Stationary Velocity Distributions in Traffic Flows

E. Ben-Naim† and P. L. Krapivsky‡
†*Theoretical Division and Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, NM 87545*
‡*Center for Polymer Studies and Department of Physics, Boston University, Boston, MA 02215*

We introduce a traffic flow model that incorporates clustering and passing. We obtain analytically the steady state characteristics of the flow from a Boltzmann-like equation. A single dimensionless parameter, $R = c_0 v_0 t_0$ with c_0 the concentration, v_0 the velocity range, and t_0^{-1} the passing rate, determines the nature of the steady state. When $R \ll 1$, uninterrupted flow with single cars occurs. When $R \gg 1$, large clusters with average mass $\langle m \rangle \sim R^{\alpha}$ form, and the flux is $J \sim R^{-\gamma}$. The initial distribution of slow cars governs the statistics. When $P_0(v) \sim v^{\mu}$ as $v \to 0$, the scaling exponents are $\gamma = 1/(\mu+2)$, $\alpha = 1/2$ when $\mu > 0$, and $\alpha = (\mu+1)/(\mu+2)$ when $\mu < 0$.
PACS numbers: 02.50-r, 05.40.+j, 89.40+k, 05.20.Dd

I. INTRODUCTION

Traffic flows are strongly interacting many-body systems. They also present a natural testbed for theories and techniques developed for physical systems such as kinetic theory and hydrodynamics. Traffic systems have been receiving much attention recently [1], and a number of approaches were suggested including fluid mechanics [2–5], cellular automata [6–13], particle hopping [14–17], and ballistic motion [18–22]. The diversity of the approaches reflects the rich phenomenology which includes shock waves, clustering, and slowing down. Traffic networks can be viewed as low dimensional systems. For example, rural traffic is intrinsically one-dimensional and urban grid traffic is two-dimensional. This important simplifying feature makes analytical treatment possible.

Ballistic models are harder to simulate than cellular automata and particle hopping models. However, they are quite realistic since time and space are treated as continuous variables. They can also prove useful for theoretical treatment. An exactly solvable clustering process shows that extremal properties of the velocity distribution determine the kinetic behavior [18]. However, it results in ever-growing and ever-slowing jams with a trivial steady state in a finite system. In this study, we investigate more realistic situations where fast cars can pass slow cars. This is motivated by and should be applicable to passing zones of one lane roadways as well as multilane highways. Our goal is to determine analytically statistical properties of the flow such as the flux, and characterize their dependence on the intrinsic velocity distribution.

We start by formulating the model. Consider a one-dimensional traffic flow with sizeless cars ("particles") moving with a constant velocity. We assume that cars have intrinsic velocities by which they would drive on an empty road. Initially, cars are randomly distributed in space and they drive with their intrinsic velocities. However, the presence of slower cars forces some cars to drive behind a slower car and therefore leads to the formation of clusters. Simple collision and escape mechanisms are implemented. When a cluster overtakes a slower cluster, a larger cluster forms. It moves with the smaller of the two velocities. Meanwhile, all cars in a given cluster may escape their respective cluster and resume driving with their intrinsic velocity (see Fig. 1). We assume a constant escape rate t_0^{-1}. The actual collision and escape times are proportional to the car size and thus set to zero (these time scales should become important in heavy traffic).

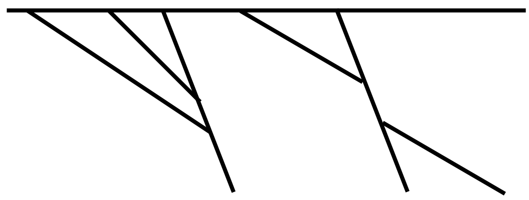

Fig.1 Space time diagram of the traffic model. Formation of a cluster with two fast cars is shown to the left and formation of a one car cluster and its breakup due to escape is shown to the right.

A heuristic argument suggests that a single dimensionless parameter underlies the steady state. Consider a state where the car concentration is c_0, and the typical intrinsic velocity range is v_0. Let the steady state cluster density be $c < c_0$, which implies the typical cluster size $\langle m \rangle = c_0/c$. If large clusters form, $\langle m \rangle \gg 1$, then the overall escape rate can be estimated by $\langle m \rangle t_0^{-1}$. Assuming that most collisions involve fast cars and slow clusters, the typical collision rate is $c v_0$. In the steady state, the number of cars joining and leaving clusters should balance and thus, $c_0/(c t_0) = v_0 c$ or $c = (c_0/v_0 t_0)^{1/2}$. This heuristic argument gives the leading behavior of the average cluster size

$$\langle m \rangle \sim R^{1/2} \qquad \text{when} \quad R \gg 1, \tag{1}$$

where R is the ratio of the two elementary time scales, the escape time $t_{\rm esc} = t_0$ and the collision time $t_{\rm col} = (c_0 v_0)^{-1}$:

$$R = \frac{t_{\rm esc}}{t_{\rm col}} = c_0 v_0 t_0. \tag{2}$$

We term this dimensionless quantity the "collision number". For large collision numbers, large clusters occur according to Eq. (1), while for small collision numbers the effect of collisions is small $\langle m\rangle \cong 1 + \text{const.} \times R$. Analysis of the master equations detailed below confirms this heuristic picture under quite general conditions.

The rest of this paper is organized as follows. In Sec. II, the master equations are used to derive analytical expressions for various velocity distributions in the steady state. The leading behavior in the limiting cases of light and heavy traffic are highlighted in Sec. III. Explicit expressions are written for the special cases of uniform initial and final velocity distributions as well as discrete distributions in Sec. IV. The theoretical predictions compare well with actual traffic data presented in Sec. V. We close with some open problems, a discussion, and possible applications.

II. THEORY

In the following, it is convenient to introduce dimensionless velocity $v/v_0 \to v$, space $xc_0 \to x$, and time $c_0 v_0 t \to t$ variables. This rescales the escape rate t_0^{-1} to the inverse collision number R^{-1}. Let $P(v,t)$ be the density of clusters moving with velocity v at time t. Initially, isolated single cars drive with their intrinsic velocities drawn from the distribution $P_0(v) \equiv P(v, t=0)$. This intrinsic velocity distribution is normalized to unity, $\int dv P_0(v) = 1$. The flow is invariant under a velocity translation, and the minimal velocity is set to zero.

Initially, the velocities and the positions of the particles are uncorrelated. Escape effectively mixes the positions and the velocities. Assuming that no spatial correlations develop, a closed master equation for the velocity distribution of clusters $P(v,t)$ can be written

$$\frac{\partial P(v,t)}{\partial t} = R^{-1}\left[P_0(v) - P(v,t)\right] - P(v,t)\int_0^v dv'(v-v')P(v',t). \tag{3}$$

The density of slowed down cars with intrinsic velocity v is $P_0(v) - P(v,t)$. Such cars escape their clusters with rate R^{-1}, and thus the escape term. Collisions occur with rate proportional to the velocity difference as well as the product of the velocity distributions. The integration limits ensure that only collisions with slower cars are taken into account.

Steady state is obtained by taking the long time limit $t \to \infty$ or $\partial/\partial t = 0$. Since we are primarily interested in the steady state, we omit the time variable $P(v) \equiv P(v, t=\infty)$. Equating the right-hand side of the master equation to zero, a relation between the intrinsic car distribution and steady state cluster distribution emerges

$$P(v)\left[1 + R\int_0^v dv'(v-v')P(v')\right] = P_0(v). \tag{4}$$

Given the intrinsic velocity distribution this relation gives the final cluster velocity distribution only implicitly. In contrast, the inverse problem is simpler as knowledge of the final distribution, the observed quantity in real traffic flows, gives explicitly the intrinsic distribution. We confirm that in the limit $R \to \infty$, all clusters move with the minimal velocity $P(v) \to \delta(v)$, while in the limit $R \to 0$, all cars move with their intrinsic velocity $P(v) \to P_0(v)$.

It is convenient to transform the integral equation (4) into a differential one. Consider the auxiliary function

$$Q(v) = R^{-1} + \int_0^v dv'(v-v')P(v'), \tag{5}$$

which gives the cluster distribution by second differentiation

$$P(v) = Q''(v). \tag{6}$$

Thence, the steady state condition (4) reduces to the second order nonlinear differential equation

$$Q(v)Q''(v) = R^{-1}P_0(v). \tag{7}$$

The boundary conditions are $Q(0) = R^{-1}$ and $Q'(0) = 0$. The cluster concentration is found from the cluster velocity distribution using

$$c = \int_0^\infty dv\, P(v), \tag{8}$$

and the average cluster mass is simply $\langle m\rangle = c^{-1}$. Furthermore, the average cluster velocity is obtained from

$$\langle v\rangle = c^{-1}\int_0^\infty dv\, vP(v). \tag{9}$$

Cars may drive with a velocity smaller than their intrinsic one, and it is natural to consider the joint velocity distribution $P(v,v')$, the density of cars of intrinsic velocity v driving with velocity v'. The master equation for the joint distribution reads

$$\begin{aligned}\frac{\partial P(v,v')}{\partial t} = &- R^{-1}P(v,v') + (v-v')P(v)P(v') \\ &- P(v,v')\int_0^{v'} dv''(v'-v'')P(v'') \\ &+ P(v')\int_{v'}^{v} dv''(v''-v')P(v,v'').\end{aligned} \tag{10}$$

The first term accounts for loss due to escape, while the rest of the terms represent changes due to collisions. For instance, the last term describes events where a v-car driving with velocity v'' is further slowed down after a collision with a v'-cluster. One can verify that the total number of v-cars,

$$P_0(v) = P(v) + \int_0^v dv' P(v,v'), \tag{11}$$

is conserved by the evolution Eqs. (3) and (10).

At the steady state, the joint distribution satisfies

$$P(v,v')Q(v') = (v-v')P(v)P(v') + Q(v,v')P(v'), \quad (12)$$

obtained using the definition of $Q(v)$ and the joint auxiliary function

$$Q(v,v') = \int_{v'}^{v} dw(w-v')P(v,w). \quad (13)$$

Although the collision number R does not appear in Eq. (12) explicitly, it enters through $Q(v)$ and $P(v)$.

Combining (12) with Eqs. (13), (6), and using the relationship $P(v,v') = \partial^2 Q(v,v')/\partial v'^2$ yields

$$\frac{\partial}{\partial v'}\left[Q^2(v')\,\frac{\partial}{\partial v'}\,\frac{Q(v,v')}{Q(v')}\right] = (v-v')P(v)P(v'). \quad (14)$$

Integrating twice over v' gives the joint auxiliary function in terms of the single variable functions

$$Q(v,v') = P(v)Q(v')\int_{v'}^{v}\frac{du}{Q^2(u)}\int_{u}^{v} dw(v-w)P(w). \quad (15)$$

The boundary conditions $Q(v,v) = \frac{\partial}{\partial v'}Q(v,v')\big|_{v'=v} = 0$ were used to obtain this expression. Furthermore, integration by parts of $\int_u^v dw(v-w)P(w) = \int_u^v dw(v-w)Q''(w)$ gives

$$Q(v,v') = P(v)\left[Q(v)Q(v')\int_{v'}^{v}\frac{du}{Q^2(u)} - (v-v')\right]. \quad (16)$$

Substituting Eq. (16) into (12) and then replacing PQ with $R^{-1}P_0$ we find a relatively simple expression for the joint velocity distribution

$$P(v,v') = \frac{P_0(v)P_0(v')}{Q(v')}\int_{v'}^{v}\frac{du}{[RQ(u)]^2}. \quad (17)$$

Another interesting quantity is the flux or the average velocity given by $J = \int dv\left[vP(v) + \int_0^v dw\, wP(v,w)\right]$. From the definition of the joint auxiliary function, the second integral is identified with $Q(v,0)$, implying

$$J = \int_0^\infty dv\,[vP(v) + Q(v,0)]\,. \quad (18)$$

The integrand can be considerably simplified using Eq. (16), $Q(0) = R^{-1}$, and Eq. (7). The term $vP(v)$ cancels and we find a useful expression for the flux

$$J = \int_0^\infty dv\,P_0(v)\int_0^v\frac{du}{[RQ(u)]^2}. \quad (19)$$

One can also ask for the actual velocity distribution of cars defined via

$$G(v) = P(v) + \int_v^\infty dw\,P(w,v). \quad (20)$$

Substituting the joint velocity distribution allows us to express the car velocity distribution via single variable distributions

$$G(v) = P(v)\left[1 + R\int_v^\infty dw\,P_0(w)\int_v^w\frac{du}{[RQ(u)]^2}\right]. \quad (21)$$

The car velocity distribution satisfies the normalization conditions $1 = \int dv\,G(v)$ and $J = \int dv\,vG(v)$.

In summary, for arbitrary intrinsic velocity distributions, the entire steady state problem is reduced to the nonlinear second order differential equation (7). Given $Q(v)$, steady state characteristics such as $P(v)$, $P(v,v')$, J, and $G(v)$ can be calculated using the explicit formulae (6), (17), (19), and (21), respectively.

III. LIMITING CASES

Although one cannot solve Eq. (7) analytically in general, it is still possible to obtain the leading behavior in the limits of $R \to 0$ and $R \to \infty$.

A. Low Collision Numbers

To analyze the flow characteristics in the collision-controlled regime, $R \ll 1$, we use Eq. (4) to write $P(v)$ as a perturbation expansion in R:

$$P(v) \cong P_0(v)\left[1 - R\int_0^v dv'(v-v')P_0(v')\right]. \quad (22)$$

In this limit, the auxiliary function is roughly constant $RQ(v) \cong 1$, and Eq. (17) gives the joint distribution to first order in R

$$P(v,v') \cong R(v-v')P_0(v)P_0(v'). \quad (23)$$

The final density and flux are

$$c \cong 1 - c_1 R, \quad J \cong J_0 - J_1 R, \quad (24)$$

with $c_1 = \int dv P_0(v)\int_0^v dv'(v-v')P_0(v')$, $J_0 = M_1$, $J_1 = M_2 - M_1^2$ (M_n are the moments of the intrinsic velocity distribution $M_n = \int dv\,v^n P_0(v)$). The coefficient $J_1 \geq 0$ equals the width of the initial velocity distribution. This gives a simple intuitive picture: the larger the initial velocity fluctuations, the smaller the flux. By either substituting the joint velocity distribution into the definition of $G(v)$, or from Eq. (21), the car velocity distribution is

$$G(v) \cong P_0(v)\left[1 + R\int_0^\infty dv'(v'-v)P_0(v')\right]. \quad (25)$$

As the integral is over the entire velocity range, the order R correction is positive for small v and negative for large v. In other words $G(v) > P_0(v)$ when $v < v_c$.

The crossover velocity equals the average intrinsic velocity $v_c = J_0 = M_1$, as seen from Eq. (25).

We conclude that the collision-controlled limit is weakly interacting, explicit expressions for the leading corrections of the steady state properties are possible.

B. Large Collision Numbers

The analysis in the complementary escape-controlled regime, $R \gg 1$, is more subtle since the condition $R\int_0^v dv'(v-v')P_0(v') \ll 1$ is satisfied only for small velocities. No matter how large R is, sufficiently slow cars are not affected by collisions, and $P(v)$ is given by Eq. (22) when $v \ll v^*$. The threshold velocity $v^* \equiv v^*(R)$ is estimated from $R\int_0^{v^*} dv(v^*-v)P_0(v) \sim 1$.

It is useful to consider algebraic intrinsic distributions

$$P_0(v) = (\mu+1)v^\mu \quad \mu > -1, \tag{26}$$

in the velocity range [0:1] with the prefactor ensuring unit normalization. For such distributions, the threshold velocity decreases with growing R according to $v^* \sim R^{-\frac{1}{\mu+2}}$. For $v \gg v^*$, the integral in Eq. (4) dominates over the constant factor and $RP(v)\int_0^v dv'(v-v')P(v') \sim v^\mu$. Anticipating an algebraic behavior for the cluster velocity distribution, $P(v) \sim R^\sigma v^\delta$ when $v \gg v^*$, gives different answers for positive and negative μ. The leading behavior for $v \gg v^*$ can be summarized as follows

$$P(v) \sim \begin{cases} (v^*)^\mu (v/v^*)^{\mu-1} & \mu < 0; \\ (v/v^*)^{-1}[\ln(v/v^*)]^{-\frac{1}{2}} & \mu = 0; \\ (v^*)^\mu (v/v^*)^{\frac{\mu}{2}-1} & \mu > 0. \end{cases} \tag{27}$$

The small and large velocity components of $P(v)$ match at the threshold velocity, $P(v^*) \sim P_0(v^*)$. Careful analysis, detailed in the following section, is needed to get the logarithmic corrections in the borderline case $\mu = 0$. Substituting the leading asymptotic behavior of Eq. (27) into Eq. (8), the average cluster size is found

$$\langle m \rangle \sim \begin{cases} R^{(\mu+1)/(\mu+2)} & \mu < 0; \\ (R/\ln R)^{1/2} & \mu = 0; \\ R^{1/2} & \mu > 0. \end{cases} \tag{28}$$

Similarly, the average cluster velocity defined in Eq. (9) is evaluated

$$\langle v \rangle \sim \begin{cases} R^{\mu/(\mu+2)} & \mu < 0; \\ 1/\ln R & \mu = 0; \\ \text{const} & \mu > 0. \end{cases} \tag{29}$$

Two distinct regimes of behavior emerge. For $\mu > 0$, car-cluster collisions dominate while for $\mu < 0$ cluster-cluster collisions dominate. The scaling argument given in the introduction assumes the former picture, and thus it does not hold in general. *A posteriori*, one can extend the scaling argument to the $\mu < 0$ regime. The argument becomes involved, and we do not present it here. Interestingly, in the cluster-cluster dominated regime, the scaling behavior for the average cluster size, $\langle m \rangle \sim R^\alpha$ with $\alpha = (\mu+1)/(\mu+2)$, is identical to the *kinetic* scaling, $\langle m \rangle \sim (c_0 v_0 t)^\alpha$ with the same α, found in the no passing limit [18]. This suggests an analogy between the dimensionless collision number $R = c_0 v_0 t_0$ and the dimensionless time $c_0 v_0 t$. On the other hand, the steady state behavior is much richer as it is characterized by two regimes of behavior and different exponents.

The flux can be evaluated in a similar fashion using Eq. (19),

$$J \sim v^* \sim R^{-\frac{1}{\mu+2}}. \tag{30}$$

Interestingly, the flux is proportional to the threshold velocity v^*. As a result, the flux exponent $\gamma = 1/(\mu+2)$ is a regular function of μ unlike the cluster size exponent α. Eq. (30) is also consistent with identification of the crossover velocity v_c with the marginal velocity v^*. No flux reduction occurs when the intrinsic distribution is dominated by fast cars, i.e., in the limit $\mu \to \infty$. In the other extreme, the maximal flux reduction $J \sim R^{-1}$ is realized when $\mu \to -1$.

The car velocity distribution is strongly enhanced in the low velocity limit, as seen by evaluating Eq. (21)

$$G(v) \sim R^{\frac{\mu+1}{\mu+2}} v^\mu (1 - \text{const.} \times v^{\mu+1}), \quad v \ll v^*. \tag{31}$$

On the other hand, for $v \gg v^*$ we get

$$\frac{G(v)}{P(v)} \sim \begin{cases} 1 + R^{\mu/(\mu+2)}\, v^{-1}(1 - v^{\mu+1}) & \mu < 0; \\ 1 + (v^{-1}-1)[\ln(v/v^*)]^{-1} & \mu = 0; \\ v^{-\mu-1} & \mu > 0. \end{cases} \tag{32}$$

As a check of self-consistency, one can easily verify that $\int dv\, G(v) \sim 1$, and $J = \int dv\, v\, G(v) \sim v^*$.

The car velocity distribution is useful for studying velocity fluctuations. More generally, one can consider the moments of the velocity distribution, $G_n = \int dv v^n G(v)$. Naively, one would expect $G_n \sim (G_1)^n$, which together with $G_1 \equiv J$ and Eq. (30) implies $G_n \sim R^{-n/(\mu+2)}$. Using Eq. (32) one computes the moments to confirm this expectation for sufficiently small n, namely for $n < 1-\mu$ if $\mu < 0$, $n < 1$ if $\mu = 0$, and $n < 1 + \mu/2$ if $\mu > 0$. When the index n exceeds above thresholds, more interesting behavior is found:

$$G_n \sim \begin{cases} R^{(\mu-1)/(\mu+2)} & -1 < \mu < 0, n > 1-\mu; \\ R^{-1/2}(\ln R)^{-3/2} & \mu = 0, n > 1; \\ R^{-1/2} & \mu > 0, n > 1 + \mu/2. \end{cases} \tag{33}$$

For sufficiently small μ, the fluctuations in flux are very large, $G_2 \gg G_1^2$. Thus in the most interesting region $-1 < \mu < 2$ fluctuations dominate.

In summary, as $R \to \infty$ the solution to the differential equation (7) exhibits a boundary layer structure. Inside the boundary layer, $v < v^*$, the cluster velocity distribution is only slightly affected by collisions, while in the outer region $v > v^*$, the cluster velocity distribution is much smaller than the intrinsic velocity distribution. The threshold velocity v^* is determined by the small velocity behavior of the intrinsic velocity distribution, and for the algebraic distributions (26) we have found $v^* \sim R^{-\frac{1}{\mu+2}} \to 0$. The behavior detailed above in the escape controlled limit is not restricted to purely algebraic distributions but is quite general. We conclude that a single parameter

$$\mu = \lim_{v\to 0} v\frac{\partial}{\partial v}\ln P_0(v) \qquad (34)$$

determines the behavior as $R \to \infty$. In short, extreme statistics underly the escape-limited flow properties. Additionally, an interesting transition between a slow and a fast velocity dominated flow occurs at $\mu = 0$.

IV. EXAMPLES

Although the above analysis is quite general, it applies only to the limiting values of R. To examine intermediate behavior, it is also useful to obtain explicit solutions for some special cases. Below, we consider two relevant cases: uniform $P_0(v)$ and $P(v)$. We also obtain explicit expressions in the case of discrete velocity distributions.

A. Uniform Intrinsic Distribution

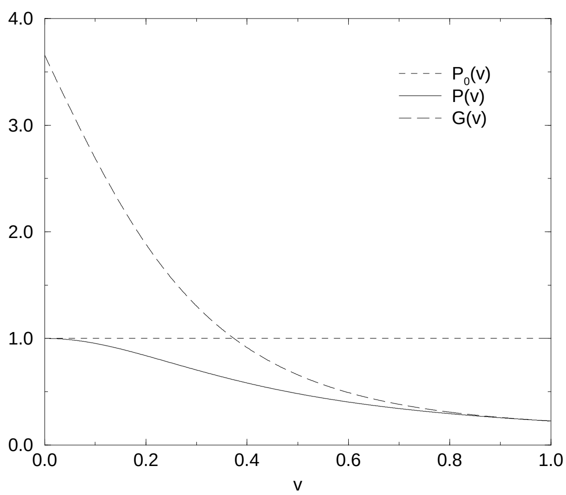

Fig.2 Velocity distributions in the case of a uniform initial distribution $P_0(v) = 1$, for $R = 10$.

We now consider the case of a uniform intrinsic distribution, $P_0(v) = 1$ for $0 < v < 1$. This case appears to be the most relevant to real traffic flows since the intrinsic velocity distribution should be regular near the minimal velocity. Integrating $QQ'' = R^{-1}$ subject to the boundary conditions $Q(0) = R^{-1}$ and $Q'(0) = 0$ gives $Q' = \sqrt{2R^{-1}\ln(RQ)}$. Second integration gives

$$\int_1^{RQ} \frac{dq}{\sqrt{2\ln q}} = v\sqrt{R}, \qquad (35)$$

and thus implicitly determines $Q(v)$. Evaluating the leading behavior when $R \gg 1$, we find

$$\langle m\rangle \simeq \sqrt{\frac{R}{\ln R}}, \quad \langle v\rangle \simeq \frac{1}{\ln R}, \quad J \simeq \sqrt{\frac{\pi}{2R}}. \qquad (36)$$

Fig. 2 shows the velocity distribution obtained numerically using Eqs. (35) and (21) for $R = 10$. For $v \ll v^*$, $G(v) \gg P_0(v)$, and for $v \gg v^*$, $G(v) \cong P(v) \ll P_0(v)$. The calculated distributions are consistent with the predictions, $G(0) \sim R^{1/2}$ and $v^* \sim R^{-1/2}$. The car velocity distribution is linear near the origin in agreement with Eq. (31).

B. Uniform Cluster Distribution

Consider the uniform final cluster distribution $P(v) = c$. This inverse problem is simple as all quantities can be obtained explicitly. From Eq. (5), the auxiliary function is $Q(v) = R^{-1} + \frac{1}{2}cv^2$ and from Eq. (7) the initial distribution reads

$$P_0(v) = c\left[1 + \frac{1}{2}Rcv^2\right]. \qquad (37)$$

The overall initial concentration is unity, thereby relating R and c via $1 = c + \frac{1}{6}Rc^2$. The flux is calculated from Eq. (19),

$$J = \frac{(3+\lambda)\sqrt{\lambda}\tan^{-1}\sqrt{\lambda} + \lambda - \ln(1+\lambda)}{3R}, \qquad (38)$$

with $\lambda = \frac{1}{2}Rc = (3/2)\left[\sqrt{1 + 2R/3} - 1\right]$. These explicit solutions agree with our low and high R predictions. For instance, when $R \gg 1$ we find $\langle m\rangle \sim R^{1/2}$ and $J \sim R^{-1/4}$. If we look at the initial distribution, $P_0(v) \cong (6/R)^{1/2} + 3v^2$, then the constant part is negligible and the distribution corresponds to the $\mu = 2$ case of the power-law distribution (26). For this case the size exponent is $\alpha = 1/2$ and the flux exponent is $\gamma = 1/4$, see (28) and (30), in agreement with our findings.

Substituting $P_0(v)$ and $Q(v)$ in Eq. (17) and performing the integration gives the joint distribution

$$P(v, v') = 2R^{-1}\lambda^2\frac{(v - v')(1 - \lambda vv')}{1 + \lambda v'^2} \qquad (39)$$
$$+2R^{-1}\lambda^{3/2}(1 + \lambda v^2)\left[\tan^{-1}(\lambda^{1/2}v) - \tan^{-1}(\lambda^{1/2}v')\right].$$

A direct integration of the joint distribution confirms the conservation law (11), thus providing a useful check of self-consistency. The joint velocity distribution is linear in the velocity difference for small v and v'. This is reminiscent of the small collision number behavior of Eq. (23). As the velocity difference increases, significant curvature develops (see Fig. 3).

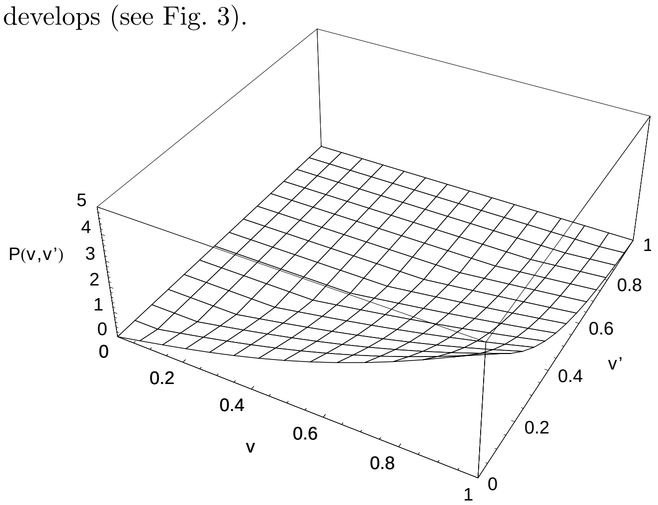

Fig.3 The joint velocity distribution for the uniform final distribution case with $R = 10$.

C. Discrete Velocity Distribution

The results formulated for continuous distributions can be used to study the special case of discrete velocity distribution as well. Here we quote the results in terms of the original (non-dimensionless) quantities. Consider the intrinsic velocity distribution

$$P_0(v) = \sum_{i=1}^{n} c_i \delta(v - v_i), \tag{40}$$

with $v_1 < v_2 < \cdots < v_n$. We denote by p_i the discrete counterpart of the cluster velocity distribution, e.g., $P(v) = \sum_{i=1}^{n} p_i \delta(v - v_i)$. The steady state condition of Eq. (4) reads

$$p_i \left[1 + t_0 \sum_{j=1}^{i-1} (v_i - v_j) p_j \right] = c_i. \tag{41}$$

Substituting the intrinsic velocity distribution and solving iteratively, we get

$$\begin{aligned} p_1 &= c_1 \\ p_2 &= \frac{c_2}{1 + c_1(v_2 - v_1)t_0} \\ p_3 &= \frac{c_3}{1 + c_1(v_3 - v_1)t_0 + \frac{c_2(v_3 - v_2)t_0}{1 + c_1(v_2 - v_1)t_0}} \end{aligned} \tag{42}$$

etc. Rather than a solution to a differential equation, the steady state solution is in the form of an explicit continued fraction. This expression involves the initial distribution and the velocity differences, and can be useful to analyze data in a histogram form. In a similar way, explicit expressions can be obtained for the rest of the steady state properties.

V. RURAL TRAFFIC OBSERVATIONS

To compare the theoretical predictions with actual traffic flows, we collected data in a rural one lane road where passing is allowed. We chose a road near Los Alamos that was as uniform as possible: over a long stretch it did not contain junctions, stop signs or stop lights. The number of cars and the number of clusters passing a given point in each direction in a fixed time interval was recorded, thereby measuring the flux and the average cluster size, respectively. The data was then histogramed, and the cluster size $\langle m \rangle$ was plotted as a function of the flux I (the observed flux I should be distinguished from the flux J which is in a reference frame moving with the slowest car). We verified that the behavior was independent of the traffic direction as well as the time of day. The former test confirms that the road is indeed uniform.

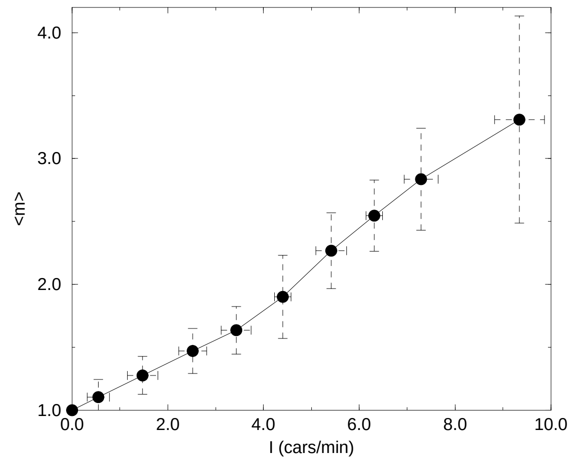

Fig.4 The average cluster size $\langle m \rangle$ as a function of the flux I. The data was obtained from 20 hours of observations of over 5,000 cars. Each data point represents an average over roughly 500 cars, and the error bars account for the standard deviation between different measurements.

We were able to collect data primarily in the dilute limit. According to Eq. (24), $\langle m \rangle \cong 1 + \text{const.} \times R$ when $R \ll 1$. The velocity range v_0 was much smaller than the minimal velocity $v_{\min}$, and consequently $I = c_0(v_{\min} + v_0 J) \propto c_0 v_{\min}$. Thus, the main quantity which varies with the flux is the concentration, and the collision number, $R = c_0 v_0 t_0 \propto c_0 \propto I$ is proportional to the flux. In other words, the theory predicts that in the low flux (or collision number) limit, the average mass grows linearly with the flux, $\langle m\rangle \cong 1+\text{const.}\times I$. The observations agree with this prediction as for sufficiently small fluxes, $I < 4$ cars/minute, there is a linear dependence between the average cluster size and the flux (see Fig. 4). We conclude that at least for low collision numbers, the theoretical predictions concerning the cluster size agree with actual traffic data.

VI. DISCUSSION

An important property, the cluster size distribution is absent from our treatment so far [23]. Naturally, the size and the velocity of a cluster are strongly correlated and one must consider $P_m(v)$, the distribution of clusters of size m and velocity v. The joint cluster size-velocity distribution obeys the master equation

$$\begin{aligned}\frac{\partial P_m(v)}{\partial t} &= R^{-1}[mP_{m+1}(v)-(m-1)P_m(v)]\\ &+ R^{-1}\delta_{m,1}[P_0(v)-P(v)] - F(v)P_m(v) \\ &+ \int_v^\infty dv'(v'-v)\sum_{j=1}^{m} P_j(v')P_{m-j}(v)\end{aligned} \tag{43}$$

which applies for all $m \geq 1$. Terms proportional to R^{-1} account for escape, while the rest represent collisions. The factor $F(v)=\int_0^\infty dv'|v-v'|P(v')$ measures the overall collision rate experienced by a v-cluster, and is reminiscent of kinetic theory. Summing Eqs. (43), one recovers the rate equation (3) for $P(v)=\sum_m P_m(v)$. On the other hand, integration over the entire velocity range does not reduce Eqs. (43) to a closed system of rate equations for the cluster size distribution $P_m = \int dv P_m(v)$. Therefore, the entire joint distribution is needed to determine P_m. Additionally, we note that in Eqs. (3) and (10), the integration limits include only slower velocities, a feature that considerably simplifies the analysis. This property is lost for Eqs. (43), therby making analytical treatment harder.

Nevertheless, a leading order analysis is still possible for low collision numbers. For example, the density of single cars is given by the expansion

$$P_1(v) = P_0(v) - RP_0(v)\int_0^\infty dv'\,|v-v'|P_0(v') + \ldots \tag{44}$$

In general, one can see that

$$P_m(v) = R^{m-1}\tilde{P}_m(v) + \mathcal{O}(R^m). \tag{45}$$

Heuristically, clusters with m cars are created by $m-1$ collisions and a factor R is generated in each collision. The perturbation expansion functions $\tilde{P}_m(v)$ can be obtained recursively using

$$(m-1)\tilde{P}_m(v) = \int_v^\infty dv'(v'-v)\sum_{j=1}^{m-1}\tilde{P}_j(v')\tilde{P}_{m-j}(v). \tag{46}$$

For example, the first term reads

$$\tilde{P}_2(v) = P_0(v)\int_v^\infty dv'(v-v')P_0(v'). \tag{47}$$

We also note that the infinite set of recursive equations (46) can be transformed into a closed equation

$$\frac{\partial^2}{\partial v^2}\frac{\partial}{\partial z}\ln\tilde{P}(z,v) = \tilde{P}(z,v), \tag{48}$$

for the generating functions $\tilde{P}(z,v) = \sum z^{m-1}\tilde{P}_m(v)$. Although the functions $\tilde{P}_m(v)$ become quite complicated, the overall prefactor R^{m-1} suggests an exponential cluster size distribution in the dilute limit.

In the special case of a bimodal velocity distribution, a solution is possible. The structure of clusters here is simple: A cluster of size m consists of a leading slow car and $m-1$ fast cars behind it. The rate equation (43) simplifies considerably, and a Poisson size distribution is found $P_m \propto e^{-f}f^{m-1}/(m-1)!$. The collision rate f is equal to the product of the escape time, the velocity difference, and the fast car concentration. This steady state distribution satisfies a detailed balance condition as the escape rate and the collision rate are equal microscopically, $(m-1)P_m = fP_{m-1}$. Thus, an equilibrium steady state is reached. However, in general, a nonequilibrium steady state is approached with the collision rate and the escape rate balancing only macroscopically. This is seen by noting that the cluster size may increase by an arbitrary number due to collisions, but can decrease only by one due to escape.

Further investigation of the collision term in the rate equation will be useful as well. In the no escape case $R^{-1}=0$, the exact Boltzmann equation

$$\frac{\partial P(v,t)}{\partial t} = -P(v,t)\int_0^v dv'(v-v')P_0(v') \tag{49}$$

is different from our master equation as $P_0(v')$ replaces $P(v',t)$ in the integrand [18]. This seemingly small difference is important as it shows that the system remembers the initial state. We argue that escape, no matter how small, induces mixing and acts to erase this memory, and therefore Eq. (3). It still remain, however, to establish quantitatively how appropriate is this mean field assumption.

The model and the results presented above can be generalized to study other traffic situations. First, a multilane flow can be treated as a system of coupled one lane flows. Escape naturally couples neighboring lanes. Second, a natural generalization is to heterogeneous situations where passing is allowed only in a fraction r of the road. We expect that for regular distribution of these passing segments the problem should reduce to the homogeneous case with a renormalized collision number R/r. The most challenging question appears to be the role played by the escape mechanism. We considered the case where all cars are equally likely to escape. This assumption simplified the master equation considerably as the escape term is linear in $P(v)$, and thus, is exact. The complementary case where only the first car in the cluster can escape is interesting as well. For low collision numbers, large clusters are unlikely, and the behavior is independent of the escape mechanism. However, for high collision numbers the escape mechanism becomes weaker and larger clusters should form. Indeed, a scaling argument along the lines of Eq. (1) gives $\langle m\rangle \sim R$ in the car-cluster dominated regime.

In conclusion, despite the simplifying assumptions made, the suggested model results in realistic behavior. The overall picture is both familiar and intuitive: due to the presence of slower cars, clusters form and the overall flux is reduced. Our theory is in qualitative agreement with rural traffic observations in the dilute case. For heavy traffic, the characteristics of the flow are solely determined by the distribution of slow cars. A single dimensionless parameter, the collision number R, ultimately determines the nature of the steady state. The stationary distributions obtained analytically provide a simple practical recipe for calculating the flow properties for arbitrary intrinsic distributions. It will be interesting to analyze velocity distributions from actual traffic data using these theoretical tools.

We thank I. Daruka and S. Redner for useful discussions. PLK thanks the CNLS for its hospitality and the ARO for financial support.

[1] See, e.g., *Transportation and Traffic Theory*, edited by N. H. Gartner and N. H. M. Wilson (Elsevier, New York, 1987); W. Leutzbach, *Introduction to the Theory of Traffic Flow* (Springer-Verlag, Berlin, 1988); *Traffic and Granular Flow*, edited by D. E. Wolf, M. Schreckenberg, and A. Bachem (World Scientific, Singapore, 1996).

[2] M. J. Lighthill and G. B. Whitham, *Proc. R. Soc. London Ser. A* **229**, 281 (1955); **229**, 317 (1955).

[3] R. Haberman, *Mathematical Models in Mechanical Vibrations, Population Dynamics, and Traffic Flow* (Prentice-Hall, Englewood Cliffs, NJ, 1977).

[4] I. Prigogine and R. Herman, *Kinetic Theory of Vehicular Traffic* (Elsevier, New York, 1971).

[5] B. S. Kerner and P. Konhäuser, *Phys. Rev. E* **48**, 2335 (1993); *Phys. Rev. E* **50**, 54 (1994).

[6] K. Nagel and M. Schreckenberg, *J. de Physique I* **2**, 2221 (1992).

[7] O. Biham, A. Middleton, and D. Levine, *Phys. Rev. A* **46**, 6124 (1992).

[8] T. Nagatani, *Phys. Rev. E* **48**, 3290 (1993).

[9] A. Schadschneider and M. Schreckenberg, *J. Phys. A* **26**, L679 (1993).

[10] K. Nagel and M. Paczuski, *Phys. Rev. E* **51**, 2909 (1995).

[11] K. Nagel, *Phys. Rev. E* **53**, 4655 (1996).

[12] J. G. Brankov, V. B. Priezzhev, A. Schadschneider, and M. Schreckenberg, *J. Phys. A* **29**, L229 (1996).

[13] D. V. Ktitarev, D. Chowdhury, and D. E. Wolf, *J. Phys. A* **30**, L221 (1997).

[14] B. Derrida and M. R. Evans, in: *Nonequilibrium Statistical Mechanics in One Dimension*, edited by V. Privman (Cambridge University Press, New York, 1997), p. 277.

[15] I. Benjamini, P. A. Ferrari, and C. Landim, *Stoch. Proc. Appl.* **61**, 181 (1996).

[16] M. R. Evans, *J. Phys. A* **30**, 5669 (1997).

[17] H. W. Lee, V. Popkov, D. Kim, cond-mat/9708006.

[18] E. Ben-Naim, P. L. Krapivsky, and S. Redner, *Phys. Rev. E* **50**, 822 (1994).

[19] M. Bando, K. Hasebe, A. Nakayama, A. Shibata, and Y. Sugiama, *Phys. Rev. E* **51**, 1035 (1995).

[20] T. S. Komatsu and S. Sasa, *Phys. Rev. E* **52**, 5574 (1996).

[21] D. Helbing, *Phys. Rev. E* **53**, 2366 (1996).

[22] T. Nagatani, *J. Phys. Soc. Jpn.* **66**, 1219 (1997).

[23] K. Gavrilov, *Phys. Rev. E* **56**, 4860 (1997).