Granular Compaction: A Theoretical Model

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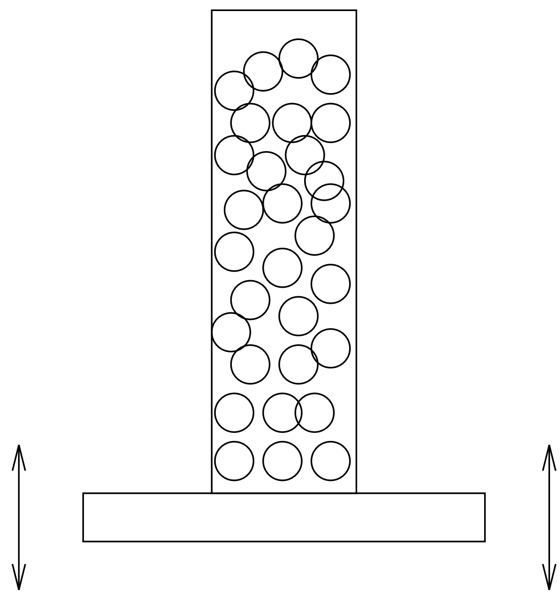


I. Experimental Observations

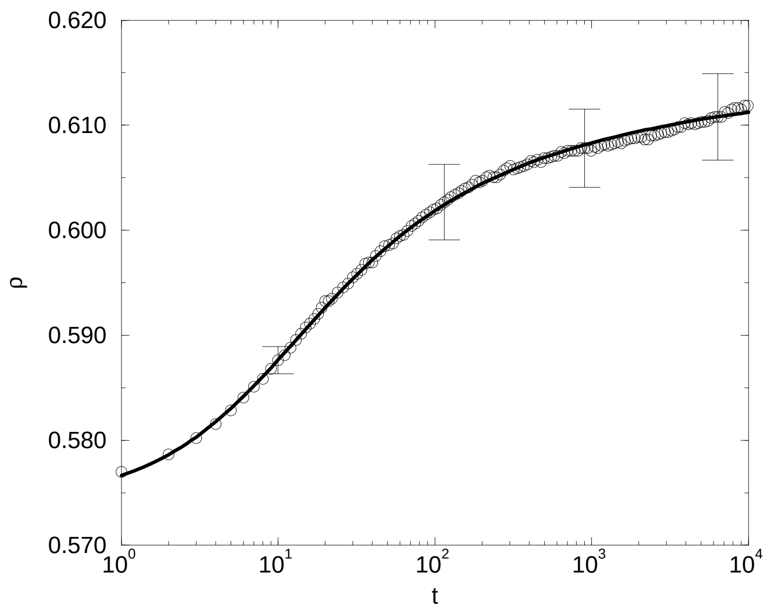


Fig. 1: Packing fraction vs. time (in units of taps) (∘) observed experimentally in [1].

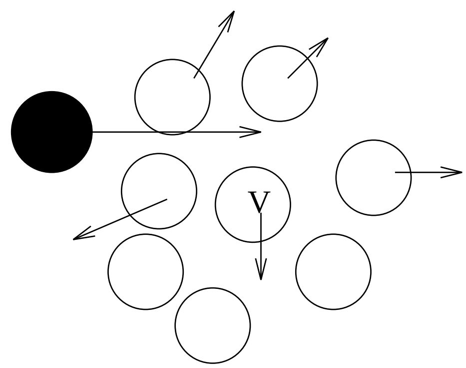
- Experimental set-up — bulk density was measured at various heights of the column of tapped (shaken) glass beads. Before every tap the beads are at rest, and time is in units of taps.

- Even after 10^5 taps, density is still increasing. The data is consistent with an inverse logarithmic time dependence.

$$\rho(t) = \rho_\infty - \frac{\rho_\infty - \rho_0}{1 + \ln(t/\tau)}$$

$\boxed{\textbf{Slow density relaxation}}$

II. A Heuristic Argument



ρ =Volume fraction, V =Particle volume, V_0 =Pore volume/particle

$$\rho = \frac{V}{V+V_0} \qquad \text{or} \qquad V_0 = V\frac{1-\rho}{\rho}$$

How many particles N move to make space for an additional particle?

$$NV_0 = V \qquad \text{or} \qquad N = \frac{\rho}{1-\rho}$$

Assuming particles move randomly, the time T for this event is

$$T = e^N = e^{\frac{\rho}{1-\rho}}$$

The density growth rate $\propto T^{-1} \propto e^{-N}$ is exponentially suppressed

$$\frac{d\rho}{dt} \propto (1-\rho)e^{-\frac{\rho}{1-\rho}}$$

long waiting time gives rise to logarithmically slow density increase

$$\rho(t) = 1 - \frac{1}{\ln t}$$

$$\boxed{\textbf{Volume exclusion causes slow relaxation}}$$

III. A Solvable Model in One Dimension

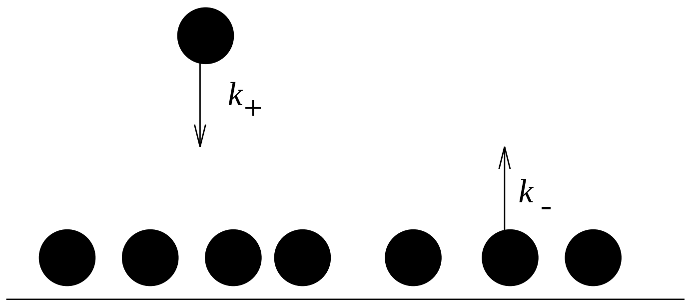


Fig. 2: The stochastic parking process.

Identical particles adsorb with rate k_+ on a continuous $1D$ substrate and desorbs with rate k_-. Adsorption subject to volume availability.

$P(x,t)$ — The distribution of voids of length x. Normalized to

$$\rho(t) = \int_0^\infty dx P(x,t) \qquad 1 = \int_0^\infty dx (x+1) P(x,t).$$

Void density master equation

$$\frac{dP(x,t)}{dt} = -2k_- P(x,t) + 2k_+ \int_{x+1}^\infty dy P(y,t)$$

$$+\theta(x-1)\left[\frac{k_-}{\rho(t)}\int_0^{x-1} dy P(y,t) P(x-1-y,t) - k_+(x-1)P(x,t)\right]$$

Density equation

$$\frac{\partial \rho(t)}{\partial t} = k_+ \int_1^\infty dx (x-1) P(x,t) - k_- \rho(t)$$

Exact Equilibrium Properties ($t = \infty$, $\partial/\partial t = 0$)

Exponential void density

$$P_\infty(x) = \beta e^{-\alpha x} \qquad \alpha e^\alpha = k \equiv \frac{k_+}{k_-}.$$

Density

$$\rho_\infty = \frac{\alpha}{1+\alpha} \cong \begin{cases} k & k \ll 1 \quad \text{(dilute)} \\ 1 - \frac{1}{\ln k} & k \gg 1 \quad \text{(dense)} \end{cases}$$

Near Equilibrium Approximation

Combining equilibrium density with the master equation gives

$$\frac{\partial \rho}{\partial t} = k_+(1-\rho)e^{-\frac{\rho}{1-\rho}} - k_-\rho.$$

Same rate reduction - heuristic argument exact in 1D

When $k = k_+/k_- \to \infty$, the asymptotic solution for density is

$$\rho(t) \cong 1 - \frac{1}{\ln t}$$

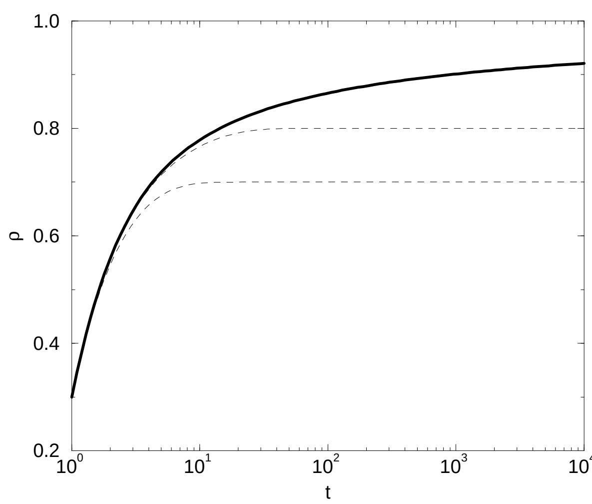


Fig. 3: Density curves for finite (- - -) and infinite (—) rate ratio k.

- **Density fluctuations** — Experimental power spectrum of the density fluctuations in the steady state are similar to those obtained by simulations of the parking process.

- **Size segregation** — The mobility of a particle decays exponentially with its volume. This explains why polydisperse grains size segregate.

Conclusions

- Interaction between grains is hard core.

- Granular ensembles relax logarithmically slow because large co-operative motion is necessary due to volume exclusion.

References

[1] J. B. Knight, C. G. Fandrich, C. N. Lau, H. M. Jaeger, and S. R. Nagel, *Phys Rev. E* **51**, 3957 (1995).

[2] E. Ben-Naim, J. B. Knight, and E. R. Nowak, H. M. Jaeger, and S. R. Nagel, *Physica D* **123**, 380 (1998).

[3] P. L. Krapivsky and E. Ben-Naim, *J. Chem. Phys.* **100**, 6778 (1994).