Multiscaling in Inelastic Collisions

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We study relaxation properties of two-body collisions on the mean-field level. We show that this process exhibits multiscaling asymptotic behavior as the underlying distribution is characterized by an infinite set of nontrivial exponents. These nonequilibrium relaxation characteristics are found to be closely related to the steady state properties of the system.
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Our understanding of the statistical mechanics of nonequilibrium systems remains incomplete, in sharp contrast with their equilibrium counterpart. The rich phenomenology associated with dynamics of far from equilibrium interacting particle systems exposes the lack of a unifying theoretical framework. Simple tractable microscopic models can therefore help us gain insight and better the description of nonequilibrium dynamics.

In this study, we focus on the nonequilibrium relaxation of an infinite particle system interacting via two body collisions. We find that a hierarchy of scales underlies the relaxation. In particular, we devise an extremely simple system which exhibits multiscaling on the mean-field level while in finite dimensions simple scaling behavior is restored. Furthermore, we show that this behavior extends to a broader class of collision processes.

In the mean-field framework, the spatial structure is ignored. Therefore, we consider an infinite number of identical particles which are characterized by a single parameter, their velocity v. Two-body collisions are realized by choosing two particles at random and changing their velocities according to $(u_1, u_2) \to (v_1, v_2)$ with

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \gamma & 1-\gamma \\ 1-\gamma & \gamma \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{1}$$

In other words, the post-collision velocities are given by a linear combination of the pre-collision velocities. Both the total momentum ($u_1 + u_2 = v_1 + v_2$) and the total number of particles are conserved by this process. In fact, the collision rule (1) is the most general linear combination which obeys momentum conservation and Galilean invariance, i.e., invariance under velocity translation $v \to v - v_0$.

Our motivation for studying this problem is inelastic collisions in granular gases [1–3]. Therefore, we restrict our attention to dissipative collisions, $0 < \gamma < 1$. While the two problems involve different collision rates, they share the same trivial final state where all velocities vanish, $P(v,t) \to \delta(v)$ when $t \to \infty$ (without loss of generality, the average velocity was set to zero by invoking the transformation $v \to v - \langle v \rangle$). We chose to describe this work in slightly more general terms since closely related dynamics were used in different contexts including voting systems [4,5], asset exchange processes [6], combinatorial processes [7], and headway distances in traffic flows [8]. We will show that multiscaling characterizes fluctuations in these problems as well.

Velocity fluctuations may be obtained via the probability distribution function $P(v,t)$ which obeys the following master equation

$$\frac{\partial P(v,t)}{\partial t} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} du_1\, du_2\, P(u_1,t)P(u_2,t) \times \left[\delta(v - \gamma u_1 - (1-\gamma)u_2) - \delta(v - u_2)\right]. \tag{2}$$

This Boltzmann equation with a velocity independent collision rate is termed the Maxwell model in kinetic theory [9]. The $\delta-$functions on the right-hand side reflect the collision rule (1) and guarantee conservation of the number of particles, $\int dv P(v,t) = 1$, and the total momentum $\int dv\, v P(v,t) = 0$. Eq. (2) can be simplified by eliminating one of the integrations

$$\frac{\partial P(v,t)}{\partial t} + P(v,t) = \frac{1}{1-\gamma}\int_{-\infty}^{\infty} du P(u,t) P\left(\frac{v-\gamma u}{1-\gamma}, t\right). \tag{3}$$

Further simplification is achieved via the Fourier transform $\hat{P}(k,t) = \int dv\, e^{ikv}\, P(v,t)$ which obeys

$$\frac{\partial}{\partial t}\hat{P}(k,t) + \hat{P}(k,t) = \hat{P}[\gamma k, t]\,\hat{P}[(1-\gamma)k, t]. \tag{4}$$

Although the integration is eliminated, this compact equation is challenging as the nonlinear term is nonlocal.

Velocity fluctuations can be quantified using the moments of the velocity distribution, $M_n(t) = \int dv\, v^n P(v,t)$. The moments obey a closed and recursive set of the ordinary differential equations. The corresponding equations can be derived by inserting the expansion $\hat{P}(k,t) = \sum_n \frac{(ik)^n}{n!} M_n(t)$ into Eq. (4) or directly from Eq. (2). The first few moments evolve according to $\dot{M}_0 = \dot{M}_1 = 0$, and

$$\begin{aligned} \dot{M}_2 &= -a_2 M_2, \\ \dot{M}_3 &= -a_3 M_3, \\ \dot{M}_4 &= -a_4 M_4 + a_{24} M_2^2, \end{aligned} \tag{5}$$

with the coefficients

$$a_n \equiv a_n(\gamma) = 1 - (1-\gamma)^n - \gamma^n, \tag{6}$$

and $a_{24} = 6\gamma^2(1-\gamma)^2$. Integrating these rate equations yields $M_0 = 1$, $M_1 = 0$ and

$$\begin{aligned} M_2(t) &= M_2(0)e^{-a_2 t} \\ M_3(t) &= M_3(0)e^{-a_3 t} \\ M_4(t) &= \left[M_4(0) + 3M_2^2(0)\right] e^{-a_4 t} - 3M_2^2(t). \end{aligned} \tag{7}$$

The asymptotic behavior of the first few moments suggests that knowledge of the RMS fluctuation $v^* \equiv M_2^{1/2}$ is not sufficient to characterize higher order moments since $M_3^{1/3}/v^*, M_4^{1/4}/v^* \to \infty$ as $t \to \infty$.

This observation extends to higher order moments as well. In general, the moments evolve according to

$$\dot{M}_n + a_n M_n = \sum_{m=2}^{n-2} \binom{n}{m} \gamma^m (1-\gamma)^{n-m} M_m M_{n-m}. \tag{8}$$

Note that for $0 < \gamma < 1$, the coefficients a_n satisfy $a_n < a_m + a_{n-m}$ when $1 < m < n-1$. This inequality can be shown by introducing $G(\gamma) = a_m(\gamma) + a_{n-m}(\gamma) - a_n(\gamma)$ which satisfies $G(0) = 0$ and $G(\gamma) = G(1-\gamma)$. Therefore, one needs to show that $G'(\gamma) = m[b_m - b_n] + (n-m)[b_{n-m} - b_n] > 0$ for $0 < \gamma < 1/2$ with $b_n \equiv b_n(\gamma) = (1-\gamma)^{n-1} - \gamma^{n-1}$. One can verify that the b_n's decrease monotonically with increasing n, $b_n \geq b_{n+1}$ for $n \geq 2$, therefore proving the desired inequality. Since moments decay exponentially, this inequality shows that the right hand side in the above equation is negligible asymptotically. Thus, the leading asymptotic behavior for all $n > 0$ is $M_n \sim \exp(-a_n t)$. Since the a_n's increase monotonically, $a_n < a_{n+1}$, the moments decrease monotonically in the long time limit, $M_n > M_{n+1}$. Furthermore, in terms of the second moment one has

$$M_n \propto M_2^{\alpha_n}, \qquad \alpha_n = \frac{1-(1-\gamma)^n - \gamma^n}{1-(1-\gamma)^2 - \gamma^2}. \tag{9}$$

While the prefactors depend on the details of the initial distribution, the scaling exponents are universal. Therefore, the velocity distribution does not follow a naive scaling form $P(v,t) \sim \frac{1}{v^*} P(\frac{v}{v^*})$. Such a distribution would imply the linear exponents $\alpha_n = \alpha_n^* = n/2$. Instead, the actual behavior is given by Eq. (9) with the exponents α_n reflecting a multiscaling asymptotic behavior with a nontrivial (non-linear) dependence on the index n. For instance, the high order exponents saturate, $\alpha_n \to a_2^{-1}$ for $n \to \infty$, instead of diverging. One may quantify the deviation from ordinary scaling via a properly normalized set of indices $\beta_n = \alpha_n/\alpha_n^*$ defined from $M_n^{1/n} \sim (v^*)^{\beta_n}$. By evaluating the $\gamma = 1/2$ case where multiscaling is most pronounced, a bound can be obtained for these indices: $7/8, 31/48 \leq \beta_n \leq 1$ for $n = 4, 6$ respectively. Furthermore, $\beta_n \to 1 - \frac{2n-3}{2}\gamma$ when $\gamma \to 0$ indicating that the deviation from ordinary scaling vanishes for weakly inelastic collisions. Thus, the multiscaling behavior can be quite subtle [10].

The above shows that a hierarchy of scales underlies fluctuations in the velocity. In parallel, a hierarchy of diverging time scales characterizes velocity fluctuations

$$M_n^{1/n} \sim \exp(-t/\tau_n), \qquad \tau_n = \frac{n}{a_n}. \tag{10}$$

These time scales diverge for large n according to $\tau_n \simeq n$. Large moments reflect the large velocity tail of a distribution. Indeed, the distribution of extremely large velocities is dominated by persistent particles which experienced no collisions up time t. The probability for such events decays exponentially with time $P(v,t) \sim P(v,0)\exp(-t)$ for $v \gg 1$ (alternatively, this behavior emerges from Eq. (3) since the gain term is negligible for the tail and hence $\dot{P} + P = 0$). This decay is consistent with the large order moment decay $M_n \sim \exp(-t)$ when $n \to \infty$.

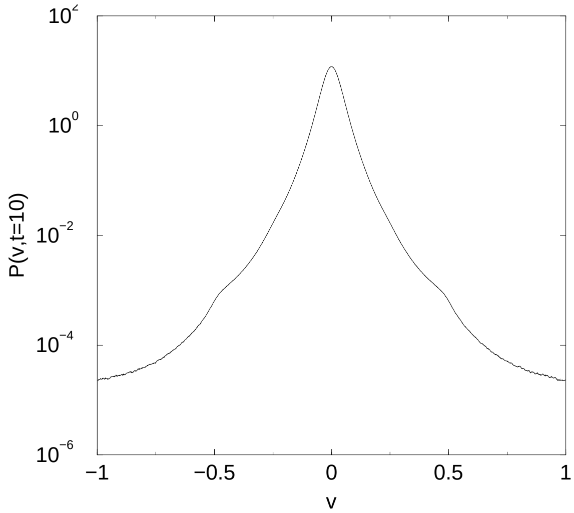

FIG. 1. Development of a singularity for a compact initial distribution. Shown is the probability distribution obtained by simulating the collision process of Eq. (1) with $\gamma = 1/2$. The data represents an average over 200 independent realization in a system with 10^7 particles, starting from a uniform distribution in the range $[-1, 1]$.

Although the leading asymptotic behavior of the moments was established, understanding the entire distribution $P(v,t)$ remains a challenge. Simulations of the $\gamma = 1/2$ process reveal an interesting structure for compact distributions. Starting from a uniform velocity distribution, $P_0(v) = 1/2$ for $-1 < v < 1$, the distribution loses analyticity at $v = \pm 1/2$. Our analysis of Eq. (4) shows that such a singularity should indeed develop at $v = \pm 1/2$ and it additionally implies the appearance of (progressively weaker and weaker) singularities at $v = \pm 1/4$, etc. In general, for an arbitrary *compact* initial distribution and an arbitrary γ, the distribution $P(v,t)$ loses analyticity for $t > 0$ and develops an infinite (countable) set of singularities whose locations depend on the arithmetic nature of γ (e.g., it is very different for rational and irrational γ's). On the other hand, unbounded distributions do not develop such singularities, and therefore, the loss of analyticity is not necessarily responsible for the multiscaling behavior.

Asymptotically, our system reaches a trivial steady state $P(v, t = \infty) = \delta(v)$. To examine the relation between dynamics and statics, a non-trivial steady state can be generated by considering the driven version of the collision process [11–13]. External forcing balances dissipation due to collisions and therefore results in a nontrivial nonequilibrium steady state. Specifically, we assume that in addition to changes due to collisions, velocities may also change due to an external forcing: $\frac{dv_j}{dt}|_{\rm heat} = \xi_j$. We assume standard uncorrelated white noise $\langle\xi_i(t)\xi_j(t')\rangle = 2D\delta_{ij}\delta(t-t')$ with a zero average $\langle\xi_j\rangle = 0$. The left hand side of the master equation (3) should therefore be modified by the diffusion term

$$\frac{\partial P(v,t)}{\partial t} \to \frac{\partial P(v,t)}{\partial t} - D\frac{\partial^2 P(v,t)}{\partial v^2}. \tag{11}$$

Of course, the addition of the diffusive term does not alter conservation of the total particle number and the total momentum, and one can safely work in a reference frame moving with the center of mass velocity.

We restrict our attention to the steady state. The Fourier transform $\hat{P}_\infty(k) \equiv \hat{P}(k, t=\infty)$ satisfies

$$(1 + Dk^2)\hat{P}_\infty(k) = \hat{P}_\infty[\gamma k]\,\hat{P}_\infty[(1-\gamma)k]. \tag{12}$$

The solution to this functional equation which obeys the conservation laws $\hat{P}_\infty(0) = 1$ and $\langle v\rangle = \hat{P}'_\infty(0) = 0$ is found recursively

$$\hat{P}_\infty(k) = \prod_{i=0}^{\infty}\prod_{j=0}^{i}\left[1 + \gamma^{2j}(1-\gamma)^{2(i-j)}Dk^2\right]^{-\binom{i}{j}}. \tag{13}$$

To simplify this double product we take the logarithm and transform it as follows

$$\begin{aligned}
\ln\hat{P}_\infty(k) &= -\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{i}{j}\ln\left[1 + \gamma^{2j}(1-\gamma)^{2(i-j)}Dk^2\right]\\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{i}{j}\sum_{n=1}^{\infty}\frac{(-Dk^2)^n\gamma^{2jn}(1-\gamma)^{2(i-j)n}}{n}\\
&= \sum_{n=1}^{\infty}\frac{(-Dk^2)^n}{n}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{i}{j}\gamma^{2nj}(1-\gamma)^{2n(i-j)}\\
&= \sum_{n=1}^{\infty}\frac{(-Dk^2)^n}{n}\sum_{i=0}^{\infty}\left[\gamma^{2n} + (1-\gamma)^{2n}\right]^i.
\end{aligned} \tag{14}$$

The second identity follows from the series expansion $\ln(1+q) = -\sum_{n\geq 1} n^{-1}(-q)^n$, and the forth from the binomial identity $\sum_{j=0}^{i}\binom{i}{j}p^j q^{i-j} = (p+q)^i$. Finally, using the geometric series $(1-x)^{-1} = \sum_{n\geq 0} x^n$, the Fourier transform at the steady state is found

$$\hat{P}_\infty(k) = \exp\left\{\sum_{n=1}^{\infty}\frac{(-Dk^2)^n}{n a_{2n}(\gamma)}\right\}, \tag{15}$$

with $a_n(\gamma)$ given by Eq. (6). The nth cumulant of the steady state distribution κ_n can be readily found from $\ln\hat{P}_\infty(k) = \sum_m \frac{(ik)^m}{m!}\kappa_m$. Therefore, the odd cumulants vanish while the even cumulants are simply proportional to the time scales characterizing the exponential relaxation of the corresponding moments:

$$\kappa_{2n} = \frac{(2n-1)!}{n}D^n\tau_{2n}. \tag{16}$$

Of course, the moments can be constructed from these cumulants. Interestingly, a direct correspondence between the steady state characteristics and the nonequilibrium relaxation time scales is established via the cumulants of the probability distribution.

None of the (even) cumulants vanish, thereby reflecting significant deviations from a Gaussian distribution. Nevertheless, for sufficiently large velocities, one may concentrate on the small wave number behavior. Using the inverse Fourier transform of (15) one finds the tail of the distribution

$$P_\infty(v) \simeq \frac{1}{\sqrt{2\pi v_0^2}}\exp\left(-\frac{v^2}{2v_0^2}\right) \qquad v \gg v_0, \tag{17}$$

with $v_0^2 = 2D/a_2$. This in particular implies the large moment behavior $M_{2n} \to (2n-1)!!\,v_0^{2n}$ as $n \to \infty$. The Gaussian tail is different than the $\exp(-\text{const.}\times v^{3/2})$ behavior obtained when the collision rate is proportional to the velocity difference [13]. In contrast, large velocities in one-dimensional inelastic gases are suppressed according to $\exp(-\text{const.}\times v^3)$ [14].

Finally, we briefly discuss a few generalizations and extensions of the basic model. Note that Eq. (9) extends to energy-generating collisions as well ($\gamma < 0$ or $\gamma > 1$), despite the fact that the limiting distribution is no longer a δ-function. Next, relaxing Galilean invariance, the most general momentum conserving collision rule is

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \gamma_1 & 1-\gamma_2 \\ 1-\gamma_1 & \gamma_2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{18}$$

Following the same steps that led to (9) shows that when $\gamma_1, \gamma_2 \neq 0, 1$ and when $M_1 = 0$ this process also exhibits multiscaling with the exponents $\alpha_n = a_n/a_2$, where $a_n(\gamma_1,\gamma_2) = \frac{1}{2}[a_n(\gamma_1) + a_n(\gamma_2)]$. When $\gamma_1 = 1-\gamma_2 = \gamma$ one recovers the model introduced by Melzak [4], and when $\gamma_1 = \gamma_2 = \gamma$ one recovers inelastic collisions. Since $a_n(\gamma) = a_n(1-\gamma)$ both models have identical multiscaling exponents. Furthermore, a multiscaling behavior with the very same exponents $\alpha_n(\gamma)$ is also found for the process $(u_1, u_2) \to (u_1 - \gamma u_1, v_1 + \gamma u_1)$ investigated in the context of asset distributions [6] and headway distributions in traffic flows [8].

One can also consider stochastic rather than deterministic collision processes by assuming that the collision (18) occurs with probability density $\sigma_1(\gamma_1, \gamma_2)$. Our findings extend to this model: the multiscaling exponents are given by the general expression $\alpha_n = a_n/a_2$ with $a_n = \int d\gamma_1 \int d\gamma_2 \sigma(\gamma_1,\gamma_2) a_n(\gamma_1,\gamma_2)$. In particular, for completely random inelastic collisions, i.e., $\sigma \equiv 1$ and $\gamma_1 = \gamma_2 = \gamma$, one finds $a_n = \frac{n-1}{n+1}$ and hence $\alpha_n = 3\frac{n-1}{n+1}$.

So far, we discussed only two-body interactions. We therefore consider N-body interactions where a collision is symbolized by $(u_1,\ldots,u_N) \to (v_1,\ldots,v_N)$. We consider a generalization of the $\gamma = \frac{1}{2}$ two-body case where the post-collision velocities are all equal. Momentum conservation implies $v_i = \bar{u} = N^{-1}\sum u_i$. The master equation is a straightforward generalization of the two-body case and we merely quote the moment equations

$$\dot{M}_n + a_n M_n = N^{-n} \sum_{n_i \neq 1} \binom{n}{n_1 \ldots n_N} M_{n_1} \cdots M_{n_N} \quad (19)$$

with $a_n = 1 - N^{1-n}$. Using a generalization of the aforementioned inequality $a_{m_1+\cdots+m_k} > a_{m_1} + \ldots + a_{m_k}$ when all $m_j > 1$, we find that the right-hand side of the above equation remains asymptotically negligible. Therefore, $M_n \sim e^{-a_n t}$ and

$$M_n \sim M_2^{\alpha_n}, \qquad \alpha_n = \frac{1 - N^{1-n}}{1 - N^{-1}}. \quad (20)$$

Thus, this N-body "averaging" process exhibits multiscaling asymptotic behavior as well.

Thus far, we investigated a mean field model. When particles reside on a d-dimensional lattice and only nearest neighbors interact, the above dynamics is equivalent to a diffusion process [15]. As a result, the underlying correlation length is diffusive, $L(t) \sim t^{1/2}$. Within this correlation length the velocities are "well mixed" and momentum conservation therefore implies that $v \sim L^{-d/2} \sim t^{-d/4}$. Indeed, the infinite dimension limit is consistent with the above exponential decay. Furthermore, an exact solution for moments of arbitrary order is possible [15]. We do not detail it here and simply quote that ordinary scaling is restored $M_n \sim t^{-n/4}$, i.e. $\alpha_n = \alpha_n^* = n/2$. Thus, spatial correlations counter the mechanism responsible for multiscaling.

In summary, we have investigated inelastic collision processes on the mean-field level. We have shown that such systems are characterized by multiscaling, or equivalently by an infinite hierarchy of diverging time scales. Multiscaling holds for several generalizations of the basic model including stochastic collision models and even processes which do not obey Galilean invariance. In this latter case, however, multiscaling is restricted to situations with zero total momentum. This perhaps explains why multiscaling asymptotic behavior was overlooked in previous studies [4,6]. Another explanation is that this behavior may be difficult to detect from numerical simulations. Indeed, in other problems such as multidimensional fragmentation [10], and in fluid turbulence, low order moments deviate only slightly from the normal scaling expectation.

Interestingly, although similarity solutions can be found for the master equation, multiscaling implies that they are unphysical. For example, Eq. (4) admits an exact Bobylev-Krook-Wu [9] scaling solution, $\hat{P}(k,t) = (1+K)e^{-K}$, where $K = Ak\, e^{-\gamma(1-\gamma)t}$ with an arbitrary constant A. In the present case, however, this BKW solution corresponds to the pathological initial condition $P_0(v) = \delta(v - iA) + iA\delta'(v - iA)$.

There are a number of extensions of this work which are worth pursuing. We have started with a kinetic theory of a 1D granular gas with a velocity independent collision rate. Within such a framework, it is sensible to approximate the collision rate with the RMS velocity fluctuation. This leads to the algebraic decay $M_n \sim t^{-2\alpha_n}$ with α_n given by Eq. (9) and in particular, Haff's cooling law $T = M_2 \sim t^{-2}$ is recovered [1]. Our numerical studies indicate that when velocity dependent collision rates are implemented, ordinary scaling behavior is restored. One may also use this model as an approximation for inelastic collisions in higher dimensions as well, following the Maxwell approximation in kinetic theory [9,16].

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[1] P. K. Haff, J. Fluid Mech. **134**, 401 (1983).
[2] S. McNamara and W. R. Young, Phys Fluids A **4**, 496 (1992).
[3] B. Bernu and R. Mazighi, J. Phys. A **23**, 5745 (1990).
[4] Z. A. Melzak, *Mathematical Ideas, Modeling and Applications, Volume II of Companion to Concrete Mathematics* (Wiley, New York, 1976), p. 279.
[5] P. A. Ferrari and L. R. G. Fontes, El. J. Prob. **3**, Paper no. 6 (1998).
[6] S. Ispolatov, P. L. Krapivsky, and S. Redner, Eur. Phys. J. B **2**, 267 (1998).
[7] D. Aldous and P. Diaconis, Prob. Theory Relat. Fields **103**, 199 (1995).
[8] J. Krug and J. Garcia, *cond-mat/9909034*.
[9] M. H. Ernst, Phys. Rep. **78**, 1 (1981).
[10] P. L. Krapivsky and E. Ben-Naim, Phys. Rev. E **50**, 3502 (1994); E. Ben-Naim and P. L. Krapivsky, Phys. Rev. Lett. **76**, 3234 (1996).
[11] D. R. M. Williams and F. C. MacKintosh, Phys. Rev. E **54**, 9 (1996).
[12] M. R. Swift, M. Boamfă, S. J. Cornell, and A. Maritan, Phys. Rev. Lett. **80**, 4410 (1998).
[13] T. P. C. van Noije and M. H. Ernst, Granular Matt. **1**, 57 (1998).
[14] E. Ben-Naim, S. Y. Chen, G. D. Doolen, and S. Redner, Phys Rev. Lett., in press (1999).
[15] E. Ben-Naim and P. L. Krapivsky, unpublished.
[16] E. Ben-Naim and P. L. Krapivsky, Phys. Rev. E **59**, 7000 (1999).