($d = 2$), Cantor cheese ($d = 3$), *etc.*. In the long-time limit, the volume of these stochastic Cantor sets is characterized by the single scale $t^{-1/2}$. The volume distribution function exhibits a weak logarithmic singularity near the origin. Similar logarithmic singularity has been observed recently in multidimensional fragmentation [3]. We have found that other geometrical characteristics such as the average length, the surface area, *etc.*, decay nonuniversally in time because of the existence of an infinite amount of different scales, *i. e.*, due to multiscaling. We have shown that the intrinsic reason for multiscaling is the existence of infinitely many hidden conservation laws. This feature again resembles multidimensional fragmentation prosesses [3,4]. Finally, we have found that the fractal dimension characterizing stochastic Cantor set in d dimensions is always smaller than its classic counterpart.

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Figure Captions

Figure 2. Realization of the process on a unit square at time $t = 1000$.