Alignment of Rods and Partition of Integers

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Poster, papers available from: http://cnls.lanl.gov/~ebn

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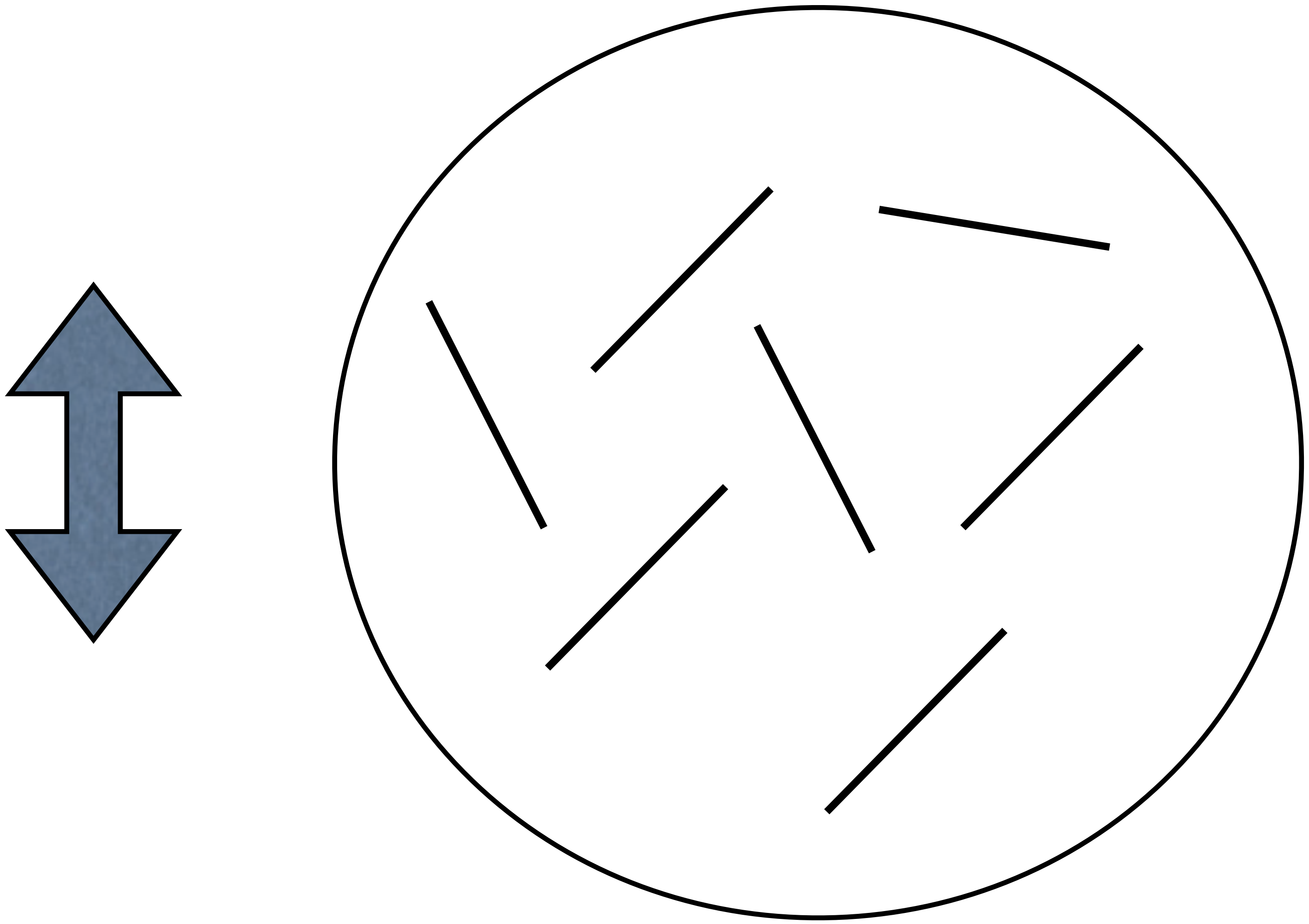
“A shaken dish of toothpicks”



Motivation

- Biology: molecular motors
- Ecology: flocking
- Granular matter: granular rods and chains
- Phase synchronization

The rod alignment model

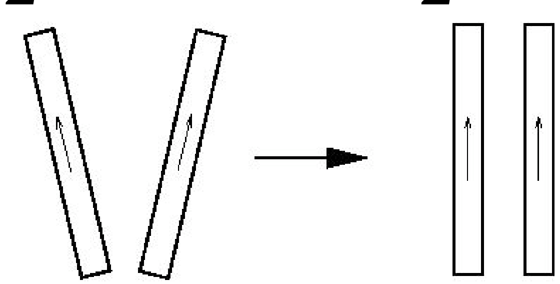
Aronson & Tsimring 05

- Each rod has an orientation

$$-\pi \leq \theta \leq \pi$$

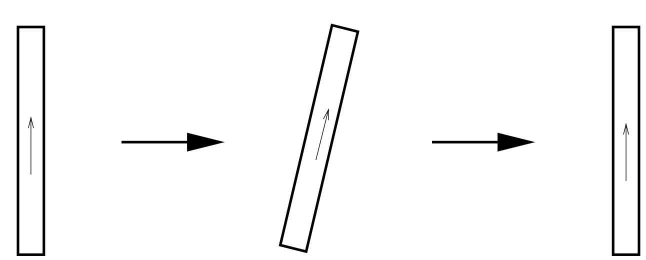
I. Alignment by pairwise interactions (nonlinear)

$$(\theta_1, \theta_2) \rightarrow \begin{cases} \left(\frac{\theta_1+\theta_2}{2}, \frac{\theta_1+\theta_2}{2}\right) & |\theta_1 - \theta_2| < \pi \\ \left(\frac{\theta_1+\theta_2+2\pi}{2}, \frac{\theta_1+\theta_2+2\pi}{2}\right) & |\theta_1 - \theta_2| > \pi \end{cases}$$



II. Diffusive wiggling (linear)

$$\frac{d\theta_j}{dt} = \eta_j(t) \qquad \langle \eta_j(t)\eta_j(t')\rangle = 2D\delta(t-t')$$



Kinetic theory

- Nonlinear integro-differential equation

$$\frac{\partial P}{\partial t} = D\frac{\partial^2 P}{\partial \theta^2} + \int_{-\pi}^{\pi} d\phi\, P\left(\theta - \frac{\phi}{2}\right) P\left(\theta + \frac{\phi}{2}\right) - P.$$

- Fourier transform

$$P_k = \langle e^{-ik\theta}\rangle = \int_{-\pi}^{\pi} d\theta e^{-ik\theta} P(\theta) \qquad P(\theta) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} P_k e^{ik\theta}$$

- Closed nonlinear equation

$$(1 + Dk^2)P_k = \sum_{i+j=k} A_{i-j} P_i P_j$$

- Coupling constants

$$A_q = \frac{\sin\frac{\pi\, q}{2}}{\frac{\pi q}{2}} = \begin{cases} 1 & q = 0 \\ 0 & q = 2, 4, \cdots \\ (-1)^{\frac{q-1}{2}}\frac{2}{\pi|q|} & q = 1, 3, \cdots \end{cases}$$

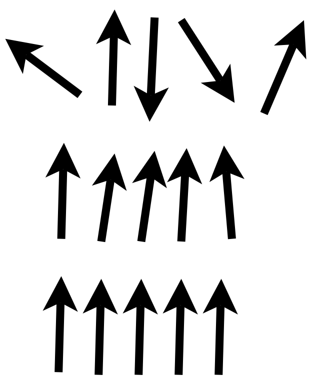
The order parameter

- Lowest order Fourier mode

$$R = |\langle e^{i\theta} \rangle| = |P_{-1}|$$

- Probes the state of the system

$$R = \begin{cases} 0 & \text{disordered} \\ 0.4 & \text{partially ordered} \\ 1 & \text{perfectly ordered} \end{cases}$$



The Fourier Equation

- Compact Form

$$P_k = \sum_{\substack{i+j=k \\ i\neq 0,\, j\neq 0}} G_{i,j} P_i \, P_j$$

- Transformed coupling constants

$$G_{i,j} = \frac{A_{i-j}}{1+D(i+j)^2-2A_{i+j}}$$

- Properties

$$\begin{aligned} G_{i,j} &= G_{j,i} \\ G_{i,j} &= G_{-i,-j} \\ G_{i,j} &= 0, \qquad \text{for} \qquad |i-j| = 2, 4, \ldots . \end{aligned}$$

Solution

- Repeated iterations (product of three modes)

$$P_k = \sum_{\substack{i+j=k \\ i\neq 0,\, j\neq 0}} \sum_{\substack{l+m=j \\ l\neq 0,\, m\neq 0}} G_{i,j}\, G_{l,m}\, P_i\, P_l\, P_m.$$

- When k=2,4,8,...

$$\begin{aligned} P_2 &= G_{1,1} P_1^2 \\ P_4 &= G_{2,2} P_2^2 = G_{2,2} G_{1,1}^2 P_1^4. \end{aligned}$$

- Generally

$$\begin{aligned} P_3 &= 2G_{1,2} P_1 P_2 + 2G_{-1,4} P_{-1} P_4 + \cdots \\ &= 2G_{1,2} G_{1,1} P_1^3 + 2G_{-1,4} G_{2,2} G_{1,1}^2 P_1^4 P_{-1} \cdots \end{aligned}$$

Partition of Integers

- Diagramatic solution

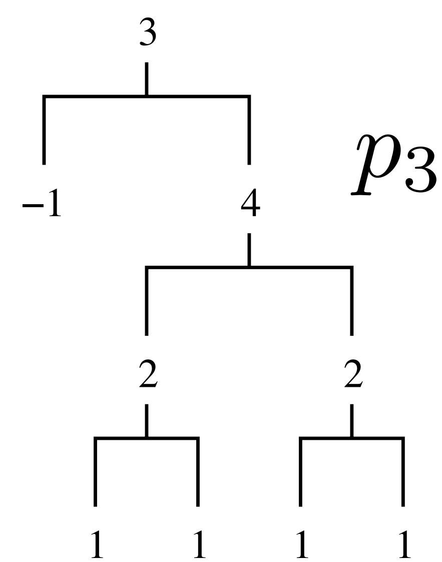
$$P_k = R^k \sum_{n=0}^{\infty} p_{k,n} \, R^{2n}$$

- Partition

$$k = \underbrace{1 + 1 + \cdots + 1 + 1}_{k+n} \underbrace{-1 - \cdots - 1}_{n} .$$

- Partitions rules

$$\begin{aligned} k &= i + j \\ i &\neq 0 \\ j &\neq 0 \\ G_{i,j} &\neq 0 \end{aligned}$$



$$p_{3,1} = 2G_{-1,4}G_{2,2}G_{1,1}^2$$

All modes expressed in terms of order parameter

The order parameter

- Infinite series solution

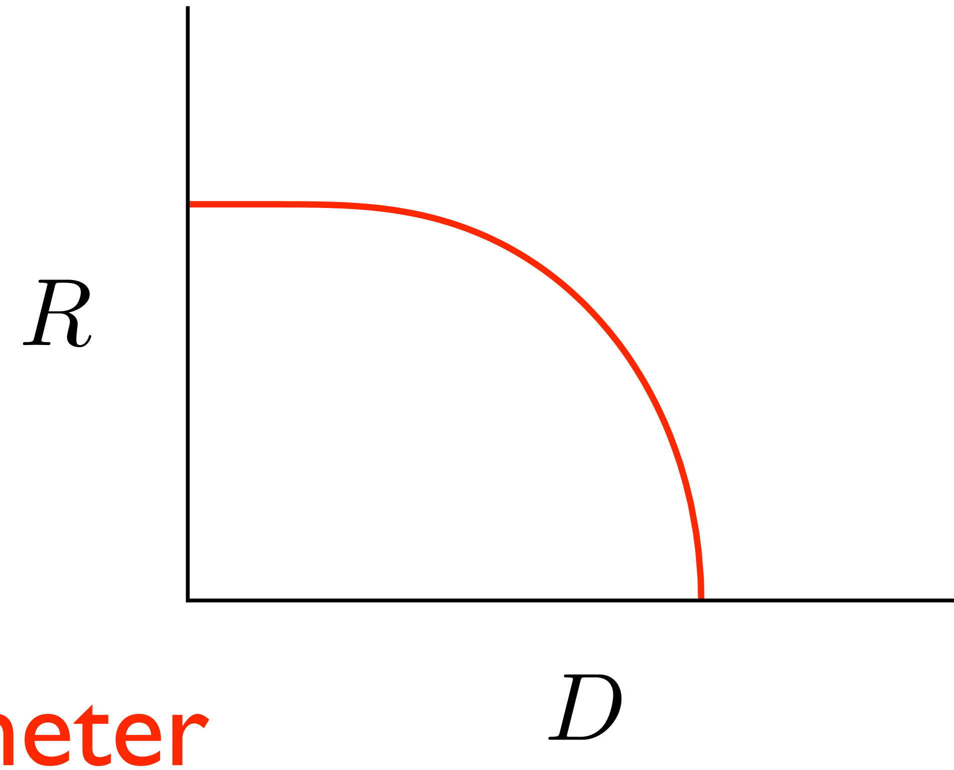
$$R = R^k \sum_{n=0}^{\infty} p_{1,n}\, R^{2n}$$

- Landau theory

$$R = \frac{C}{D_c - D} R^3 + \cdots$$

- Critical diffusion constant

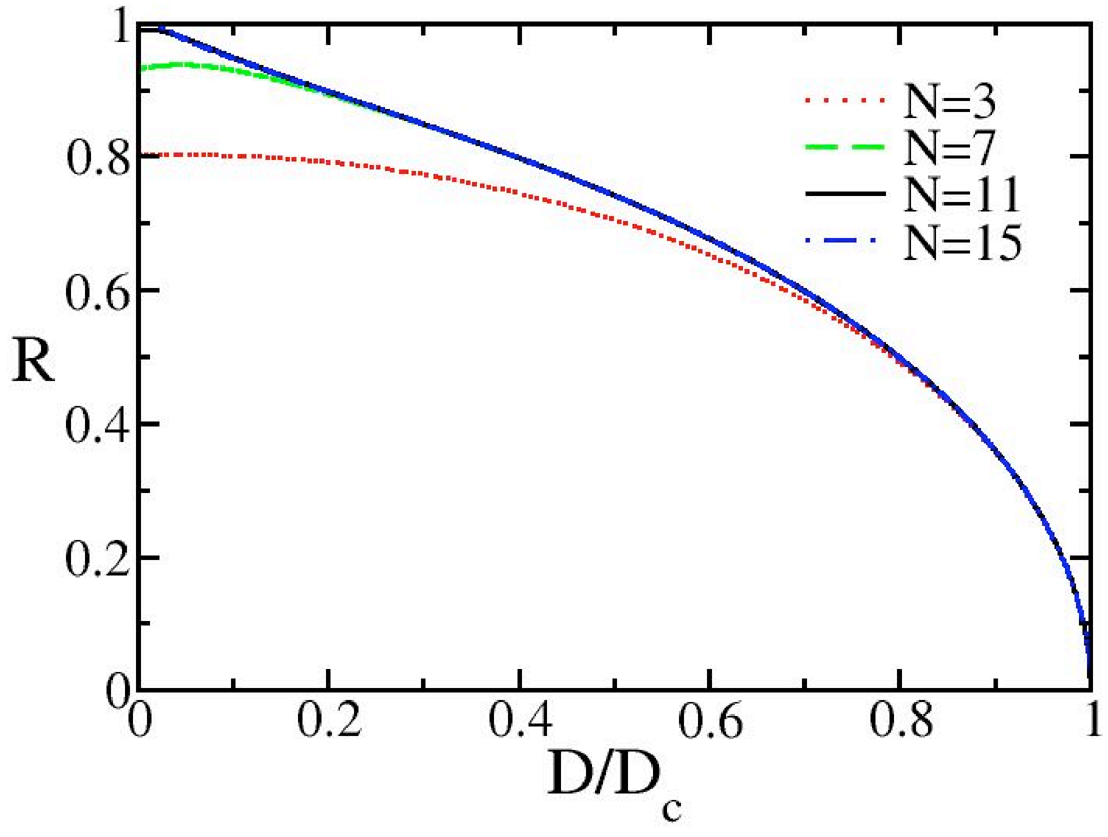
$$D_c = \frac{4}{\pi} - 1$$



Close equation for order parameter

Nonequilibrium phase transition

- Critical diffusion constant $D_c = \frac{4}{\pi} - 1$
- Subcritical: ordered phase $R > 0$
- Supercritical: disordered phase $R = 0$
- Critical behavior $R \sim (D_c - D)^{1/2}$

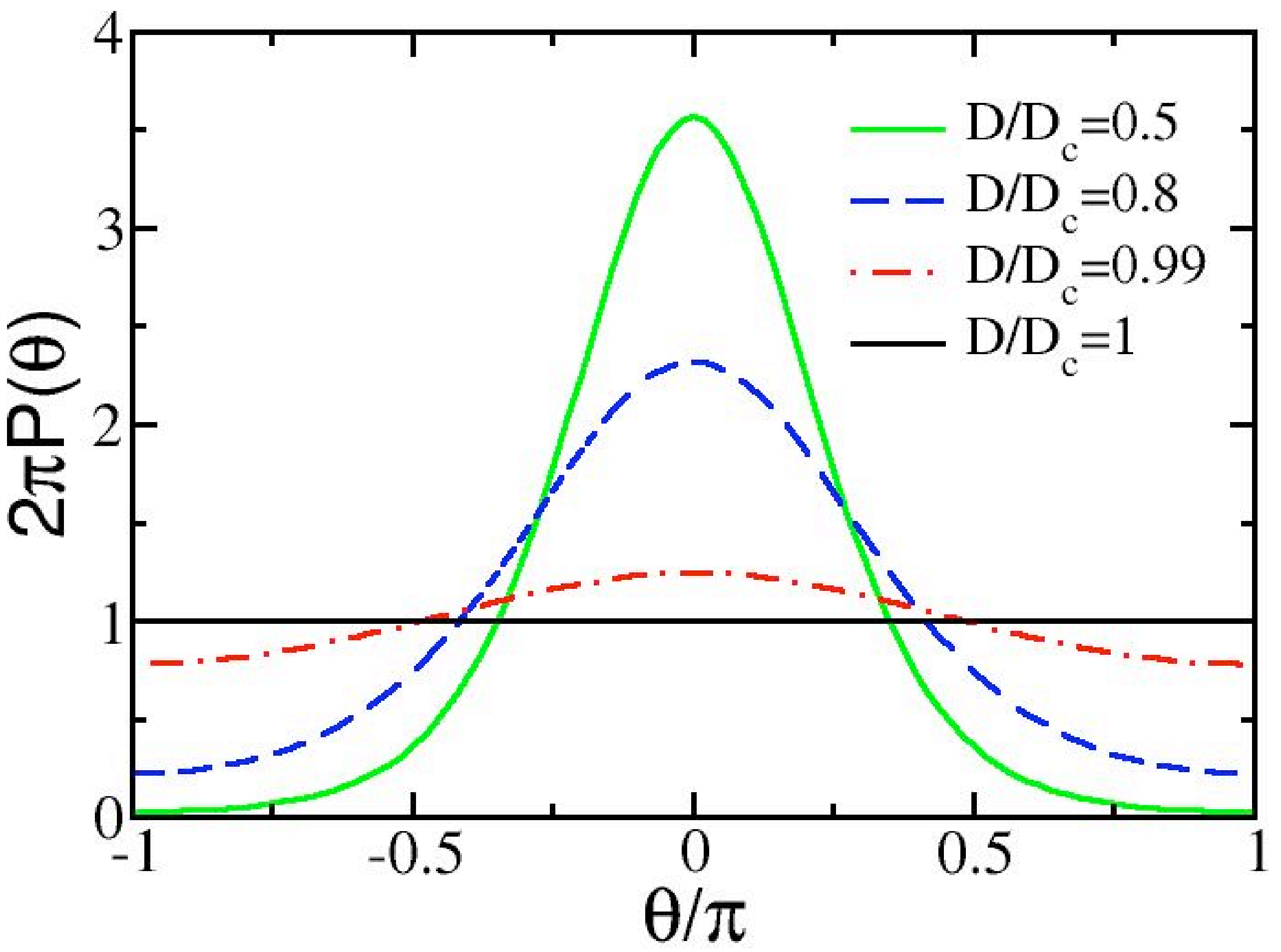


Distribution of orientation

- Fourier modes decay exponentially with R

$$P_k \sim R^k$$

- Small number of modes sufficient in practice



$$P(\theta) = \frac{1}{2\pi}\left[1 + 2R\cos\theta + 2G_{1,1}R^2\cos(2\theta) + 4G_{1,2}G_{1,1}R^3\cos(3\theta) + \cdots\right]$$

General alignment rates

- Alignment rate

$$K(|\theta_1 - \theta_2|)$$

- Diagramatic solution holds
- Hard-rods

$$K(\phi) \propto |\sin \phi| \qquad D_c = \frac{1}{3}$$

- Hard-spheres: system always disordered

$$K(\phi) \propto |\phi|$$

Boltzmann equation can be solved!
Phase transition may or may not exist

Experiments

Conclusions

- Nonequilibrium phase transition
- Weak noise: ordered phase (nematic)
- Strong noise: disordered phase
- Solution relates to iterated partition of integers
- Only when Fourier spectrum is discrete: exact solution possible for arbitrary averaging rates

E. Ben-Naim and P.L. Krapivsky, Phys. Rev. E **73**, 031109 (2006)