Steady State Properties of Traffic Flows

E. Ben-Naim† and P. L. Krapivsky‡
† *Theoretical Division and Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, NM 87545, USA*
‡ *Center for Polymer Studies and Department of Physics, Boston University, Boston, MA 02215, USA*

The role of the passing mechanism in traffic flows is examined. Specifically, we consider passing rates that are proportional to the difference between the velocities of the passing car and the passed car. From a Boltzmann equation approach, steady state properties of the flow such as the flux, the average cluster size, and the velocity distributions are found analytically. We show that a single dimensionless parameter determines the nature of the flow and helps distinguish between dilute and dense flows. For dilute flows, perturbation expressions are obtained, while for dense flows, a boundary layer analysis is carried out. In the latter case, extremal properties of the initial velocity distribution underly the leading scaling asymptotic behavior. For dense flows, the stationary velocity distribution exhibits a rich "triple deck" boundary layer structure. Furthermore, in this regime fluctuations in the flux may become extremely large.
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I. INTRODUCTION

Traffic flows display a variety of cooperative behaviors similar to nonequilibrium driven systems such as gas and granular flows [1–4]. Typically, only a few major characteristics of the interparticle interaction are responsible for such collective phenomena. Therefore, it is important to use models that are as simple as possible. Such a strategy has proven useful in studying traffic flows, where the ultimate goal is understanding complex phenomena such as slowing down, traffic jams, synchronized flows, and phase transitions [1,5].

Theoretical approaches to modeling traffic flows are quite diverse and include fluid mechanics [2,3,6,7], cellular automata [8–16], particle hopping [17–20], kinetic theory [4,7,21], and ballistic motion [22–27]. In the hydrodynamic description, space and time are both continuous variables, while in cellular automata they are both discrete. Moreover, the former is a macroscopic approach, while the latter is microscopic. Kinetic theory and especially ballistic motion models can help bridge this gap as they are formulated on a microscopic level, but lead naturally to a macroscopic theory. In our previous studies, a Boltzmann equation was derived for traffic flows in no-passing zones of one lane roadways [24], and then generalized to passing zones as well [25]. A transition from a low-density "laminar" flow to a high-density "congested" flow was generally found. This transition as well as other statistical properties are well described by a single dimensionless parameter, R, termed the collision number.

In our former study [25], a constant passing rate was assumed. However, a passing rate that increases with velocity difference is more realistic as faster drivers tend to pass more often than slower drivers. Therefore, we study the complementary case where the passing rate is linear in the velocity difference. Our goal is to examine the role played by the passing mechanism. While the governing equations become more complicated, a formal analytical solution is still possible. Although certain quantitative features such as the scaling exponents change, the qualitative picture remains the same. Interestingly, this accelerated passing mechanism leads to a triple deck (a boundary layer accompanied by an additional inner layer) structure of the car velocity distribution.

II. THEORY

In the ballistic motion model, fluctuations in the velocity of a single isolated car are neglected. We thus consider a one-dimensional traffic flow where size-less cars ("particles") move with a constant velocity. Initially, cars are randomly distributed in space and they move with their intrinsic velocities. The presence of slower cars forces some cars to drive behind a slower car and therefore leads to the formation of clusters. When a cluster overtakes a slower cluster, a larger cluster forms and its velocity is the smaller of the two velocities. Meanwhile, all cars in a given cluster may escape their respective clusters and resume driving with their intrinsic velocity. We consider the case where the escape rate is proportional to Δv, the difference between the velocity of the passing car and lead (slowest) car in the cluster, i.e., $\ell^{-1}\Delta v$, where ℓ has dimensions of length.

It is convenient to introduce dimensionless velocity $v/v_0 \to v$, space $xc_0 \to x$, and time $c_0 v_0 t \to t$ variables. Consequently, the escape rate $\ell^{-1}\Delta v$ becomes $R^{-1}\Delta v$. The dimensionless number $R = t_{\rm esc}/t_{\rm col}$ is simply equal to the ratio of the two elementary time scales, the escape time scale $t_{\rm esc} = \ell/v_0$ and the collision time scale $t_{\rm col} = (c_0 v_0)^{-1}$ [25]. We term $R = c_0\ell$ the "collision number".

Let $P(v,t)$ be the density of clusters moving with velocity v at time t. Initially, isolated single cars drive with their intrinsic velocities drawn from the distribution $P_0(v) \equiv P(v, t = 0)$. The dimensionless intrinsic velocity distribution is thus normalized to unity, $\int dv P_0(v) = 1$.

The flow is invariant under a finite velocity translation, and the minimal velocity is set to zero.

Neglecting correlations between the velocities and the positions of particles, a mean-field equation for the cluster velocity distribution $P(v)$ can be written

$$\begin{aligned}\frac{\partial P(v)}{\partial t} &= R^{-1}\int_0^v dv'(v-v')P(v,v') \\ &- P(v)\int_0^v dv'(v-v')P(v'),\end{aligned} \tag{1}$$

where the time variable t has been suppressed. This master equation involves $P(v,v')$, the density of cars of intrinsic velocity v driving with actual velocity $v' < v$. Such slowed down cars escape their clusters with rate proportional to the velocity difference, $R^{-1}(v-v')$, and thus the escape term. Collisions occur with rate proportional to the velocity difference as well as the product of the velocity distributions. The integration limits ensure that only collisions with slower cars are taken into account.

The master equation is complemented by a second rate equation corresponding to the joint velocity distribution $P(v,v')$,

$$\begin{aligned}\frac{\partial P(v,v')}{\partial t} = &- R^{-1}(v-v')P(v,v') + (v-v')P(v)P(v') \\ &- P(v,v')\int_0^{v'} dv''(v'-v'')P(v'') \\ &+ P(v')\int_{v'}^{v} dv''(v''-v')P(v,v'').\end{aligned} \tag{2}$$

In this equation as well, the escape term is proportional to the velocity difference. A useful check of self-consistency is that the total density of v-cars, $P_0(v) = P(v) + \int_0^v dv' P(v,v')$ is conserved by these master equations.

We restrict attention to the steady state properties, which can be obtained by taking the limit $t \to \infty$ or $\partial/\partial t \equiv 0$. We will express the joint distribution via the single cluster densities, and then insert it into (1) to get a *closed* equation for the cluster velocity distribution. Once the cluster distribution is found, average quantities such as the average cluster concentration, c, and the average cluster velocity, $\langle v \rangle$, will easily follow

$$c = \int_0^\infty dv\, P(v), \qquad \langle v \rangle = c^{-1}\int_0^\infty dv\, vP(v). \tag{3}$$

Furthermore, knowledge of the joint velocity distribution, $P(v,v')$, will enable computation of J, the average flux

$$J = \int_0^\infty dv \left[vP(v) + \int_0^v dw\, wP(v,w) \right], \tag{4}$$

and $G(v)$, the actual car velocity distribution

$$G(v) = P(v) + \int_v^\infty dw\, P(w,v). \tag{5}$$

This latter quantity satisfies the normalization conditions $1 = \int dv\, G(v)$ and $J = \int dv\, vG(v)$.

We turn now to solving the steady state master equations. It proves useful to define two auxiliary functions, $Q(v,v')$ and $T(v,v')$:

$$\begin{aligned}Q(v,v') &= R^{-1}(v-v') + \int_0^{v'} dv''(v'-v'')P(v''), \\ T(v,v') &= \int_{v'}^{v} dv''(v''-v')P(v,v'').\end{aligned} \tag{6}$$

The densities $P(v')$ and $P(v,v')$ can be obtained from these auxiliary functions by differentiation

$$P(v') = \frac{\partial^2 Q(v,v')}{\partial v'^2}, \qquad P(v,v') = \frac{\partial^2 T(v,v')}{\partial v'^2}. \tag{7}$$

To obtain $P(v,v')$ let us first re-write (2) as

$$P(v,v')Q(v,v') = (v-v')P(v)P(v') + T(v,v')P(v'). \tag{8}$$

Using the definition of the auxiliary functions (6), this equation is rewritten as

$$\frac{\partial}{\partial v'}\left[Q^2(v,v')\,\frac{\partial}{\partial v'}\,\frac{T(v,v')}{Q(v,v')} \right] = (v-v')P(v)P(v'). \tag{9}$$

Integrating twice over v' gives the joint auxiliary function $T(v,v')$ in terms of $Q(v,v')$ and the single variable functions:

$$\frac{T(v,v')}{P(v)} = Q(v,v')\int_{v'}^{v}\frac{du}{Q^2(v,u)}\int_u^v dw(v-w)P(w). \tag{10}$$

The boundary conditions $T(v,v) = \frac{\partial}{\partial v'}T(v,v')\big|_{v'=v} = 0$ were used to obtain (10). Replacing $P(w)$ with $\frac{\partial^2 Q(v,w)}{\partial w^2}$ and integrating by parts gives

$$\frac{T(v,v')}{P(v)} = Q(v)Q(v,v')\int_{v'}^{v}\frac{du}{Q^2(v,u)} - (v-v'). \tag{11}$$

Here, $Q(v) = Q(v,v) = \int_0^v dv'(v-v')P(v')$. Substituting Eq. (11) into (8), we find a relatively simple expression for the joint velocity distribution

$$P(v,v') = P(v)P(v')Q(v)\int_{v'}^{v}\frac{du}{Q^2(v,u)}. \tag{12}$$

Furthermore, combining this joint velocity distribution with the normalization condition $P_0(v) = P(v) + \int_0^v dv' P(v,v')$, we arrive at

$$P_0(v) = P(v)\left[1 + Q(v)\int_0^v dv' P(v')\int_{v'}^{v}\frac{du}{Q^2(v,u)} \right]. \tag{13}$$

Additional simplification can be achieved by using the relationship $P(v')=\frac{\partial^2 Q(v,v')}{\partial v'^2}$ and performing the integration by parts. This yields

$$P_0(v)=Q(v)Q''(v)\left[\frac{R}{v}+\frac{1}{R}\int_0^v\frac{du}{Q^2(v,u)}\right]. \tag{14}$$

Since $Q(v,u)=R^{-1}(v-u)+Q(u)$, the function appearing in the integrand contains only $Q(u)$; therefore, the integro-differential equation (14) is a closed equation for $Q(v)$. Given the auxiliary function $Q(v)$, the cluster velocity distribution is calculated from

$$P(v)=Q''(v). \tag{15}$$

Eq. (5) then gives the actual car velocity distribution

$$G(v)=P(v)\left[1+\int_v^\infty dwP(w)Q(w)\int_v^w\frac{du}{Q^2(w,u)}\right]. \tag{16}$$

We now compute the flux, or the average car velocity given by Eq. (4). From the definition of the joint auxiliary function, the second integral in (4) equals $T(v,0)$, thereby implying $J=\int_0^\infty dv\,[vP(v)+T(v,0)]$. The integrand can be further simplified by using Eq. (11). The term $vP(v)$ cancels and we find a useful expression for the flux

$$J=\int_0^\infty dv\,P(v)Q(v)\frac{v}{R}\int_0^v\frac{du}{Q^2(v,u)}. \tag{17}$$

We conclude that for arbitrary intrinsic velocity distributions, $P_0(v)$, the entire steady state problem is reduced to the closed integro-differential equation (14) for the auxiliary function $Q(v)$. Once this function is known, steady state characteristics such as $P(v)$, $P(v,v')$, $G(v)$, and J are given by Eqs. (15), (12), (16), and (17), respectively. Despite the complicated nature of the equations, a leading order analysis is still generally possible, as detailed below.

III. LEADING BEHAVIOR

In the physically relevant limits of small and large collision numbers, a leading order analysis is possible. When $R\ll 1$, a perturbation expansion in the small parameter R shows that the flow is almost uninterrupted. In the complementary case, $R\gg 1$, a boundary layer analysis is needed for the velocity distributions. The leading scaling behavior shows that a transition from a “laminar” low-density flow into a “congested” high-density flow occurs.

A. Laminar Flows

In the collision-controlled regime, the velocity distributions can be obtained perturbatively. Let us write $P(v)=\sum_{n\geq 0}R^nP^{(n)}(v)$ and $P(v,v')=\sum_{n\geq 0}R^nP^{(n)}(v,v')$. The zeroth order terms are trivial $P^{(0)}(v)=P_0(v)$ and $P^{(0)}(v,v')=0$. The first order correction for the joint velocity distribution can be computed by inserting the zeroth order approximation in the right-hand side of Eq. (12). This gives

$$P^{(1)}(v,v')=\begin{cases}RP_0(v)P_0(v') & v-v'\gg R;\\ RP_0(v)P_0(v')\frac{(v-v')}{v-v'+RQ_0(v)} & v-v'\ll R;\end{cases}$$

where the notation $Q_0(v)=\int_0^v dv'(v-v')P_0(v')$ was used. One can check that $P^{(1)}(v,v')$ satisfies the master Eq. (2) to first order in R. This joint velocity distribution together with Eq. (5) determines the cluster and car velocity distributions to first order

$$P(v)\cong P_0(v)\left[1-R\int_0^v dv'P_0(v')\right], \tag{18}$$
$$G(v)\cong P_0(v)\left[1-R\int_0^v dv'P_0(v')+R\int_v^\infty dv'P_0(v')\right].$$

Furthermore, the cluster density and the flux can be evaluated to first order in R by integrating Eq. (18)

$$c\cong 1-\frac{R}{2},\qquad J\cong J^{(0)}-J^{(1)}R, \tag{19}$$

with $J^{(0)}=\int dv\,vP_0(v)$ and $J^{(1)}=\int dv\,P_0(v)Q_0(v)$. The order R corrections are obtained by writing the integrand using derivatives of $Q_0(v)$ and integrating by parts. For example, $c=\int dv\,P(v)\cong 1-R\int dv\,Q_0''(v)Q_0'(v)$, and similarly for the flux. Interestingly, the first order correction to the density is universal in that it does not depend on the details of the intrinsic velocity distribution. We conclude that as the collision-controlled limit is weakly interacting, explicit expressions for the leading behavior are possible for the steady state properties.

B. Congested Flows

When $R\gg 1$, slow and fast cars exhibit very different behaviors. Small enough velocities are not affected by the presence of faster cars, and the perturbative expression (18) holds for $P(v)$, i.e., $P(v)\cong P_0(v)$ for $v\ll v^*$. At the threshold velocity v^*, the correction to the initial velocity distribution in Eq. (18) becomes of order unity, $1\sim R\int_0^{v^*}dv'P_0(v')$.

We consider (without loss of generality) algebraic intrinsic velocity distributions

$$P_0(v)=(\mu+1)v^\mu\quad \mu>-1, \tag{20}$$

in the velocity range [0:1]. Thus, the threshold velocity decreases with growing R according to

$$v^*\sim R^{-1/(\mu+1)}. \tag{21}$$

We expect that the velocity distributions are strongly suppressed for velocities much larger than this threshold velocity. We further assume an algebraic behavior, $P(v) \sim R^\delta v^\sigma$. As $P(v) = Q''(v)$, the following auxiliary function $Q(v) \sim R^\delta v^{\sigma+2}$ is implied. Substituting this expression into Eq. (14) and noting that the term R/v dominates over the integral in the square brackets, we find $\delta = -1/2$ and $\sigma = (\mu - 1)/2$. The behavior of the cluster velocity distribution is therefore

$$P(v) \sim \begin{cases} v^\mu & v \ll v^*; \\ R^{-1/2} v^{(\mu-1)/2} & v \gg v^*. \end{cases} \tag{22}$$

Of course, the small and large velocity components of the cluster velocity distribution match at the threshold velocity, $P(v^*) \sim P_0(v^*)$. Notice that behavior of the cluster velocity distribution is different than the one found in the case of constant escape rates [25] where $\sigma = \mu - 1$ when $\mu < 0$ and $\sigma = \mu/2 - 1$ when $\mu > 0$. We conclude that in the case of linear escape rate the scaling exponents change, and the behavior becomes generic in μ.

Using the leading behavior of $P(v)$, we determine the average cluster concentration, $c = \int dv P(v) \sim R^{-1/2}$, and the average number of cars per cluster, $\langle m \rangle = c^{-1} \sim R^{1/2}$. This behavior is in accordance with the following heuristic argument. Let the initial car concentration be c_0, the final cluster concentration be $c \ll c_0$, and the typical velocity be v_0. Thus, the typical cluster size is $\langle m \rangle = c_0/c$. When large clusters form, $\langle m \rangle \gg 1$, and the overall escape rate can be estimated by $\langle m \rangle \ell^{-1} v_0$. On the other hand, the typical collision rate is cv_0. In the steady state the number of cars entering and leaving clusters must balance, and therefore the overall collision and escape rates are equal, $(c_0/c) \times \ell^{-1} v_0 = v_0 c$. Therefore, $c \sim (c_0/\ell)^{1/2}$ and $\langle m \rangle = c_0/c \sim \sqrt{c_0 \ell} \sim R^{1/2}$ is recovered.

In the case of velocity independent escape rates, similar behavior occurred only when the intrinsic velocity distribution did not have a strong fast car component, namely when $\mu > 0$. In the present case, the flow is dominated by large and slow clusters. We conclude that as fast cars are more likely to pass when the escape rate is linear, the cluster velocities are only slightly reduced due to collisions. This is consistent with the fact that the average cluster velocity $\langle v \rangle$, defined in Eq. (3), remains of order unity.

To study the actual velocity distribution $G(v)$, the integral $I = \int_v^w du [R^{-1}(w-u) + Q(u)]^{-2}$ should be evaluated. We note that at $v = v^{**} \sim R^{-1/(\mu+3)}$, the two terms in the integrand become comparable. The leading order behavior is thus $I \sim w^{-2} R^2 v^{**}$ for $v \ll v^{**}$, and $I \sim R v^{-(\mu+2)}$ for $v \gg v^{**}$. Combining this leading behavior with $P(v)$ given by Eq. (22) and substituting into Eq. (16) one finds $G(v) \sim R^{(\mu+2)/(\mu+3)} P(v)$ for $v \ll v^{**}$, and $G(v) \sim v^{-\mu-2} P(v)$ for $v \gg v^{**}$. Using Eq. (22), we arrive at the leading order behavior of the actual car velocity distribution

$$G(v) \sim \begin{cases} R^{-1/2} v^{-(\mu+5)/2} & v^{**} \ll v; \\ R^{(\mu+1)/(2\mu+6)} v^{(\mu-1)/2} & v^* \ll v \ll v^{**}; \\ R^{(\mu+2)/(\mu+3)} v^\mu & v \ll v^*. \end{cases} \tag{23}$$

It is simple to verify that these expressions match at the two threshold velocities, v^* and v^{**}. In the case of constant escape rates, only a single threshold velocity was found. Here, in contrast, an interesting triple deck structure with two marginal velocities emerges. Besides the threshold at $v^* \sim R^{-1/(\mu+1)}$ we observe an additional threshold located at $v^{**} \sim R^{-1/(\mu+3)}$. The latter threshold velocity is proportional to the flux, obtained from $J = \int dv\, v G(v)$,

$$J \sim v^{**} \sim R^{-1/(\mu+3)}. \tag{24}$$

Unlike the scaling behavior of the average cluster mass, the flux scaling law depends on μ. No flux reduction occurs in the limit where the intrinsic velocity distribution is dominated by fast cars ($\mu \to \infty$), while maximal flux reduction, $J \sim R^{-1/2}$, occurs in the limit when the distribution is dominated by slow cars ($\mu \to -1$).

It is instructive to study velocity fluctuations by evaluating the moments of the actual velocity distribution defined via $G_n = \int dv\, v^n G(v)$. These moments are calculated from Eq. (23) to give

$$G_n \sim \begin{cases} R^{-1/2} & n > (\mu+3)/2; \\ R^{-n/(\mu+3)} & n < (\mu+3)/2. \end{cases} \tag{25}$$

Interestingly, fluctuations in the flux become very large ($G_2 \gg G_1^2$) when $-1 < \mu < 1$. This suggests that slowing down due to collisions dominates when the intrinsic velocity distribution has a strong small velocity component. This results in a broadening of the actual velocity distribution.

One may wonder whether this behavior is restricted to purely algebraic distributions. Since both of the threshold velocities approach zero as the collision number diverges, for sufficiently large R, a vanishingly small component of the velocity distribution is sampled. Thus, if the limit $\mu = \lim_{v\to 0} v \frac{\partial}{\partial v} \ln P_0(v)$ exists, then μ determines the behavior when $R \to \infty$ as detailed above. We conclude that our previous analysis applies to a broad range of intrinsic distributions, not only purely algebraic ones.

We also see that when $R \to \infty$ the solution to Eq. (14) exhibits a two layer structure, a result reminiscent of classical boundary layer theory [28]. Inside the boundary layer, $v \ll v^*$, the cluster velocity distribution is only slightly affected by collisions, while in the outer region, $v \gg v^*$, the cluster velocity distribution is much smaller than the intrinsic velocity distribution. The threshold velocity v^* is determined by the condition that below v^* the intrinsic velocity distribution remains unaffected, $P(v) \cong P_0(v)$. The actual velocity distribution, however, exhibits a richer three layer structure. The first layer (referred to as the lower deck in fluid mechanics) is followed by the middle deck $v^* \ll v \ll v^{**}$ and then the upper deck completes the structure. Such triple deck structures were originally found within the framework of classical fluid dynamics and applied mathematics [28].

IV. CONCLUSIONS

In summary, we have studied traffic flows with a passing mechanism which favors faster cars over slower cars. Despite the complicated structure of the master Boltzmann equation, a reduction to an integro-differential equation and a complete formal solution is possible. The overall behavior agrees qualitatively with the constant escape rate case. For example, a single dimensionless parameter, the collision number R, ultimately determines the nature of the flow in the steady state. In the "laminar" flow regime, $R \ll 1$, corrections due to collisions are of order R. Here, large clusters are rare, and the exact form of the passing mechanism plays a secondary role. In agreement with rural traffic observations, the average cluster mass grows linearly with flux in this dilute limit. In the "congested" regime, i.e., when $R \gg 1$, we found a boundary layer structure for the cluster velocity distribution. The flux is reduced by an algebraic function of R.

However, quantitative differences do arise including changes in the flux reduction exponent, and the behavior of the velocity distribution in the limit of large velocities. Furthermore, the actual velocity distribution exhibits a triple deck structure with two threshold velocities in contrast with the constant escape rate. However, as both of the threshold velocities vanish when $R \to \infty$, extreme statistics of the velocity distribution still dominate the flow in this congested phase. This latter result, valid for a broad class of intrinsic velocity distributions, is reminiscent of intermittency in turbulent flows. Additionally, we showed under what conditions do fluctuations in flux can become significant.

As one would intuitively expect, a linear escape mechanism more effectively suppresses clustering because extremely fast cars will spend less time in clusters. Indeed, we found that the average cluster velocity remains of order unity which implies that situations where the flow consists solely of slow clusters are avoided.

It is remarkable that the underlying master equations are still solvable even under more complicated escape rules. Unlike in Boltzmann equations arising in kinetic theory, the integration limits are restricted thereby enabling a reduction to an ordinary differential equation and a formal solution. We expect that further analytical progress may be possible. For example, it will be useful to apply the above formalism to other realistic situations such as multi-lane traffic.

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